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Smoothing property of an evolution process associated with semilinear heat equation with delay on an interval with moving ends

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Abstract. The goal of this work is to prove the smoothing property of the evolution process associated with the semilinear heat equation with delay, which is defined on a one-dimensional moving boundary domain. Then, as a consequence of the smoothing property, we can estimate the fractal dimension of the pullback attractors associated with this parabolic problem.

Keywords: Nonlinear heat equation, delay differential equations, moving boundary, pullback attractors, fractal dimension.

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1 Introduction

Let $\alpha, \beta : \mathbb{R} \to \mathbb{R}$ be two functions such that $\beta(t) > \alpha(t)$ for all $t \in \mathbb{R}$. Let us denote by $\mathcal{I}_t := \big(\alpha(t), \beta(t)\big)$ the open interval at the time $t \in \mathbb{R}$, with ends $\alpha(t)$ and $\beta(t)$, and whose length is given by the function $\gamma(t) = |\mathcal{I}_t| = \beta(t) - \alpha(t) > 0$. Thus, on the functions α, β and γ consider the following hypotheses:

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(H1) $\alpha, \beta \in C^2(\mathbb{R})$ and $0 < \gamma_0 < \gamma(t) < \gamma_1$ for all $t \in \mathbb{R}$;

(H2) $\alpha', \beta' \in L^\infty(\mathbb{R})$.

The following hypothesis refers to the delay term:

(H3) For $h > 0$ fixed, let $\delta(t)$ be a C^1-function (or continuously differentiable function) such that $\delta(t) \geq 0$ for all $t \in \mathbb{R}$, $h = \sup_{t\in\mathbb{R}} \delta(t) > 0$ and $\delta_* = \sup_{t\in\mathbb{R}} \delta'(t) < 1$.

And, for the pullback asymptotic analysis the following conditions will be necessary:

(H4) We will assume that

$$\mathbf{I}_t := \bigcup_{s\leq t} \mathcal{I}_s = \bigcup_{s\leq t} \big(\alpha(s), \beta(s)\big) \text{ is bounded for any } t \in \mathbb{R}.$$

Given $\tau, T \in \mathbb{R}$ with $\tau \leq T$, we define the non-cylindrical regions $Q_{\tau,T}$ and Q_τ as

$$Q_{\tau,T} := \bigcup_{t\in(\tau,T)} \mathcal{I}_t \times \{t\} \qquad \text{and} \qquad Q_\tau := \bigcup_{t\in(\tau,+\infty)} \mathcal{I}_t \times \{t\}.$$

Finally, let us denote by $\mathcal{I} = (0,1)$ and for each $t \in \mathbb{R}$ consider the map $r_t = r(t) : \mathcal{I}_t \to \mathcal{I}$ defined as

$$r_t(x) = r(x,t) = \frac{x - \alpha(t)}{\gamma(t)}, \quad \forall x \in \mathcal{I}_t,$$

and for each $t \in \mathbb{R}$ let us denote by $\widetilde{r}_t := r^{-1}_{t-\delta(t)} \circ r_t : \mathcal{I}_t \to \mathcal{I}_{t-\delta(t)}$.

Given $\tau \in \mathbb{R}$, the heat transfer equation with delay on the non-cylindrical region Q_τ and with homogeneous Cauchy-Dirichlet boundary conditions, denoted by (DHE), is:

$$\begin{cases} \dfrac{\partial u}{\partial t} - c_0 \dfrac{\partial^2 u}{\partial x^2} + g(u) = f(t) + u\big(\widetilde{r}_t, t - \delta(t)\big) \quad \text{in } Q_\tau, \\ u(\alpha(t), t) = u(\beta(t), t) = 0 \quad \forall t \in [\tau, +\infty), \\ u(\tau) = u^\tau \quad \text{in } \mathcal{I}_\tau, \\ u(\tau + s) = \phi(s) \quad \text{in } \mathcal{I}_{\tau+s} \text{ and } s \in (-h, 0), \end{cases} \tag{DHE}$$

where $c_0 > 0$ is the thermal conductivity, u is the temperature function, u^τ is the temperature in the initial time $t = \tau$, ϕ is the initial condition with memory defined on interval $(-h,0)$ with $h>0$, and $f\in L^1_{loc}(Q_\tau)$ is the heat source, and $g\in C^1(\mathbb{R})$ is a given function for which there exist non-negative constants $\alpha_0,\alpha_1,\beta_0$ and l, and $p\geq 2$ such that

$$-\beta_0+\alpha_0|s|^p\leq g(s)s\leq \beta_0+\alpha_1|s|^p \quad \forall s\in\mathbb{R},$$

and

$$g'(s)\geq -l \qquad \forall s\in\mathbb{R}. \tag{1.1}$$

Before commenting further on this, let us do a timeline of what has already known about the problems related to the system (DHE). Between the years 2008 and 2009, the n-dimensional case of the problem (DHE) was treated in [2, 3], but without a memory term. In their paper is proved the existence of strong and weak solutions, and the existence of pullback attractors on tempered universes. In the year 2022, in [5] the authors worked on problem (DHE), proving the existence of strong and weak solutions, and the existence of a pullback attractor with finite fractal dimension. To estimate the fractal dimension of the pullback attractor, the Lyapunov exponent method was used, and for this, it was necessary to assume that the nonlinear function $g\in C^2(\mathbb{R})$. Recently in [6] the n-dimensional case of problem (DHE) was studied, with a more general memory term. The authors proved the existence of strong and weak solutions, and the finite fractal dimension of the pullback attractor. The last one was possible by assuming that the nonlinear term $g\in C^2(\mathbb{R})$ is globally Lipschitz, and with this guarantee that the evolution process satisfies the smoothing property.

The main objective of this work is to study the fractal dimension of the pullback attractors of (DHE) via the smoothing property of the evolution process associated. For this, it is sufficient to assume that the nonlinear function $g\in C^1(\mathbb{R})$, which is an improvement compared to the paper [6].

This paper is organized as follows: In Section 2 we present some results and definitions that are fundamental to have a better understanding of the paper, for example, a special class of Bochner spaces and results about fractal dimension associated with evolution processes acting on families of Banach spaces. We also remember the results of the existence, regularity, and uniqueness of weak solutions, and the existence of pullback attractors associated with (DHE). Section 3 is dedicated to obtaining results on the continuous dependence on initial data taken in the sections of the pullback attractor. Finally, the Section 4 is devoted to estimating the fractal dimension of each section (or fiber) of the pullback attractors associated with the problem (DHE).

2 Abstract Setting of the Problem and Known Results

2.1 Functional spaces

In order to state the problem in the correct framework, let us consider the following time-dependent Banach spaces: For each $t \in \mathbb{R}$ let us denote by $|\cdot|_{p,t}$ the norm of the Banach space $L^p(\mathcal{I}_t)$. For the special case $p = 2$, the norm of the Hilbert space $L^2(\mathcal{I}_t)$ will be denoted by $|\cdot|_t$ and its inner product by $(\cdot,\cdot)_t$. The norm for the Sobolev space $H_0^1(\mathcal{I}_t)$ will be denoted by $\|\cdot\|_t$. Finally, for each $t \in \mathbb{R}$, denote by $\langle\cdot,\cdot\rangle_{-1,t}$ the duality product between $H^{-1}(\mathcal{I}_t)$ and $H_0^1(\mathcal{I}_t)$.

On the other hand, in the same way as [3, 5, 6], we are going to consider a special class of Bochner spaces that are defined on time-dependent Banach spaces. Then, for each $\tau \in \mathbb{R}$, let us define by:

$$C_{L^2}(\mathcal{I}_\tau) := \Big\{\phi \in C\big([-h,0];\cup_{s\in[-h,0]}L^2(\mathcal{I}_{\tau+s})\big) : \phi(s) \in L^2(\mathcal{I}_{\tau+s})\ \forall s \in [-h,0]\Big\},$$

with norm defined by $\|\phi\|_{C_{L^2}(\mathcal{I}_\tau)} := \sup_{s\in[-h,0]} |\phi(s)|_{\tau+s}$, for all $\phi \in C_{L^2}(\mathcal{I}_\tau)$.

$$L^2_{L^2}(\mathcal{I}_\tau) := \Big\{\phi \in L^2\big(-h,0;\cup_{s\in[-h,0]}L^2(\mathcal{I}_{\tau+s})\big) : \phi(s) \in L^2(\mathcal{I}_{\tau+s})\ a.e.\ s \in [-h,0]\Big\},$$

with norm $\|\phi\|^2_{L^2_{L^2}(\mathcal{I}_\tau)} := \int_{-h}^0 |\phi(s)|^2_{\tau+s}ds$, for all $\phi \in L^2_{L^2}(\mathcal{I}_\tau)$.

$$C_{H_0^1}(\mathcal{I}_\tau) := \Big\{\phi \in C\big([-h,0];\cup_{s\in[-h,0]}H_0^1(\mathcal{I}_{\tau+s})\big) : \phi(s) \in H_0^1(\mathcal{I}_{\tau+s})\ \forall s \in [-h,0]\Big\},$$

with norm $\|\phi\|_{C_{H_0^1}(\mathcal{I}_\tau)} := \sup_{s\in[-h,0]} \|\phi(s)\|_{\tau+s}$, for all $\phi \in C_{L^2}(\mathcal{I}_\tau)$, and

$$L^2_{H_0^1}(\mathcal{I}_\tau) := \Big\{\phi \in L^2\big(-h,0;\cup_{s\in[-h,0]}H_0^1(\mathcal{I}_{\tau+s})\big) : \phi(s) \in H_0^1(\mathcal{I}_{\tau+s})\ a.e.\ s \in [-h,0]\Big\},$$

with norm $\|\phi\|^2_{L^2_{H_0^1}(\mathcal{I}_\tau)} := \int_{-h}^0 \|\phi(s)\|^2_{\tau+s}ds$, for all $\phi \in L^2_{H_0^1}(\mathcal{I}_\tau)$.

Observe that $C_{L^2}(\mathcal{I}_\tau) \hookrightarrow L^2_{L^2}(\mathcal{I}_\tau)$ and $C_{H_0^1}(\mathcal{I}_\tau) \hookrightarrow L^2_{H_0^1}(\mathcal{I}_\tau) \hookrightarrow L^2_{L^2}(\mathcal{I}_\tau)$ for all $\tau \in \mathbb{R}$. On the other hand, we consider

$$M^2_{L^2}(\mathcal{I}_\tau) := L^2(\mathcal{I}_\tau) \times L^2_{L^2}(\mathcal{I}_\tau) \quad \text{with norm} \quad \|(u,\phi)\|^2_{M^2_{L^2}(\mathcal{I}_\tau)} := |u|^2_\tau + \|\phi\|^2_{L^2_{L^2}(\mathcal{I}_\tau)}.$$

and

$$M^2_{H_0^1}(\mathcal{I}_\tau) := H_0^1(\mathcal{I}_\tau) \times L^2_{H_0^1}(\mathcal{I}_\tau) \quad \text{with norm} \quad \|(u^\tau,\phi)\|^2_{M^2_{H_0^1}(\mathcal{I}_\tau)} := \|u^\tau\|^2_\tau + \|\phi\|^2_{L^2_{H_0^1}(\mathcal{I}_\tau)}.$$

Note that $L^2(\mathcal{I}_\tau)\times C_{L^2}(\mathcal{I}_\tau) \hookrightarrow M^2_{L^2}(\mathcal{I}_\tau)$ and $M^2_{H^1_0}(\mathcal{I}_\tau) \hookrightarrow M^2_{L^2}(\mathcal{I}_\tau)$ for all $\tau \in \mathbb{R}$.

On the other hand, let $\{X_t\}_{t\in\mathbb{R}}$ be a family of Banach spaces and consider the interval $[-h, 0]$ with $h > 0$. Given a function u defined on $\mathbb{R}$, such that $u(t) \in X_t$ for each $t \in \mathbb{R}$, then for $t \in \mathbb{R}$ we denote by u_t the function:

$$\begin{aligned} u_t :&[-h,0] \to \{X_{t+s} : s \in [-h,0]\} \\ &s \quad \mapsto \quad u_t(s) = u(t+s) \in X_{t+s}. \end{aligned}$$

For example, let $\tau \in \mathbb{R}$ and $h > 0$. Thus, if $u \in L^2(\tau - h, \tau; L^2(\mathcal{I}_t))$, then $u_t \in L^2_{L^2}(\mathcal{I}_\tau)$.

2.2 Fractal dimension

In this section we introduce the definition of fractal dimension of compact sets, e.g. [1, 7]. We also state a recent result, see [6, Theorem 4.11], to estimate the fractal dimension of families of time-dependent compact sets, which are associated with a family of time-dependent maps that satisfy the smoothing property.

Definition 2.1. Let X be a metric space and K a compact subset of X. The fractal dimension of K is defined by

$$dim_f(K, X) = \limsup_{r\to 0} \frac{\log N_X(K, r)}{-\log r},$$

where $N_X(K, r)$ is the minimum number of balls of radius r that cover K.

Proposition 2.2. *(cf. [1, Lemma 4.2]) Let X, Y be two normed spaces. Consider $\mathscr{C} \subset X$, and $f : \mathscr{C} \to Y$ a Hölder continuous function with exponent θ, $\theta \in (0, 1]$, i.e. there exists an $L > 0$ such that*

$$\|f(x) - f(y)\|_Y \leq L\|x - y\|^\theta_X,$$

for all $x, y \in \mathscr{C}$. Then

$$dim_f(f(\mathscr{C}), Y) \leq \frac{dim_f(\mathscr{C}, X)}{\theta}.$$

Theorem 2.3. *(cf. [6, Theorem 4.11]) Let $\{X_t\}_{t\in\mathbb{R}}$ and $\{Y_t\}_{t\in\mathbb{R}}$ families of normed vector spaces such that Y_t is compactly embedded in X_t for every $t \in \mathbb{R}$. Assume that $\{\mathscr{C}_t\}_{t\in\mathbb{R}}$ is a family of bounded subsets of $\{X_t\}_{t\in\mathbb{R}}$ and there exists* *a non-decreasing function* $\varrho : \mathbb{R} \to (0, +\infty)$ *such that for each* $t \in \mathbb{R}$ *there exists* $u_t \in \mathscr{C}_t$ *satisfying*

$$\mathscr{C}_t \subset B_{X_t}(u_t, \varrho(t)) \quad \text{for all} \quad t \in \mathbb{R}. \tag{2.1}$$

Assume that there exists a family of operators $\{L_t : X_{t-1} \to Y_t\}_{t \in \mathbb{R}}$ *such that*

- **Negative invariance** $\mathscr{C}_t \subset L_t \mathscr{C}_{t-1}$ *for all* $t \in \mathbb{R}$;
- **Smoothing property** *there exists a function* $\kappa : \mathbb{R} \to (0, +\infty)$ *such that*

$$\|L_t x - L_t y\|_{Y_t} \leq \kappa(t) \|x - y\|_{X_{t-1}}, \ \forall x, y \in \mathscr{C}_{t-1}, \ \forall t \in \mathbb{R};$$

- **Entropy control** *also assume that there exists a discrete function* $\mathcal{N} : \mathbb{R} \to \mathbb{N}$ *such that*

$$\sup_{s \leq t} N^s_{1/4\kappa(s)} \leq \mathcal{N}(t),$$

where $N^t_\epsilon := N_{X_t}(B_{Y_t}(0, 1), \epsilon)$ *for any* $\epsilon > 0$ *and* $t \in \mathbb{R}$.

Then,

$$\sup_{s \leq t} dim_f(\mathscr{C}_s, X_s) \leq \frac{\log \mathcal{N}(t)}{\log 2} \quad \text{for all} \quad t \in \mathbb{R}.$$

Remark 2.4. Theorem 2.3 can be formulated in the discrete framework, i.e., $\{X_m\}_{m \in \mathbb{Z}}$, $\{Y_m\}_{m \in \mathbb{Z}}$, $\{L_m\}_{m \in \mathbb{Z}}$ and $\{\mathscr{C}_m\}_{m \in \mathbb{Z}}$ such that $Y_m \hookrightarrow\hookrightarrow X_m$ (Y_m is compactly embedded in X_m), $\mathscr{C}_m \subset L_m \mathscr{C}_{m-1}$ for all $m \in \mathbb{Z}$, and all conditions of Theorem 2.3 are satisfied. On the other hand, keep in mind also that Theorem 2.3 can be formulated not for all time, but for we can assume the existence of a maximum time $t_0 \in \mathbb{R}$ or $\mathbb{Z}$, such that all the conditions of the theorem are valid for all $t \leq t_0$.

Lemma 2.5. *(cf. [6, Lemma 4.13]) Let* $\mathcal{X}, \mathcal{Y}$ *be normed vector spaces and* $F : \mathcal{X} \to \mathcal{Y}$ *an isometry, that is,*

$$\|F(u) - F(v)\|_{\mathcal{Y}} = \|u - v\|_{\mathcal{X}}, \ \forall u, v \in \mathcal{X}.$$

Assume that W' *is a totally bounded subset of* $\mathcal{Y}$ *such that*

$$F(W) \subset W'. \tag{2.2}$$

Then, W *is totally bounded in* $\mathcal{X}$ *and for any* $\epsilon > 0$ *we have*

$$N_{\mathcal{X}}[W, \epsilon] \leq N_{\mathcal{Y}}[W', \epsilon].$$

2.3 Weak solutions to (DHE)

This section is devoted to recalling the existence, uniqueness and regularity of weak solutions of the problem (DHE). It should be noted that the demonstrations of these results are done via the Galerkin method, which can be consulted in [5, Theorem 3.2], also see [2, 3, 6, 7].

For $\tau < T$, let us start defining the space of the test functions as

$$\mathcal{U}_{\tau,T} = \left\{\varphi \in L^2(\tau, T; H_0^1(\mathcal{I}_t)) \cap L^p(Q_{\tau,T}) : \frac{\partial \varphi}{\partial t} \in L^2(Q_{\tau,T}),\ \varphi(\tau) = \varphi(T) = 0\right\}.$$

Definition 2.6. Given $f \in L^2(\tau, T; H^{-1}(\mathcal{I}_t))$, by weak solution of (DHE) associated to the initial condition $(u^\tau, \phi) \in M^2_{L^2}(\mathcal{I}_\tau)$, we understand a function u, belonging to the class

$$\begin{cases} u \in L^2(Q_{\tau-h,T}) \cap L^p(Q_{\tau,T}) \cap C([\tau, T]; L^2(\mathcal{I}_t)) \cap L^2(\tau, T; H_0^1(\mathcal{I}_t)), \\ \dfrac{\partial u}{\partial t} \in L^2(\tau, T; H^{-1}(\mathcal{I}_t)) + L^q(Q_{\tau,T}), \end{cases} \tag{2.3}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, which satisfies the weak formulation

$$\begin{aligned} \int_{Q_{\tau,T}} -u(x,t)\frac{\partial \varphi}{\partial t}(x,t) + c_0 \frac{\partial u}{\partial x}(x,t)\frac{\partial \varphi}{\partial x}(x,t) + g(u(x,t))\varphi(x,t)dxdt \\ = \int_\tau^T \langle f(t), \varphi(t)\rangle_{-1,t} dt + \int_{Q_{\tau,T}} u(\widetilde{r}_t(x), t-\delta(t))\varphi(x,t)dxdt, \end{aligned} \tag{2.4}$$

for all $\varphi \in \mathcal{U}_{\tau,T}$ and

$$u(\tau) = u^\tau \text{ and } u(\tau + s) = \phi(s) \text{ a.e. } s \in (-h, 0). \tag{2.5}$$

And by strong solution, associated to the initial condition $(u^\tau, \phi) \in M^2_{L^2}(\mathcal{I}_\tau)$, we understand a function u satisfying (2.4) and (2.5), belonging to the class

$$\begin{cases} u \in L^2(Q_{\tau-h,T}) \cap L^2(\tau, T; H^2(\mathcal{I}_t)) \cap C([\tau, T]; H_0^1(\mathcal{I}_t)) \cap L^\infty(\tau, T; L^p(\mathcal{I}_t)), \\ \dfrac{\partial u}{\partial t} \in L^2(\tau, T; L^2(\mathcal{I}_t)). \end{cases}$$

Now, we are in position to state the result of existence, regularity and uniqueness of weak solution for (DHE). The proof of this result is via Galerkin method.

Theorem 2.7. *(Existence, regularity and uniqueness, cf. [5, Theorem 3.2, Theorem 3.5]) Suppose that conditions (H1)-(H3) hold. Let us consider τ, T with $\tau < T$, $(u^\tau, \phi) \in M^2_{L^2}(\mathcal{I}_\tau)$, and $f \in L^2(\tau, T; H^{-1}(\mathcal{I}_t))$. Then, there exists a unique weak solution of the problem (DHE).*

• *If* $u^\tau \in H_0^1(\mathcal{I}_\tau)$, $\phi \in L^2_{L^2}(\mathcal{I}_\tau)$, *and* $f \in L^2(\tau,T;L^2(\mathcal{I}_t))$. *Then, the weak solution of the problem* (DHE) *belongs to*

$$u \in L^2(\tau,T;H^2(\mathcal{I}_t)) \cap L^\infty(\tau,T;H_0^1(\mathcal{I}_t)) \text{ with } \frac{\partial u}{\partial t} \in L^2(\tau,T;L^2(\mathcal{I}_t)).$$

Remark 2.8. If the initial condition with memory $\phi \in C_{L^2}(\mathcal{I}_\tau)$ with $\phi(0) = u^\tau$, then the weak solution of (DHE) u belongs to $C([\tau-h,T];L^2(\mathcal{I}_t))$. If u is strong solution and $\phi \in C_{H_0^1}(\mathcal{I}_\tau)$ with $\phi(0) = u^\tau$, then $u \in C([\tau-h,T];H_0^1(\mathcal{I}_t))$.
Note that any weak solution u to (DHE) can be taken as test function in the weak formulation (2.4). Thus, we obtain the *First Energy Equality*

$$\frac{1}{2}\frac{d}{dt}|u(t)|_t^2 + c_0\|u(t)\|_t^2 + (g(u),u)_t = \langle f,u\rangle_{-1,t} + \big(u(\widetilde{r}_t, t-\delta(t)),u\big)_t, \qquad (2.6)$$

a.e. $t \in (\tau,T)$.
• If u is a strong solution of (DHE), then we obtain the *Second Energy Equality*

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_t^2 + c_0\Big|\frac{\partial^2 u}{\partial x^2}\Big|_t^2 - \Big(g(u),\frac{\partial^2 u}{\partial x^2}\Big)_t = -\Big(f,\frac{\partial^2 u}{\partial x^2}\Big)_t - \Big(u\big(\widetilde{r}_t, t-\delta(t)\big),\frac{\partial^2 u}{\partial x^2}\Big)_t,$$

a.e. $t \in (\tau,T)$.

2.4 Pullback attractor associated with (DHE)

In this section we are interested in presenting the result about the existence of the pullback attractors associated with (DHE), on the families of Banach spaces $\{C_{L^2}(\mathcal{I}_t)\}_{t\in\mathbb{R}}$ and $\{M^2_{L^2}(\mathcal{I}_t)\}_{t\in\mathbb{R}}$.

Under the hypotheses (H1)-(H4) and $f \in L^2_{loc}(\mathbb{R};H^{-1}(\mathcal{I}_t))$, Theorem 2.7 guarantees the existence of the continuous evolution processes $\{U(t,\tau) : (t,\tau) \in \mathbb{R}^2_d\}$ and $\{S(t,\tau) : (t,\tau) \in \mathbb{R}^2_d\}$ defined as

$$U(t,\tau) : C_{L^2}(\mathcal{I}_\tau) \to C_{L^2}(\mathcal{I}_t) \quad \text{by} \quad U(t,\tau)\phi = u_t(\cdot;\tau,\phi), \qquad (2.7)$$

where $u(\cdot) = u(\cdot;\tau,\phi)$ is the unique weak solution of (DHE) associated to the initial condition $\phi \in C_{L^2}(\mathcal{I}_\tau)$ for any $(t,\tau) \in \mathbb{R}^2_d$, and

$$S(t,\tau) : M^2_{L^2}(\mathcal{I}_\tau) \to M^2_{L^2}(\mathcal{I}_t) \quad \text{by} \quad S(t,\tau)(u^\tau,\phi) = \big(u(t;\tau,u^\tau,\phi), u_t(\cdot;\tau,u^\tau,\phi)\big), \qquad (2.8)$$

where $u(\cdot) = u(\cdot;\tau,u^\tau,\phi)$ is the unique weak solution of (DHE) associated to the initial condition $(u^\tau,\phi) \in M^2_{L^2}(\mathcal{I}_\tau)$ for any $(t,\tau) \in \mathbb{R}^2_d$.

In what follows we are going to define the universes where we will look for pullback attractors associated with the evolution processes $U(\cdot,\cdot)$ and $S(\cdot,\cdot)$.

These universes will have a tempered condition associated with the dissipation of evolution processes, which are associated with a positive non-increasing function $\eta : \mathbb{R} \to (0, +\infty)$, such that for each $t \in \mathbb{R}$ we denote by $\eta_t := \eta(t)$. An example of this type of function is $\eta_t = \lambda_{1,t}$ for all $t \in \mathbb{R}$, that is the first eigenvalue of $-\frac{\partial^2}{\partial x^2}$ on $H_0^1(\mathbf{I}_t)$, i.e.

$$\lambda_{1,t} := \min_{w \in H_0^1(\mathbf{I}_t), w \neq 0} \frac{\left\| \frac{\partial w}{\partial x} \right\|^2_{L^2(\mathbf{I}_t)}}{\|w\|^2_{L^2(\mathbf{I}_t)}},$$

such that, under the assumption of hypothesis (H4), $\lambda_{1,(\cdot)} : \mathbb{R} \to (0, +\infty)$ with $t \mapsto \lambda_{1,t}$, is a non-increasing function, i.e. if $s \leq t$, then $\lambda_{1,t} \leq \lambda_{1,s}$.

Definition 2.9. (Tempered universes) Given a non-increasing function $\eta : \mathbb{R} \to (0, +\infty)$, we define by

1. $\mathcal{D}_\eta(C_{L^2})$ the class of all families of nonempty subsets $\widehat{D} = \{D(t) : t \in \mathbb{R},\ D(t) \subset C_{L^2}(\mathcal{I}_t) \text{ and } D(t) \neq \emptyset\}$ such that, for all $t \in \mathbb{R}$

$$\lim_{\tau \to -\infty} e^{\eta_t \tau} \sup_{\phi \in D(\tau)} \|\phi\|^2_{C_{L^2}(\mathcal{I}_\tau)} = 0.$$

2. $\mathcal{D}_\eta(M^2_{L^2})$ the class of all families of nonempty subsets $\widehat{D} = \{D(t) : t \in \mathbb{R},\ D(t) \subset M^2_{L^2}(\mathcal{I}_t) \text{ and } D(t) \neq \emptyset\}$ such that, for all $t \in \mathbb{R}$

$$\lim_{\tau \to -\infty} e^{\eta_t \tau} \sup_{(v,\phi) \in D(\tau)} \|(v, \phi)\|^2_{M^2_{L^2}(\mathcal{I}_\tau)} = 0.$$

Given $\eta : \mathbb{R} \to (0, +\infty)$ let us denote by $\mathcal{I}_*^{2,\eta}$ the set of all functions $f \in L^2_{loc}(\mathbb{R}; H^{-1}(\mathcal{I}_t))$ such that

$$\int_{-\infty}^t e^{\eta_t s} \|f(s)\|^2_{-1,s} ds < +\infty \qquad \forall t \in \mathbb{R}.$$

Theorem 2.10. *(cf. [5, Theorem 5.12, Remark 5.14]) Let $p > 2$ and suppose that conditions (H1)-(H4) hold. Assume that there exists a non-increasing bounded function $\eta : \mathbb{R} \to (0, +\infty)$ such that $f \in \mathcal{I}_*^{2,\eta}$. Then, the families $\widehat{B}_{\eta, M^2_{L^2}} = \{B_{M^2_{L^2}(\mathcal{I}_t)}[0, \mathcal{R}(t)] : t \in \mathbb{R}\}$ and $\widehat{B}_{\eta, C_{L^2}} = \{B_{C_{L^2}(\mathcal{I}_t)}[0, \mathcal{R}(t)] : t \in \mathbb{R}\}$ are pullback $\mathcal{D}_\eta(M^2_{L^2})$-absorbing and pullback $\mathcal{D}_\eta(C_{L^2})$-absorbing, for the processes $S(\cdot, \cdot)$ and $U(\cdot, \cdot)$ respectively, where*

$$\mathcal{R}^2(t) = 1 + (1+h) e^{\eta_t h} \int_{-\infty}^t e^{-\eta_t (t-\theta)} \big[\widehat{H}_p(t) + c_0^{-1} \|f(\theta)\|^2_{-1,\theta}\big] d\theta, \tag{2.9}$$

where $\widehat{H}_p : \mathbb{R} \to (0,+\infty)$ *is a non-decreasing positive function. Moreover,* $\widehat{B}_{\eta,M^2_{L^2}} \in \mathcal{D}_\eta(M^2_{L^2})$ *and* $\widehat{B}_{\eta,C_{L^2}} \in \mathcal{D}_\eta(C_{L^2})$.

Then, there exist the minimal pullback $\mathcal{D}_\eta(C_{L^2})$*-attractor* $\mathcal{A}_{\mathcal{D}_\eta(C_{L^2})} \in \mathcal{D}_\eta(C_{L^2})$ *and the minimal pullback* $\mathcal{D}_\eta(M^2_{L^2})$*-attractor* $\mathcal{A}_{\mathcal{D}_\eta(M^2_{L^2})} \in \mathcal{D}_\eta(M^2_{L^2})$ *for the processes* $U(\cdot,\cdot)$ *and* $S(\cdot,\cdot)$ *respectively, and the following relationships hold*

$$\mathcal{A}_{\mathcal{D}_\eta(M^2_{L^2})}(t) \subset B_{M^2_{L^2}(\mathcal{I}_t)}[0,\mathcal{R}(t)] \quad \textit{and} \quad \mathcal{A}_{\mathcal{D}_\eta(C_{L^2})}(t) \subset B_{C_{L^2}(\mathcal{I}_t)}[0,\mathcal{R}(t)],$$

for all $t \in \mathbb{R}$. *Moreover, the following relationship between* $\mathcal{A}_{\mathcal{D}_\eta(M^2_{L^2})}$ *and* $\mathcal{A}_{\mathcal{D}_\eta(C_{L^2})}$ *which is:*

$$\mathcal{J}_t(\mathcal{A}_{\mathcal{D}_\eta(C_{L^2})}(t)) = \mathcal{A}_{\mathcal{D}_\eta(M^2_{L^2})}(t), \quad \textit{for all } t \in \mathbb{R}, \tag{2.10}$$

where $\mathcal{J}_t : C_{L^2}(\mathcal{I}_t) \to M^2_{L^2}(\mathcal{I}_t)$, *defined by* $\mathcal{J}_t(\phi) = (\phi(0),\phi)$, *is the canonical injection map.*

3 Continuous dependence with initial data on the pullback attractor.

In this section we study the continuous dependence on the chosen initial data in the sections of the pullback attractor $\mathcal{A}_{\mathcal{D}_\eta(M^2_{L^2})}$. For this, in the same way as [4], we are going to consider $f \in L^2_{loc}(\mathbb{R}; L^2(\mathcal{I}_t))$ satisfying

$$M_f(t) := \sup_{s\leq t} \int_{s-1}^{s} \|f(r)\|^2_{-1,r} dr < +\infty, \qquad \text{for all } t \in \mathbb{R}. \tag{3.1}$$

Note that, if f satisfies (3.1) then $f \in \mathcal{I}^{2,\eta}_*$ and the existence of a pullback $\mathcal{D}_\eta(M^2_{L^2})$-attractor is still guaranteed by Theorem 2.10. Note that we take $f \in L^2_{loc}(\mathbb{R}; L^2(\mathcal{I}_t))$ to guarantee the regularity on the solutions of (DHE), given in Theorem 2.7.

On the other hand, it follows from Theorem 2.10 that

$$\mathcal{A}_{\mathcal{D}_\eta(M^2_{L^2})}(t) \subset B_{M^2_{L^2}(\mathcal{I}_t)}[0,\mathcal{R}(t)],$$

for all $t \in \mathbb{R}$, i.e.

$$\|(u,\phi)\|^2_{M^2_{L^2}(\mathcal{I}_t)} \leq 1 + (1+h)e^{\eta_t h} \int_{-\infty}^{t} e^{-\eta_t(t-s)} \big[\widehat{H}_p(t) + 2\|f(r)\|^2_{-1,r}\big] ds, \tag{3.2}$$

for all $(u,\phi) \in \mathcal{A}_{\mathcal{D}_\eta(M^2_{L^2})}(t)$, $t \in \mathbb{R}$. Then, note that using (3.1), we have

$$\begin{aligned}
e^{-\eta_t t}\int_{-\infty}^{t} e^{\eta_t r}\big[\widehat{H}_p(t) + 2\|f(r)\|^2_{-1,r}\big]dr = \\
&= e^{-\eta_t t}\sum_{n=0}^{\infty}\int_{t-(n+1)}^{t-n} e^{\eta_t r}\big[\widehat{H}_p(t) + 2\|f(r)\|^2_{-1,r}\big]dr \\
&\leq e^{-\eta_t t}\sum_{n=0}^{\infty} e^{\eta_t (t-n)}\int_{t-(n+1)}^{t-n} \big[\widehat{H}_p(t) + 2\|f(r)\|^2_{-1,r}\big]dr \\
&\leq (1-e^{-\eta_t})^{-1}\big[\widehat{H}_p(t) + 2M_f(t)\big] \\
&\leq (1+\eta_t^{-1})\big[\widehat{H}_p(t) + 2M_f(t)\big].
\end{aligned}$$

Therefore, making $\eta = \sup_{t\in\mathbb{R}} \eta_t$, we deduce from (3.2) that

$$\|(u,\phi)\|_{M^2_{L^2}(\mathcal{I}_t)} \leq \mathcal{R}_p(t) \text{ for all } (u,\phi) \in \mathcal{A}_{\mathcal{D}_\eta(M^2_{L^2})}(t),\ t \in \mathbb{R}, \tag{3.3}$$

where $\mathcal{R}_p : \mathbb{R} \to (0,+\infty)$ is a non-decreasing positive function, given as

$$\mathcal{R}_p^2(t) = 1 + (1+h)e^{\eta h}(1+\eta_t^{-1})\big[\widehat{H}_p(t) + 2M_f(t)\big].$$

On the other hand, we are going to show an estimative on the delay term of (DHE), $u(\widetilde{r}_s, s-\delta(s))$, that will be used in this work. Observe that, by hypotheses (H1) and (H3) and applying the change of variable with respect to the spatial variable, we have

$$\begin{aligned}
\int_\tau^t |u(\widetilde{r}_s, s-\delta(s))|_s^2 ds &= \int_\tau^t\int_{\alpha(s)}^{\beta(s)} |u(r^{-1}_{s-\delta(s)}\circ r_s(x), s-\delta(s))|^2 dxds \\
&= \int_\tau^t\int_0^1 |u(r^{-1}_{s-\delta(s)}(y), s-\delta(s))|^2\gamma(s)dyds \\
&= \int_\tau^t\int_{\alpha(s-\delta(s))}^{\beta(s-\delta(s))} |u(x, s-\delta(s))|^2 \frac{\gamma(s)}{\gamma(s-\delta(s))}dxds \qquad (3.4) \\
&\leq \frac{\gamma_1}{\gamma_0(1-\delta_*)}\int_{\tau-\delta(\tau)}^{t-\delta(t)}\int_{\alpha(s)}^{\beta(s)} |u(x,s)|^2 dxds \\
&\leq c_\delta^2\int_{\tau-h}^{t} |u(s)|_s^2 ds,
\end{aligned}$$

for all $\tau \leq t$, where $c_\delta^2 = \frac{\gamma_1}{\gamma_0(1-\delta_*)}$.

Proposition 3.1. *Let $p > 2$ and suppose that conditions (H1)-(H4) hold. Assume that there exists a non-increasing bounded function $\eta : \mathbb{R} \to (0,+\infty)$ such* *that* $f \in L^2_{loc}(\mathbb{R}; L^2(\mathcal{I}_t))$ *satisfying* (3.1)*. Then there exists a non-decreasing function* $\varrho : \mathbb{R} \to (0, +\infty)$*, such that*

$$\|u(s)\|_{L^\infty(\mathcal{I}_s)} \le \varrho(t) \quad \forall\ u(s) \in \mathrm{Proj}_{L^2(\mathcal{I}_s)}\left[\mathcal{A}_{\mathcal{D}_\eta(M^2_{L^2})}(s)\right],\ s \le t,$$

where $\mathrm{Proj}_{L^2(\mathcal{I}_t)} : M^2_{L^2}(\mathcal{I}_t) \to L^2(\mathcal{I}_t)$ *is a projection defined as* $\mathrm{Proj}_{L^2(\mathcal{I}_t)}(u, \phi) = u$*, for all* $(u, \phi) \in M^2_{L^2}(\mathcal{I}_t)$*. Note that, in particular, we have*

$$\mathrm{Proj}_{L^2(\mathcal{I}_t)}\left[\mathcal{A}_{\mathcal{D}_\eta(M^2_{L^2})}(t)\right] \subset L^\infty(\mathcal{I}_t) \quad \textit{for all } t \in \mathbb{R}. \tag{3.5}$$

Proof. It follows from (3.3) that

$$|u(s)|_s \le \mathcal{R}_p(t) \quad \text{for all } u(s) \in \mathrm{Proj}_{L^2(\mathcal{I}_s)}\left[\mathcal{A}_{\mathcal{D}_\eta(M^2_{L^2})}(s)\right],\ s \le t. \tag{3.6}$$

Now, let $u(\cdot)$ be a trajectory of (DHE) such that $u(t) \in \mathcal{A}_{\mathcal{D}_\eta(M^2_{L^2})}(t)$ for all $t \in \mathbb{R}$. Then, using the inequalities given in [5, Lemma 5.3] and (3.4), we have that

$$\begin{aligned} c_0 \int_{s-1}^{s} \|u(\theta)\|^2_\theta d\theta + 2\alpha_0 \int_{s-1}^{s} |u(\theta)|^p_{p,\theta} d\theta &\le |u(s-1)|^2_{s-1} + 2\beta_0\gamma_1 \\ &\quad + \frac{1}{c_0}\int_{s-1}^{s} \|f(\theta)\|^2_{-1,\theta} d\theta + \int_{s-1}^{s} |u(\widetilde{r}_\theta, \theta - \delta(\theta))|^2_\theta d\theta + \int_{s-1}^{s} |u(\theta)|^2_\theta d\theta \\ &\le 2\mathcal{R}^2_p(t) + 2\beta_0\gamma_1 + \frac{1}{c_0} M_f(t) + \int_{s-1}^{s} |u(\widetilde{r}_\theta, \theta - \delta(\theta))|^2_\theta d\theta \\ &\le 2\mathcal{R}^2_p(t) + 2\beta_0\gamma_1 + \frac{1}{c_0} M_f(t) + c_\delta^2 \int_{s-1-h}^{s-1} |u(\theta)|^2_\theta d\theta \\ &\le \left(2 + hc_\delta^2\right)\mathcal{R}^2_p(t) + 2\beta_0\gamma_1 + \frac{1}{c_0} M_f(t), \end{aligned}$$

for all $s \le t$. Then, we deduce that

$$\int_{s-1}^{s} \|u(\theta)\|^2_\theta d\theta + \int_{s-1}^{s} |u(\theta)|^p_{p,\theta} d\theta \le \widehat{\mathcal{R}}_p(t) \quad \text{for all } s \le t, \tag{3.7}$$

where $\widehat{\mathcal{R}}_p : \mathbb{R} \to (0, +\infty)$ is a non-decreasing positive function defined as

$$\widehat{\mathcal{R}}_p(t) := \frac{(2 + hc_\delta^2)\mathcal{R}^2_p(t) + 2\beta_0\gamma_1 + c_0^{-1} M_f(t)}{\min\{c_0, 2\alpha_0\}} \quad \text{for all } t \in \mathbb{R}.$$

On the other hand, by [5, Lemma 5.5] we know that $u(\cdot)$ satisfies

$$\begin{aligned} \|u(s)\|^2_s \le \int_{s-1}^{s} \|u(\theta)\|^2_\theta d\theta + \frac{2\widetilde{\alpha}_1}{c_0}\int_{s-1}^{s} |u(\theta)|^p_{p,\theta} d\theta + \frac{4\widetilde{\beta}_0\gamma_1}{c_0} \\ + \frac{1}{c_0}\int_{s-1}^{s} |u(\tilde{r}_\theta, \theta - \delta(\theta))|^2_\theta d\theta + \frac{1}{c_0}\int_{s-1}^{s} |f(\theta)|^2_\theta d\theta, \end{aligned}$$

for all $s \leq t$. Thus, it follows from (3.4), (3.6) and (3.7) that

$$\|u(s)\|_s^2 \leq \frac{4\widetilde{\beta}_0\gamma_1}{c_0} + \Big[1 + \frac{2\widetilde{\alpha}_1}{c_0}\Big]\widehat{\mathcal{R}}_p(t) + \frac{hc_\delta^2}{c_0}\mathcal{R}_p^2(t) + \frac{1}{c_0}M_f(t) \quad \text{for all } s \leq t.$$

Now, we know that, by hypothesis (H3) and $H_0^1(\mathcal{I}_s) \hookrightarrow L^\infty(\mathcal{I}_s)$, there exists $\kappa_t > 0$ such that $\|u(s)\|_{L^\infty(\mathcal{I}_s)} \leq k_t\|u(s)\|_s$ for all $s \leq t$. Then, we obtain

$$\|u(s)\|_{L^\infty(\mathcal{I}_s)}^2 \leq \kappa_t^2\Big\{\frac{4\widetilde{\beta}_0\gamma_1}{c_0} + \Big[1 + \frac{2\widetilde{\alpha}_1}{c_0}\Big]\widehat{\mathcal{R}}_p(t) + \frac{hc_\delta^2}{c_0}\mathcal{R}_p^2(t) + \frac{1}{c_0}M_f(t)\Big\},$$

for all $s \leq t$, and for any $u(s) \in \mathrm{Proj}_{L^2(\mathcal{I}_s)}\big[\mathcal{A}_{\mathcal{D}_\eta(M_{L^2}^2)}(s)\big]$. □

Theorem 3.2. *Under the conditions (H1)-(H4), let $p > 2$, $\tau \in \mathbb{R}$, and $f \in L^2_{loc}\big(\mathbb{R}; L^2(\mathcal{I}_t)\big)$ that satisfies (3.1), and $(u^\tau, \phi_1), (v^\tau, \phi_2) \in \mathcal{A}_{\mathcal{D}_\eta(M_{L^2}^2)}(\tau)$. Let us denote by $u(t) = u(t; u^\tau, \phi_1)$ and $v(t) = v(t; v^\tau, \phi_2)$ the weak solutions of (DHE) corresponding to initial values (u^τ, ϕ_1) and (v^τ, ϕ_2), respectively. Then, for all $t \geq \tau$, the following inequalities hold*

$$|u(t) - v(t)|_t^2 \leq \kappa_{1,t}e^{\kappa_{2,t}(t-\tau)}\|(u^\tau, \phi_1) - (v^\tau, \phi_2)\|_{M_{L^2}(\mathcal{I}_\tau)}^2, \tag{3.8}$$

$$\int_\tau^t \|u(s) - v(s)\|_s^2 ds \leq \kappa_{3,t,t-\tau}\|(u^\tau, \phi_1) - (v^\tau, \phi_2)\|_{M_{L^2}(\mathcal{I}_\tau)}^2. \tag{3.9}$$

where $\kappa_{1,t}, \kappa_{2,t}$ are non-decreasing positive functions and $\kappa_{3,t,t-\tau} := \frac{1}{c_0}\big[k_{1,t}\big(1 + k_{2,t}e^{k_{2,t}(t-\tau)}(t-\tau)\big)\big]$. Moreover, if $\tau + 3h \leq t$, we deduce that

$$\|u_s - v_s\|_{C_{H_0^1}(\mathcal{I}_s)}^2 \leq \kappa_{5,t,t-\tau}\|(u^\tau, \phi_1) - (v^\tau, \phi_2)\|_{M_{L^2}(\mathcal{I}_\tau)}^2, \tag{3.10}$$

for all $s \in [t-h, t]$, and

$$\int_{t-h}^t \|u_s' - v_s'\|_{L_{L^2}^2(\mathcal{I}_s)}^2 ds \leq \kappa_{6,t,t-\tau}\|(u^\tau, \phi_1) - (v^\tau, \phi_2)\|_{M_{L^2}(\mathcal{I}_\tau)}^2. \tag{3.11}$$

where $\kappa_{5,t,t-\tau}$, and $\kappa_{6,t,t-\tau}$ are positive functions and non-decreasing respect to the first and second variables.

Proof. It follows from the weak formulation (2.4) and using $w(\cdot) := u(\cdot) - v(\cdot)$ as a test function, that

$$\frac{d}{dt}|w|_t^2 + 2c_0\|w\|_t^2 + 2(g(u) - g(v), w)_t = 2\big(w(\widetilde{r}_t, t - \delta(t)), w(t)\big)_t.$$

Thus, by Hölder's inequality and Condition (1.1), we have

$$\frac{d}{dt}|w|_t^2 + c_0\|w\|_t^2 \leq 2l|w|_t^2 + \frac{1}{c_0\lambda_{1,t}}|w(\widetilde{r}_t, t - \delta(t))|_t^2.$$

Now, integrating from τ to t, we obtain

$$|w(t)|_t^2+c_0\int_\tau^t \|w\|_s^2 ds \le |w(\tau)|_\tau^2+2l\int_\tau^t |w(s)|_s^2 ds+\frac{1}{c_0\lambda_{1,t}}\int_\tau^t |w(\widetilde{r}_s, s-\delta(s))|_s^2 ds. \tag{3.12}$$

Note that, in the same way as (3.4), we conclude

$$\int_\tau^t |w(\widetilde{r}_s, s-\delta(s))|_s^2 ds \le c_\delta^2 \int_{\tau-h}^t |w(s)|_s^2 ds. \tag{3.13}$$

Therefore, from (3.12) and (3.13), we deduce that

$$|w(t)|_t^2 \le \kappa_{1,t}\|(u^\tau,\phi_1)-(v^\tau,\phi_2)\|^2_{M_{L^2}(\mathcal{I}_\tau)}+\kappa_{2,t}\int_\tau^t |w(s)|_s^2 ds, \tag{3.14}$$

for all $t \ge \tau$, where $\kappa_{1,t} = \max\left\{1, \frac{c_\delta^2}{c_0\lambda_{1,t}}\right\}$ and $\kappa_{2,t} = \left(2l + \frac{c_\delta^2}{c_0\lambda_{1,t}}\right)$. Then, applying Gronwall's inequality, we conclude the first part of the proof.

Regarding the second part this theorem, let us consider $t \ge \tau+3h$, $\theta \in (-h, 0)$, and $s \in (t-2h, t)$. Now, using w' as a test function in the weak formulation (2.4), we obtain

$$\begin{aligned}|w'(s+\theta)|_{s+\theta}^2+\frac{c_0}{2}\frac{d}{ds}\|w(s+\theta)\|_{s+\theta}^2 &= -\big(g(u)-g(v), w'\big)_{s+\theta} \\ &\quad + \big(w(\widetilde{r}_{s+\theta}, s+\theta-\delta(s+\theta)), w'(s+\theta)\big)_{s+\theta} \\ &\le |g'(\xi)||w|_{s+\theta}|w'|_{s+\theta}+|w(\widetilde{r}_{s+\theta}, s+\theta-\delta(s+\theta))|_{s+\theta}|w'|_{s+\theta},\end{aligned}$$

where $\xi \in [u(x,\theta+s), v(x,\theta+s)]$. Now, by Proposition 3.1 we know that

$$\mathrm{Proj}_{L^2(\mathcal{I}_r)}\left[\mathcal{A}_{\mathcal{D}_\eta(M^2_{L^2})}(r)\right] \subset [-\varrho(t), \varrho(t)] \text{ for all } r \le t.$$

Then, taking $\kappa_{t,g} := \sup_{r\in[-\varrho(t),\varrho(t)]} g'(r)$ and applying the Hölder inequality, we deduce that

$$|w'(s+\theta)|_{s+\theta}^2+c_0\frac{d}{ds}\|w(s+\theta)\|_{s+\theta}^2 \le \kappa_{t,g}^2|w|_{s+\theta}^2+|w(\widetilde{r}_{s+\theta}, s+\theta-\delta(s+\theta))|_{s+\theta}^2. \tag{3.15}$$

Now, integrating from r to s, with $[r,s]\subset[t-2h,t]$ we derive

$$\begin{aligned} c_0\|w(s+\theta)\|^2_{s+\theta} &\le c_0\|w(r+\theta)\|^2_{s+\theta}+\kappa^2_{t,g}\int_r^s |w|^2_{\ell+\theta}d\ell \\ &\qquad + \int_r^s |w(\widetilde{r}_{\ell+\theta},\ell+\theta-\delta(\ell+\theta))|^2_{\ell+\theta}d\ell \\ &= c_0\|w(r+\theta)\|^2_{s+\theta}+\kappa^2_{t,g}\int_{r+\theta}^{s+\theta}|w|^2_\ell d\ell+\int_{r+\theta}^{s+\theta}|w(\widetilde{r}_\ell,\ell-\delta(\ell))|^2_\ell d\ell \\ &\le c_0\|w(r+\theta)\|^2_{s+\theta}+\kappa^2_{t,g}\int_\tau^t|w|^2_\ell d\ell+\int_\tau^t|w(\widetilde{r}_\ell,\ell-\delta(\ell))|^2_\ell d\ell, \end{aligned}$$

since $\tau\le t-3h$. Using (3.8) and (3.13), we obtain

$$\begin{aligned} c_0\|w(s+\theta)\|^2_{s+\theta} &\le c_0\|w(r+\theta)\|^2_{s+\theta}+\kappa^2_{t,g}\int_\tau^t|w(\ell)|^2_\ell d\ell+c^2_\delta\int_{\tau-h}^t|w(\ell)|^2_\ell d\ell \\ &\le c_0\|w(r+\theta)\|^2_{s+\theta}+\kappa_{4,t,t-\tau}\|(u^\tau,\phi_1)-(v^\tau,\phi_2)\|^2_{M_{L^2}(\mathcal{I}_\tau)}, \end{aligned}$$

for all $[r,s]\subset[t-2h,t]$, where $\kappa_{4,t,t-\tau}:=\left[c^2_\delta+(c^2_\delta+\kappa^2_{t,g})\kappa_{1,t}e^{\kappa_{2,t}(t-\tau)}(t-\tau)\right]$.

Now, for $s\in[t-h,t]$, again integrating in r, from $t-2h$ to s with , we have

$$\begin{aligned} c_0\|w(s+\theta)\|^2_{s+\theta} &\le \frac{c_0}{s-t+2h}\int_{t-2h+\theta}^{s+\theta}\|w(r)\|^2_r dr \\ &\qquad + 2h\kappa_{4,t,t-\tau}\|(u^\tau,\phi_1)-(v^\tau,\phi_2)\|^2_{M_{L^2}(\mathcal{I}_\tau)} \\ &\le \frac{c_0}{h}\int_\tau^t\|w(r)\|^2_r dr+2h\kappa_{4,t,t-\tau}\|(u^\tau,\phi_1)-(v^\tau,\phi_2)\|^2_{M_{L^2}(\mathcal{I}_\tau)}, \end{aligned}$$

for all $s\in[t-h,t]$. Thus, using (3.9) we deduce that

$$\|w(s+\theta)\|^2_{s+\theta}\le\kappa_{5,t,t-\tau}\|(u^\tau,\phi_1)-(v^\tau,\phi_2)\|^2_{M^2_{L^2}(\mathcal{I}_\tau)} \tag{3.16}$$

for all $s\in[t-h,t]$, where $\kappa_{5,t,t-\tau}:=\left(\kappa_{3,t,t-\tau}+\frac{h\kappa_{4,t,t-\tau}}{c_0}\right)$. Taking the maximum in $\theta\in[-h,0]$, we have

$$\|w_s\|^2_{C_{H^1_0}(\mathcal{I}_s)}\le\kappa_{5,h,t-\tau}\|(u^\tau,\phi_1)-(v^\tau,\phi_2)\|^2_{M^2_{L^2}(\mathcal{I}_\tau)}\quad\text{for all } s\in[t-h,t].$$

Moreover, it follows from (3.15) and (3.16), that

$$\int_{t-h}^t\|w'_s\|^2_{L^2_{L^2}(\mathcal{I}_s)}ds\le\kappa_{6,t,t-\tau}\|(u^\tau,\phi_1)-(v^\tau,\phi_2)\|^2_{M^2_{L^2}(\mathcal{I}_\tau)},$$

for all $t\ge\tau+3h$. □

Remark 3.3. (Smoothness property associated with a sequence of points) By regularity, if we consider $f \in W^{1,2}_{loc}(\mathbb{R}; L^2(\mathcal{I}_t))$, it is possible to prove that $u'' \in L^2_{loc}(\mathbb{R}; H^{-1}(\mathcal{I}_t))$ and $u' \in L^2_{loc}(\mathbb{R}; H^1_0(\mathcal{I}_t))$, then u' has continuous representation. Therefore, it follows from (3.11) and the mean value theorem that there exists $\hat{t} \in [t-h, t]$ such that

$$h\|u'_{\hat{t}} - v'_{\hat{t}}\|^2_{L^2_{L^2}(\mathcal{I}_{\hat{t}})} \leq \kappa_{6,h,t-\tau}\|(u^\tau, \phi_1) - (v^\tau, \phi_2)\|^2_{M^2_{L^2}(\mathcal{I}_\tau)}, \tag{3.17}$$

for all $\tau \leq t - 3h$.

First step: Let us consider $t_0 = 0$, then exists $\hat{t}_0 \in [t_0 - h, t_0]$ such that

$$h\|u'_{\hat{t}_0} - v'_{\hat{t}_0}\|^2_{C_{L^2}(\mathcal{I}_{\hat{t}_0})} \leq \kappa_{6,t_0,t_0-\tau}\|(u^\tau, \phi_1) - (v^\tau, \phi_2)\|^2_{M^2_{L^2}(\mathcal{I}_\tau)}, \tag{3.18}$$

for all $\tau \leq t_0 - 3h$.

Second step: Now, let us consider $t_1 = t_0 - 3h$, then there exists $\hat{t}_1 \in [t_1 - h, t_1]$ such that

$$h\|u'_{\hat{t}_1} - v'_{\hat{t}_1}\|^2_{L^2_{L^2}(\mathcal{I}_{\hat{t}_1})} \leq \kappa_{6,t_1,t_1-\tau}\|(u^\tau, \phi_1) - (v^\tau, \phi_2)\|^2_{M^2_{L^2}(\mathcal{I}_\tau)}, \tag{3.19}$$

for all $\tau \leq t_1 - 3h$. Thus, taking $\tau = \hat{t}_1$ in (3.18) we obtain

$$h\|u'_{\hat{t}_0} - v'_{\hat{t}_0}\|^2_{L^2_{L^2}(\mathcal{I}_{\hat{t}_0})} \leq \kappa_{6,t_0,t_0-\hat{t}_1}\|(u^{\hat{t}_1}, \phi_1) - (v^{\hat{t}_1}, \phi_2)\|^2_{M^2_{L^2}(\mathcal{I}_{\hat{t}_1})}, \tag{3.20}$$

and taking into account that $\kappa_{6,t_0,(\cdot)}$ is a non-decreasing function, and $t_0 - \hat{t}_1 \leq 4h$, we have

$$h\|u'_{\hat{t}_0} - v'_{\hat{t}_0}\|^2_{L^2_{L^2}(\mathcal{I}_{\hat{t}_0})} \leq \kappa_{6,t_0,4h}\|(u^{\hat{t}_1}, \phi_1) - (v^{\hat{t}_1}, \phi_2)\|^2_{M_{L^2}(\mathcal{I}_{\hat{t}_1})}. \tag{3.21}$$

Third step: Again, let $t_2 = t_1 - 3h$, then there exists $\hat{t}_2 \in [t_2 - h, t_2]$ such that

$$h\|u'_{\hat{t}_2} - v'_{\hat{t}_2}\|^2_{L^2_{L^2}(\mathcal{I}_{\hat{t}_2})} \leq \kappa_{6,t_2,t_2-\tau}\|(u^\tau, \phi_1) - (v^\tau, \phi_2)\|^2_{M^2_{L^2}(\mathcal{I}_\tau)}, \tag{3.22}$$

for all $\tau \leq t_2 - 3h$. Thus, in the same way as (3.21), making $\tau = \hat{t}_2$ in (3.19) we have

$$h\|u'_{\hat{t}_1} - v'_{\hat{t}_1}\|^2_{L^2_{L^2}(\mathcal{I}_{\hat{t}_1})} \leq \kappa_{6,t_0,4h}\|(u^{\hat{t}_2}, \phi_1) - (v^{\hat{t}_2}, \phi_2)\|^2_{M^2_{L^2}(\mathcal{I}_{\hat{t}_2})}. \tag{3.23}$$

Therefore, repeating the previous steps, there exists a family of point $\{\hat{t}_m\}_{m=0}^\infty$ such that $\hat{t}_m \to -\infty$ as $m \to \infty$, and

$$\|u'_{\hat{t}_m} - v'_{\hat{t}_m}\|^2_{L^2_{L^2}(\mathcal{I}_{\hat{t}_m})} \leq \hat{\kappa}_h\|(u^{\hat{t}_{m+1}}, \phi_1) - (v^{\hat{t}_{m+1}}, \phi_2)\|^2_{M^2_{L^2}(\mathcal{I}_{\hat{t}_{m+1}})},$$

for all $m \geq 0$, where $\hat{\kappa}_h := \kappa_{6,t_0,4h}/h$.

Finally, it should be noted that the regularity obtained regarding the history of the solution, i.e., u_t, is when the final time $t \in \mathbb{R}$ is located at a minimum distance of size $h > 0$ with respect to the initial time τ, that is, $t \geq \tau + h$.

4 Finite fractal dimension of the pullback attractor

In this section, we will focus on showing that the pullback attractors $\mathcal{A}_{\mathcal{D}_\eta(M^2_{L^2})}$ and $\mathcal{A}_{\mathcal{D}_\eta(C_{L^2})}$, given in Theorem 2.10, have finite fractal dimension. When the non-increasing function $\eta : \mathbb{R} \to (0, +\infty)$ is bounded and the external force $f \in W^{1,2}_{loc}(\mathbb{R}; L^2(\mathcal{I}_t))$ satisfies the condition (3.1).

Given $t \in \mathbb{R}$ let us consider the following Banach space

$$Y_{L^2;H^1_0}(\mathcal{I}_t) = \Big\{\phi \in L^2_{H^1_0}(\mathcal{I}_t) : \phi' \in L^2_{L^2}(\mathcal{I}_t)\Big\},$$

with norm $\|\phi\|^2_{Y_{L^2;H^1_0}(\mathcal{I}_t)} := \|\phi\|^2_{L^2_{H^1_0}(\mathcal{I}_t)} + \|\phi'\|^2_{L^2_{L^2}(\mathcal{I}_t)}$.

Now, let us define $W_{L^2;H^1_0}(\mathcal{I}_t) = H^1_0(\mathcal{I}_t) \times Y_{L^2;H^1_0}(\mathcal{I}_t)$. Then, it follows from Lemma of *Aubin-Lions-Simon*, e.g. [6, Theorem 4,15, Lemma 4.16], that

$$W_{L^2;H^1_0}(\mathcal{I}_t) \hookrightarrow\hookrightarrow M^2_{L^2}(\mathcal{I}_t) \quad \text{for all } t \in \mathbb{R}. \tag{4.1}$$

Corollary 4.1. *Under the hypothesis (H4), given $\varepsilon > 0$ we have*

$$\sup_{s\le t} N_{M^2_{L^2}(\mathcal{I}_s)}\Big(B_{W_{L^2;H^1_0}(\mathcal{I}_s)}(0,1), \varepsilon\Big) \le N_{M^2_{L^2}(\mathbf{I}_t)}\Big(B_{W_{L^2;H^1_0}(\mathbf{I}_t)}(0,1), \varepsilon\Big). \tag{4.2}$$

Proof. Given $f : \mathcal{I}_s \to \mathbb{R}$ and $\varphi : Q_{s-h,s} \to \mathbb{R}$, consider the extensions

$$\begin{aligned} \mathbb{E}_{s,t}(f) : \mathbf{I}_t &\longrightarrow \mathbb{R} \\ x &\longmapsto \mathbb{E}_{s,t}(f)(x) = \begin{cases} f(x), & \text{if } x \in \mathcal{I}_s, \\ 0, & \text{if } x \notin \mathcal{I}_s, \end{cases} \end{aligned}$$

and

$$\begin{aligned} \widehat{\mathbb{E}}_{s,t}(\varphi) : \mathbf{I}_t \times (-h,0) &\longrightarrow \mathbb{R} \\ (x,\theta) &\longmapsto \widehat{\mathbb{E}}_{s,t}(\varphi)(x,\theta) = \begin{cases} \varphi(x,\theta), & \text{if } (x,\theta) \in Q_{s-h,s}, \\ 0, & \text{if } (x,\theta) \notin Q_{s-h,s}. \end{cases} \end{aligned}$$

If $(f,\varphi) \in M^2_{L^2}(\mathcal{I}_s) = L^2(\mathcal{I}_s) \times L^2_{L^2}(\mathcal{I}_s)$, then $\mathbb{E}_{s,t} \times \widehat{\mathbb{E}}_{s,t}(f,\varphi) \in M^2_{L^2}(\mathbf{I}_t)$ and

$$\|(f,\varphi)\|_{M^2_{L^2}(\mathcal{I}_s)} = \|\mathbb{E}_{s,t} \times \widehat{\mathbb{E}}_{s,t}(f,\varphi)\|_{M^2_{L^2}(\mathbf{I}_t)}.$$

Hence the map

$$\begin{aligned} \mathbb{E}_{s,t} \times \widehat{\mathbb{E}}_{s,t} : M^2_{L^2}(\mathcal{I}_s) &\longrightarrow M^2_{L^2}(\mathbf{I}_t) \\ (f,\varphi) &\longmapsto \mathbb{E}_{s,t} \times \widehat{\mathbb{E}}_{s,t}(f,\varphi) \end{aligned}$$

is well-defined and it is an isometry. By hypothesis (H4) we know that $\mathbf{I}_t := \cup_{s\leq t}\mathcal{I}_s$ is a bounded subset of $\mathbb{R}$, then in the same way as (4.1), if we consider $W_{L^2;H_0^1}(\mathbf{I}_t)$, we obtain $W_{L^2;H_0^1}(\mathbf{I}_t) \hookrightarrow\hookrightarrow M^2_{L^2}(\mathbf{I}_t)$. Therefore (4.2) follows directly from Lemma 2.5. □

Theorem 4.2. *(Fractal Dimension) Suppose that conditions (H1)-(H4) hold, and $g \in C^1(\mathbb{R})$ and $p > 2$. Assume that there exists a non-increasing bounded function $\eta : \mathbb{R} \to (0, +\infty)$ such that $f \in W^{1,2}_{loc}(\mathbb{R}; L^2(\mathcal{I}_t))$ satisfies (3.1). Then, the pullback attractors $\mathcal{A}_{\mathcal{D}_\eta(M^2_{L^2})}$ and $\mathcal{A}_{\mathcal{D}_\eta(C_{L^2})}$ given in Theorem 2.10 have finite fractal dimension i.e.*

$$dim_f\left(\mathcal{A}_{\mathcal{D}_\eta(M^2_{L^2})}(t), M^2_{L^2}(\mathcal{I}_t)\right) < \infty \quad \text{and} \quad dim_f\left(\mathcal{A}_{\mathcal{D}_\eta(C_{L^2})}(t), C_{L^2}(\mathcal{I}_t)\right) < \infty$$

for all $t \in \mathbb{R}$. Furthermore, the following relationship holds

$$\begin{aligned}\sup_{t\in\mathbb{R}} dim_f\left(\mathcal{A}_{\mathcal{D}_\eta(C_{L^2})}(t), C_{L^2}(\mathcal{I}_t)\right) &\leq \sup_{t\in\mathbb{R}} dim_f\left(\mathcal{A}_{\mathcal{D}_\eta(M^2_{L^2})}(t), M^2_{L^2}(\mathcal{I}_t)\right) \\ &\leq \frac{\log\left[N_{M^2_{L^2}(\mathbf{I}_{\hat{t}_0})}\left(B_{W_{L^2;H_0^1}(\mathbf{I}_{\hat{t}_0})}(0,1), \frac{1}{4\tilde{\kappa}_h}\right)\right]}{\log 2}.\end{aligned}$$

Proof. For $t = 0$, by Theorem 3.2 and Remark 3.3, we have that there exists a sequence of points $\{\hat{t}_{-m}\}_{m\leq 0}$ such that $\hat{t}_{-m} \to -\infty$ and

$$\|u'_{\hat{t}_{-m}} - v'_{\hat{t}_{-m}}\|^2_{L^2_{L^2}(\mathcal{I}_{\hat{t}_{-m}})} \leq \hat{\kappa}_h \|(u^{\hat{t}_{-m+1}}, \phi_1) - (v^{\hat{t}_{-m+1}}, \phi_2)\|^2_{M^2_{L^2}(\mathcal{I}_{\hat{t}_{-m+1}})}, \tag{4.3}$$

Now, denoting $S_m = S(\hat{t}_{-m}, \hat{t}_{-m+1})$ for all $m \leq 0$. Then, $S_m : M^2_{L^2}(\mathcal{I}_{\hat{t}_{-m+1}}) \to M^2_{L^2})(\mathcal{I}_{\hat{t}_{-m}})$ satisfies the smoothing property, i.e.,

$$\|S_m(u,\phi) - S_m(v,\varphi)\|_{W_{L^2;H_0^1}(\mathcal{I}_{\hat{t}_{-m}})} \leq \tilde{\kappa}_h \|(u,\phi) - (v,\varphi)\|_{M^2_{L^2}(\mathcal{I}_{\hat{t}_{-m+1}})}, \tag{4.4}$$

for all $(u, \phi) \in \mathcal{A}_{\mathcal{D}_\eta(M^2_{L^2})}(\hat{t}_{-m+1})$, $m \leq 0$, where, by (3.10), $\tilde{\kappa}_h := \max\{\kappa_{5,t_0,4h}, \hat{\kappa}_h\}$.

On the other hand, for each $s \in \mathbb{R}$ and $\varepsilon > 0$ let us denote by

$$N^s_\varepsilon := N_{M^2_{L^2}(\mathcal{I}_s)}\left(B_{W_{L^2;H_0^1}(\mathcal{I}_s)}(0,1), \varepsilon\right).$$

Then, for $\varepsilon = \frac{1}{4\tilde{\kappa}_h}$, it follows from Corollary 4.1 that

$$\sup_{s\leq t} N^s_{\frac{1}{4\tilde{\kappa}_h}} \leq N_{M^2_{L^2}(\mathbf{I}_t)}\left(B_{W_{L^2;H_0^1}(\mathbf{I}_t)}(0,1), \frac{1}{4\tilde{\kappa}_h}\right). \tag{4.5}$$

We also have by (3.3) that there exists a non-decreasing positive function $\mathcal{R}_p : \mathbb{R} \to (0, +\infty)$ such that

$$\mathcal{A}_{\mathcal{D}_\eta(M^2_{L^2})}(t) \subset B_{M^2_{L^2}(\mathcal{I}_t)}\left[0, \mathcal{R}_p(t)\right], \quad \text{for all} \quad t \in \mathbb{R}. \tag{4.6}$$

Then, it follows from (4.4), (4.5) and (4.6) that $\left\{S_m, \mathcal{A}_{\mathcal{D}_\eta(M^2_{L^2})}(\hat{t}_m)\right\}_{m\leq 0}$ satisfies all conditions of Theorem 2.3, therefore

$$\sup_{m\leq 0} dim_f \left(\mathcal{A}_{\mathcal{D}_\eta(M^2_{L^2})}(\hat{t}_m), M^2_{L^2}(\mathcal{I}_{\hat{t}_m})\right) \leq \frac{\log\left[N_{M^2_{L^2}(\mathbf{I}_{\hat{t}_0})}\left(B_{W_{L^2;H^1_0}(\mathbf{I}_{\hat{t}_0})}(0,1), \frac{1}{4\tilde{\kappa}_h}\right)\right]}{\log 2}.$$

Since the pullback attractor is invariant and, by the continuous dependence, the evolution process $S(\cdot,\cdot)$ is Lipschitz, it follows from Proposition 2.2 that

$$\sup_{t\in\mathbb{R}} dim_f \left(\mathcal{A}_{\mathcal{D}_\eta(M^2_{L^2})}(t), M^2_{L^2}(\mathcal{I}_t)\right) \leq \frac{\log\left[N_{M^2_{L^2}(\mathbf{I}_{\hat{t}_0})}\left(B_{W_{L^2;H^1_0}(\mathbf{I}_{\hat{t}_0})}(0,1), \frac{1}{4\tilde{\kappa}_h}\right)\right]}{\log 2}. \tag{4.7}$$

Now, from the estimate of the fractal dimension of each section of the pullback attractor $\mathcal{A}_{\mathcal{D}_\eta(M^2_{L^2})}$, we will be able to estimate the fractal dimension of each section of the pullback attractor $\mathcal{A}_{\mathcal{D}_\eta(C_{L^2})}$. Indeed, for each $t\in\mathbb{R}$, let us consider the projection $\mathrm{Proj}_{L^2_{L^2}(\mathcal{I}_t)} : M^2_{L^2}(\mathcal{I}_t) \to L^2_{L^2}(\mathcal{I}_t)$ such that

$$\mathrm{Proj}_{L^2_{L^2}(\mathcal{I}_t)}(u,\phi) = \phi \quad \text{for all } (u,\phi)\in M^2_{L^2}(\mathcal{I}_t).$$

It follows from the identity (2.10), given in Theorem 2.10, that

$$\mathcal{A}_{\mathcal{D}_\eta(C_{L^2})}(t) = \mathrm{Proj}_{L^2_{L^2}(\mathcal{I}_t)}\left(\mathcal{A}_{\mathcal{D}_\eta(M^2_{L^2})}(t)\right), \quad \text{for all } t\in\mathbb{R}.$$

Then, since for each $t\in\mathbb{R}$ the projection $\mathrm{Proj}_{L^2_{L^2}(\mathcal{I}_t)}$ is a continuous linear map, it follows from Proposition 2.2 and the estimate of the fractal dimension of $\mathcal{A}_{\mathcal{D}_\eta(M^2_{L^2})}$ given in (4.7), that

$$\sup_{t\in\mathbb{R}} dim_f \left(\mathcal{A}_{\mathcal{D}_\eta(C_{L^2})}(t), C_{L^2}(\mathcal{I}_t)\right) \leq \frac{\log\left[N_{M^2_{L^2}(\mathbf{I}_{\hat{t}_0})}\left(B_{W_{L^2;H^1_0}(\mathbf{I}_{\hat{t}_0})}(0,1), \frac{1}{4\tilde{\kappa}_h}\right)\right]}{\log 2}.$$

□

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