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Four types of problems with variable exponents: Existence of global attractors

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Abstract. The purpose of this survey is to present results about existence of global attractors for four types of autonomous evolution problems under homogenous Dirichlet boundary conditions with a variable exponent and a bounded smooth domain, more precisely, a reaction diffusion equation with a globally Lipschitz reaction term, a PDE with a non globally Lipschitz perturbation, a partial differential inclusion and a coupled system of inclusions.

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1 Introduction

Autonomous dynamical systems have been investigated intensively for over a century, being applied in the field of ordinary and partial differential equations where the great interest is the study of the asymptotic behavior of solutions. When the dynamical system is dissipative, the asymptotic

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behavior of solutions is concentrated on the question of existence of global attractors. The study of attractors for infinite dimensional dynamical systems has been one of the most active areas in dynamical system theory in the last decades. The global attractors attract trajectories in the entire state space and describe the forward asymptotic dynamics of the system.

"The study of dynamical systems of infinite dimensional has been a very active area in pure and applied mathematics including the existence of global attractors for systems in mathematical physics and mechanics. The study of nonlinear dynamics is a fascinating question which is at the very heart of the understanding of many important problems of the natural sciences. In contrast to linear systems, the evolution of nonlinear systems obeys complicated laws that, in general, cannot be arrived at by pure intuition or by elementary calculations" [36].

A class of nonlinear problems with variable exponents have emerged from recent developments in science and technology such as chemical fluid dynamics, more precisely, electrorheological fluids, which are fluids characterized by the ability to drastically change the mechanical properties under the influence of exterior electromagnetic field because of the presence of some small solid substances mixed with liquid (see [14, 23–25]). The study of existence of attractors for parabolic problems with spatially variable exponents is a very recent research issue, to the best of our knowledge, the first result was published in [28].

The paper is organized as follows. In Section 2 we present the preliminaries results on semigroups, multivalued semigroups, and generalized Lebesgue and Sobolev spaces. In Section 3 we consider a reaction diffusion equation with a globally Lipschitz reaction term. In Section 4 we consider a PDE with a non globally Lipschitz perturbation. Section 5 is used to study a partial differential inclusion and Section 6 is devoted to a coupled system of inclusions.

2 Preliminaries

Let (Z, d) be a complete metric space. Let us introduce some notations:
$P(Z) := \{A \subset Z : A$ is a nonempty set in Z$\}$;
$K(Z) := \{K \subset Z : K$ is a nonempty compact set in Z$\}$;
$\mathfrak{B}(Z) := \{B \subset Z : B$ is a nonempty bounded set in Z$\}$.

Definition 2.1. We denote by *dist* the Hausdorff semi-distance in Z between the nonempty sets A and B, i.e.,

$$dist(A, B) := \sup_{a \in A} d(a, B) = \sup_{a \in A} \inf_{b \in B} d(a, b),$$

and by $dist_H$ the Hausdorff distance in Z, i.e.,

$$dist_H(A, B) := \max\{dist(A, B), dist(B, A)\}.$$

2.1 Semigroups

In this subsection we remind some definitions and results which will be useful in sections 3 and 4.

Definition 2.2. A semigroup is a family of single-valued continuous operators $T(t) : Z \to Z$ depending on a parameter $t \in \mathbb{R}^+$ and enjoying the semigroup property:

$$T(t_1)T(t_2)(x) = T(t_1 + t_2)(x), \text{ for all } t_1, t_2 \in \mathbb{R}^+ \text{ and } x \in Z;$$

and $T(0) = I_d$.

Definition 2.3. Let $B \in P(Z)$. We say that B is **invariant by the semigroup T** if $T(t)B = B, \ \forall\, t \geq 0$.

Definition 2.4. Let A and M be subsets of Z. We say that A attracts M or M is attracted to A by semigroup $\{T(t)\}_{t\geq 0}$ if for every $\epsilon > 0$ there exists a $t_1(\epsilon, M) \in \mathbb{R}^+$ such that $T(t)M \subset O_\epsilon(A) := \{x \in Z; d(x, A) < \epsilon\}$ for all $t \geq t_1(\epsilon, M)$. The set $A \subset Z$ attracts the point $x \in Z$ if A attracts the one-point set $\{x\}$.

Definition 2.5. [18] A nonempty set $\mathcal{A} \subseteq Z$ is called a **global attractor** for the semigroup T if

(i) it is compact;

(ii) it is invariant by the semigroup T;

(iii) it attracts each bounded subset of Z, in other words, $\mathcal{A}$ is a global $B-$attractor.

Remark 2.6. In general we are interested in the compact set $\mathcal{A}$ which is the minimal closed invariant global $B-$attractor, i.e., if there is another invariant and closed set C which attracts bounded sets of Z, then $\mathcal{A} \subset C$. In [13], "global attractor" already mean the maximal compact invariant global $B-$attractor.

Proposition 2.7. *[29] The maximal compact invariant global $B-$attractor coincides with the compact set which is the minimal closed and invariant global $B-$attractor.*

Definition 2.8. A semigroup is called bounded dissipative or B-dissipative if it has a bounded global B-attractor.

Definition 2.9. A semigroup $\{T(t)\}_{t\geq 0}$ belongs to the class $\mathcal{K}$ if there exists $t_0 \geq 0$ such that for each $t > t_0$ the operator $T(t)$ is compact, i. e., for any bounded set $B \subset Z$ its image $T(t)B$ is precompact.

Theorem 2.10. *[18] Let $\{T(t) : Z \to Z, t \geq 0\}$ be a semigroup of class $\mathcal{K}$. If it is B-dissipative, then $\{T(t) : Z \to Z, t \geq 0\}$ has a minimal closed global B-attractor $\mathcal{M}$, which is compact and invariant.*

We refer the reader to [13, 18, 19, 36] to see more details on semigroup theory.

2.2 Multivalued semigroups

Here we reminisce some definitions and results that will be useful in the Section 6.

Definition 2.11. [3,30] A ***generalized semiflow*** $\mathcal{G}$ on a complete metric space (Z,d) is a family of maps $\varphi:[0,\infty)\to Z$ satisfying the conditions:

(H1) For each $z\in Z$ there exists at least one $\varphi\in\mathcal{G}$ with $\varphi(0)=z$.

(H2) If $\varphi\in\mathcal{G}$ and $\tau\geq 0$, then $\varphi^{\tau}\in\mathcal{G}$, where $\varphi^{\tau}(t):=\varphi(t+\tau), \forall t\in[0,\infty)$.

(H3) If $\varphi,\psi\in\mathcal{G}$, and $\psi(0)=\varphi(t)$ for some $t\geq 0$, then $\theta\in\mathcal{G}$, where

$$\theta(\tau):=\begin{cases}\varphi(\tau) & \text{for } \ \tau\in[0,t]\\ \psi(\tau-t) & \text{for } \ \tau\in(t,\infty)\end{cases}$$

(H4) If $\{\varphi_j\}_{j=1}^{\infty}\subset\mathcal{G}$ and $\varphi_j(0)\to z$, then there exists a subsequence $\{\varphi_\mu\}$ of $\{\varphi_j\}$ and $\varphi\in\mathcal{G}$ with $\varphi(0)=z$ such that $\varphi_\mu(t)\to\varphi(t)$ for each $t\geq 0$.

Definition 2.12. A ***multivalued semigroup*** $\{T(t)\}_{t\geq 0}$ ***defined by*** $\mathcal{G}$ is a family of multivalued operators $T(t):P(Z)\to P(Z)$ such that, for each $t\geq 0$, $T(t)E:=\{\varphi(t);\ \varphi\in\mathcal{G} \text{ with } \varphi(0)\in E\}$.

Definition 2.13. Let be $A,E\in P(Z)$. We say that A ***attracts*** E if for any $\varepsilon>0$ there exists $\tau=\tau(\varepsilon,E)\geq 0$ such that $T(t)E\subset O_\varepsilon(A)$ for all $t\geq\tau$, or equivalently, $\operatorname{dist}(T(t)E,A)\to 0$ as $t\to+\infty$.

Definition 2.14. A subset $\mathcal{A}$ is a ***global B-attractor*** if it attracts all bounded subsets of Z, and is a ***global point attractor*** if it attracts all points of Z.

Definition 2.15.
(a) $\mathcal{G}$ is ***bounded dissipative or B-dissipative*** if there is a bounded global B-attractor for $\mathcal{G}$.
(b) $\mathcal{G}$ is ***point dissipative*** if there is a bounded global point attractor for $\mathcal{G}$.
(c) We say that $\mathcal{G}$ is φ***-dissipative*** if there is a bounded set B_0 such that, for any $\varphi\in\mathcal{G}$, $\varphi(t)\in B_0$ for all sufficiently large t.

Remark 2.16. Bounded dissipative $\Rightarrow$ point dissipative $\Rightarrow$ φ-dissipative. (φ**-dissipativity** is called **point dissipativity** in [3]).

Definition 2.17. $\mathcal{G}$ is ***asymptotically compact*** if, for any sequence $\{\varphi_j\} \subset \mathcal{G}$ with $\{\varphi_j(0)\} \in B(Z)$, and for any sequence $\{t_j\}$, $t_j \to +\infty$, the sequence $\{\varphi_j(t_j)\}$ has a convergent subsequence.

Theorem 2.18. *[3] Let $\mathcal{G}$ be a generalized semiflow. If $\mathcal{G}$ has a compact invariant global B-attractor, then $\mathcal{G}$ is φ-dissipative and asymptotically compact. Reciprocally, if $\mathcal{G}$ is φ-dissipative and asymptotically compact, then $\mathcal{G}$ has a compact invariant global B-attractor $\mathcal{A}$. Furthermore $\mathcal{A}$ is the maximal compact invariant subset of Z, and $\mathcal{A}$ is minimal among all closed global B-attractors.*

We refer the reader to [3, 16, 20, 30] to see more details on multivalued semigroup theory.

2.3 Generalized Lebesgue and Sobolev Spaces

The four problems which will be considered concern about variable exponents. So, let us review here the definitions of the Lebesgue and Sobolev Spaces with variable exponents and some results about these spaces.

The generalized Lebesgue spaces $L^{p(x)}(\Omega)$ are defined by $L^{p(x)}(\Omega) = \Big\{u : \Omega \to \mathbb{R} : u \text{ is measurable}; \int_\Omega |u(x)|^{p(x)}dx < \infty\Big\}$, where $\Omega \subset \mathbb{R}^N$, $N \geq 1$, is a measurable set and $p \in L^\infty(\Omega)$, with $p \geq 1$. For $u \in L^{p(x)}(\Omega)$ and $p \in L^\infty_+(\Omega) := \{q \in L^\infty(\Omega) : \text{ess inf } q \geq 1\}$, denote

$$\rho(u) := \int_\Omega |u(x)|^{p(x)}dx,$$

$$p^- = \text{ess inf } p \quad \text{and} \quad p^+ = \text{ess sup } p.$$

$L^{p(x)}(\Omega)$ is a Banach space with the norm

$$\|u\|_{p(x)} := \inf\Big\{\lambda > 0 : \rho\big(\frac{u}{\lambda}\big) \leq 1\Big\}.$$

The generalized Lebesgue-Sobolev spaces are defined by

$$W^{1,p(x)}(\Omega) = \Big\{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\Big\}.$$

We know that $W^{1,p(x)}(\Omega)$ is a Banach Space with the norm

$$\|u\|_* := \|u\|_{p(x)} + \|\nabla u\|_{p(x)}.$$

The space $W_0^{1,p(x)}(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. It is know that $\|\nabla u\|_{p(x)}$ and $\|u\|_*$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$.

Lemma 2.19. *[10, 11] If $u \in L^{p(x)}(\Omega)$. Then,*
(i) $\|u\|_{p(x)} < 1 (= 1; > 1)$ if and only if $\rho(u) < 1 (= 1; > 1)$;
(ii) If $\|u\|_{p(x)} > 1$, then $\|u\|_{p(x)}^{p^-} \leq \rho(u) \leq \|u\|_{p(x)}^{p^+}$;
(iii) If $\|u\|_{p(x)} < 1$, then $\|u\|_{p(x)}^{p^+} \leq \rho(u) \leq \|u\|_{p(x)}^{p^-}$.

Theorem 2.20. *[10, 11] (i) The space $(L^{p(x)}(\Omega), \|\cdot\|_{p(x)})$ is separable.*
(ii) If $p^- > 1$, then $L^{p(x)}(\Omega)$ is reflexive.
(iii) If $p^- > 1$, then $W^{1,p(x)}(\Omega)$ is separable and reflexive.

It follows immediately from the definition of $W_0^{1,p(x)}(\Omega)$ and properties of $W^{1,p(x)}(\Omega)$ that $W_0^{1,p(x)}(\Omega)$ is a reflexive and separable Banach space.

Theorem 2.21 ([10,11])**.** *Let Ω be a bounded domain of $\mathbb{R}^N$ and let $p, q \in L_+^\infty(\Omega)$. Then*

$$L^{p(x)}(\Omega) \subset L^{q(x)}(\Omega)$$

if and only if, $q(x) \leq p(x)$ for a.e. $x \in \Omega$, and in this case, the imbedding is continuous.

Theorem 2.22 ([10])**.** *Let Ω be a bounded domain of $\mathbb{R}^N$ and let $p, q \in C(\overline{\Omega})$ such that $p^-, q^- \geq 1$. Assume that*

$$q(x) < p^*(x) := \begin{cases} Np(x)/(N - p(x)), & p(x) < N \\ \infty, & p(x) \geq N \end{cases},$$

for all $x \in \overline{\Omega}$. Then

$$W^{1,p(x)}(\Omega) \subset L^{q(x)}(\Omega)$$

and the imbedding is continuous and compact.

In particular, it follows that $W^{1,p(x)}(\Omega) \subset L^{p(x)}(\Omega)$ with continuous and compact imbedding since $p(x) < p^*(x)$ for all $x \in \overline{\Omega}$.

From now on we consider $\Omega \subset \mathbb{R}^N$ a bounded smooth domain, $\mathcal{H} := L^2(\Omega)$ and $X := W_0^{1,p(x)}(\Omega)$, where $p(x)$ is continuous in $\overline{\Omega}$, non-constant and moreover $p(x) > 2$ a.e. in Ω. By Theorem 2.22 and Theorem 2.21 it follows that

$$W_0^{1,p(x)}(\Omega) \subset W^{1,p(x)}(\Omega) \subset L^{p(x)}(\Omega) \subset L^2(\Omega).$$

So $X \subset \mathcal{H}$ and the imbedding is continuous and compact. Also we have $X \subset \mathcal{H} \subset X^*$ with continuous and dense imbeddings.

We refer the reader to [7–12, 15, 21, 33] to see other properties of the Lebesgue and Sobolev spaces with variable exponents.

3 A reaction-diffusion equation with a globally Lipschitz reaction term

This section is based on the paper [28] and concerns the study of the asymptotic behavior of the following $p(x)$-Laplacian parabolic problem under homogeneous Dirichlet boundary conditions

$$\begin{cases} \frac{du}{dt}(t) - \Delta_{p(x)}(u(t)) = B(u(t)), \ \ t > 0 \\ u(0) = u_0 \in \mathcal{H} := L^2(\Omega), \end{cases} \tag{3.1}$$

where $-\Delta_{p(x)}(u) := -div(|\nabla u|^{p(x)-2}\nabla u)$, Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 1$, $p(x) \in C(\bar{\Omega})$ and $p(x) > 2$ a.e. in Ω and $B : \mathcal{H} \to \mathcal{H}$ is a globally Lipschitz map with Lipschitz constant $L \geq 0$.

In [33] it was considered the operator $A_1 : X \to X^*$ defined by

$$A_1 u(v) = \int_\Omega |\nabla u(x)|^{p(x)-2} \nabla u(x) \nabla v(x) \, dx,$$

for each $u, v \in X := W_0^{1,p(x)}(\Omega)$. Using Lemma 2.19, the authors proved the following estimates on this operator

Lemma 3.1. *Let $p(x) \in C(\bar{\Omega})$ with $p(x) > 2$ a.e. in Ω and $\tilde{p}(x) := p(x) - 1$.*

a) If $\|v\|_X := \|\nabla v\|_{p(x)} \leq 1$ then
(i) $\langle A_1 v, v\rangle_{X^*,X} \geq \|v\|_X^{p^+}$;
(ii) $\|A_1 v\|_{X^*} \leq \|v\|_X^{p^-} + \frac{1}{2}$;
(iii) $\|A_1 v\|_{X^*} \leq 2\|v\|_X^{\tilde{p}^-}$.

b) If $\|v\|_X \geq 1$ then
(iv)$\langle A_1 v, v\rangle_{X^*,X} \geq \|v\|_X^{p^-}$;
(v)$\|A_1 v\|_{X^*} \leq \|v\|_X^{p^+} + \frac{1}{2}$;
(vi)$\|A_1 v\|_{X^*} \leq 2\|v\|_X^{\tilde{p}^+}$.

Observe that $-\Delta_{p(x)}$ is the realization, named $A_{\mathcal{H}}$, of the operator A_1 at $\mathcal{H}$. Also, it was proved in [33] that $-\Delta_{p(x)}$ is a maximal monotone operator in $\mathcal{H}$ if $p(x) \in C(\bar{\Omega})$ and $p(x) > 2$ a.e. in Ω. One can use the theory of maximal monotone operators to guaranty existence of a unique solution to the problem (3.1) (see for example [5, 6]). By Proposition 1 at p.695 in [6] the problem (3.1) determines a continuous semigroup of nonlinear operators $\{T(t) : \mathcal{H} \to \mathcal{H},\ t \geq 0\}$, where for each $u_0 \in \mathcal{H}$, $t \mapsto T(t)u_0$ is a weak global solution to the problem (3.1) beginning at u_0. This semigroup is such that

$$\mathbb{R}^+ \times cl_{\mathcal{H}}(\mathcal{D}(-\Delta_{p(x)})) \to cl_{\mathcal{H}}(\mathcal{D}(-\Delta_{p(x)})) = \mathcal{H},\ (t, u_0) \mapsto T(t)u_0$$

is a continuous map and, if $u_0 \in \mathcal{D}(-\Delta_{p(x)}u)$, then $u(\cdot) := T(\cdot)u_0$ is a Lipschitz continuous strong global solution to the problem (3.1). We show in this work how to prove the existence of the minimal closed global B-attractor to the problem (3.1).

Let us first review the constant exponent case in order to see some differences. Let us consider the following conditions:
$(H1)$: (i) Let H be a Hilbert space and V a reflexive Banach space such that $V \subset H \subset V^*$ with continuous inclusions and V^* denoting the topological dual of V. Assume in addition that V is dense in H.
(ii) Let $A : V \to V^*$ (defined on all of V) be a nonlinear, monotone, coercive and hemicontinuous operator.

$(H2)$: There are constants $w_1, w_2 > 0$, $c_1 \in \mathbb{R}$ and $p \geq 2$ such that for all $v \in V$ the following two conditions hold:

$$\langle Av, v\rangle_{V^*,V} \geq w_1\|v\|_V^p + c_1 \tag{3.2}$$

$$\|Av\|_{V*} \leq w_2(1 + \|v\|_V^{p-1}). \tag{3.3}$$

Under the hypothesis $(H1)$, the operator A_H, defined by,

$$\begin{cases} D(A_H) = \{v \in V;\ Av \in H\} \\ A_H(v) = Av, \text{ se } \ v \in D(A_H), \end{cases} \tag{3.4}$$

is a maximal monotone operator in H [6].

Lemma 3.2. *(Lema 1, p. 696, [6]) If (H1) and (H2) hold, then the domain $D(A_H)$ is dense in H, i.e., $\overline{D(A_H)} = H$.*

Although it was not possible to check (H2), the author of [33] followed the ideas in proof of the Lemma 3.2 to prove that $cl_{\mathcal{H}}(\mathcal{D}(-\Delta_{p(x)})) = \mathcal{H} := L^2(\Omega)$ for $p(x) \in C(\bar{\Omega})$ and $2 < p(x) \leq 3 - \delta$, for a.a. $x \in \Omega$, $\delta > 0$ arbitrarily small and fixed. In 2012, the authos of [32] proved that the operator $-\Delta_{p(x)}$ is of the subdifferential type and as a consequence they obtained that $cl_{\mathcal{H}}(\mathcal{D}(-\Delta_{p(x)})) = \mathcal{H} := L^2(\Omega)$ for $p(x) \in C(\bar{\Omega})$ and $p(x) > 2$ a.e. in Ω. More precisely, $-\Delta_{p(x)}$ is the subdifferential of a proper, convex and lower semicontinuos function $\varphi : H \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\varphi(u) := \begin{cases} \displaystyle\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx, & \text{if } u \in V \\ +\infty, & \text{otherwise} \end{cases}.$$

In [28] the author proved that $-\text{div}(|\nabla u|^{p(x)-2}\nabla u)$ generates a compact semigroup.

Theorem 3.3. *[6] If the hypotheses $(H1)$ and $(H2)$ hold and $p > 2$, then for any $u_0 \in D(A_H)$ and all $T > 0$ we have that:*

$$\int_0^T \|u(s)\|_V^p ds \leq \mathcal{C}_1(\|u_0\|_H, T) \tag{3.5}$$

$$\int_0^T \|\frac{du}{dt}(s)\|_{V^*}^{p'} ds \leq \mathcal{C}_2(\|u_0\|_H, T), \tag{3.6}$$

where $\mathcal{C}_1$ and $\mathcal{C}_2$ are locally bounded functions and u is the solution of the problem $\frac{du}{dt}(t)+A_H(u(t)) = B(u(t))$, $u(0) = u_0$. Moreover, if the inclusion $V \subset H$ is compact, then (the semigroup map) $T(t) : H \to H$ is a compact map for each $t > 0$.

Although it was not possible to check (H2) for the variable exponent case, the author proved in [28] the following result:

Theorem 3.4. *Let $u_0 \in \mathcal{D}(-\Delta_{p(x)})$ and $u(\cdot) = T(\cdot)u_0$ be the global solution of (3.1). For all $T > 0$ we have*

(i) $\int_0^T \|u(s)\|_X^{p^-} ds \leq \mathcal{C}_1(\|u_0\|_{\mathcal{H}}, T)$;

(ii) $\|u\|_{L^\infty(0,T;\mathcal{H})} \leq \mathcal{C}_2(\|u_0\|_{\mathcal{H}}, T)$,

where $\mathcal{C}_1$ and $\mathcal{C}_2$ are locally bounded functions.

To guarantee the existence of a global attractor we have to prove that the semigroup is $B-$dissipative and of class $\mathcal{K}$ in order to apply Theorem 2.10. To prove the next result it was necessary to invent an average technique.

Theorem 3.5. *[28] Let $\{T(t)\}$ be the semigroup associated with the problem (3.1) on $\mathcal{H}$. Then $T(t) : \mathcal{H} \to \mathcal{H}$ is of class $\mathcal{K}$.*

Proof. Let $B \subset \mathcal{H}$ a bounded set and $t > 0$. Since $\mathcal{D}(-\Delta_{p(x)})$ is dense in $\mathcal{H}$, then by Lemma 2, p. 697, in [6], it is sufficient to check compactness of the semigroup considering initial data u_0 from $\mathcal{D}(-\Delta_{p(x)})$, that is, it is sufficient to prove that $T(t)\Big(B \cap \mathcal{D}(-\Delta_{p(x)})\Big)$ is precompact in $\mathcal{H}$. Take any sequence $\{u_{0n}\} \subset B \cap \mathcal{D}(-\Delta_{p(x)})$ and consider the sequence $\{T(t)u_{0n}\}$. Clearly, $\|u_{0n}\|_{\mathcal{H}} \leq r$, $\forall\ n \in \mathbb{N}$. Consider $T > t$, $\delta_1 > 0$ and $\delta_2 > 0$ such that $0 < \delta_1 < \delta_2 < t < T$. We denote $u_n(\cdot) = T(\cdot)u_{0n}$. Using *(i)* of

Theorem 3.4 we have for all $E \subset (0,T)$ with $M(E) \geq \delta_1$ that

$$\frac{1}{m(E)} \int_E \|u_n(s)\|_X^{p^-} ds \leq \frac{1}{\delta_1}\Big(\|u_{0n}\|_{\mathcal{H}}^2 + C_{10}\Big) \leq \frac{1}{\delta_1}\Big(r^2 + C_{10}\Big) =: K_0, \tag{3.7}$$

where $K_0 > 0$ is a constant and $m(E) = |E|$ is the Lebesgue measure of E. Now, consider $K_1 > 0$ a constant such that $K_1 > K_0$.

Statement. For each $n \in \mathbb{N}$, there exists $s_n \in (0, \delta_2)$ such that $\|u_n(s_n)\|_X^{p^-} \leq K_1$.

In fact, if $\|u_n(s)\|_X^{p^-} > K_1,\ \forall\, s \in (0, \delta_2)$, then

$$\frac{1}{\delta_2} \int_0^{\delta_2} \|u_n(s)\|_X^{p^-} ds \geq \frac{1}{\delta_2}\, K_1\, \delta_2 = K_1.$$

But, since that $m\big((0,\delta_2)\big) = \delta_2 > \delta_1$, we have by (3.7) that

$$\frac{1}{\delta_2} \int_0^{\delta_2} \|u_n(s)\|_X^{p^-} ds \leq K_0,$$

what is a contradiction. So, there exists $s_n \in (0, \delta_2)$ such that $\|u_n(s_n)\|_X^{p^-} \leq K_1$ and the proof of the statement is completed.

Now, since that $X \subset \mathcal{H}$ compactly, there is a subsequence $\{u_{n_j}(s_{n_j})\}$ of $\{u_n(s_n)\}$ such that $T(s_{n_j})u_{0n_j} = u_{n_j}(s_{n_j}) \to v_0$ in $\mathcal{H}$ as $j \to +\infty$. As $s_{n_j} \in (0, \delta_2) \subset [0, \delta_2]$, there is a subsequence, which we do not relabel, such that $s_{n_j} \to s_0 \in [0, \delta_2]$. Since that the semigroup is such that

$$\mathbb{R}^+ \times cl_{\mathcal{H}}(\mathcal{D}(-\Delta_{p(x)})) \ni (t, u_0) \mapsto T(t)u_0 \in cl_{\mathcal{H}}(\mathcal{D}(-\Delta_{p(x)})) = \mathcal{H}$$

is a continuous map and $\Big(t - s_{n_j}, T(s_{n_j})u_{0n_j}\Big) \to (t - s_0, v_0)$ as $j \to +\infty$, we have that $T(t)u_{0n_j} = T(t - s_{n_j})T(s_{n_j})u_{0n_j} \to T(t-s)v_0$ in $\mathcal{H}$ as $j \to +\infty$. The proof is completed. □

Theorem 3.6. *[28] Let $\{T(t)\}$ be the semigroup associated with the problem (3.1) on $\mathcal{H}$. Then $\{T(t)\}$ is bounded dissipative in $\mathcal{H}$.*

Proof. It is sufficient to consider initial data $u_0 \in \mathcal{D}(-\Delta_{p(x)})$. Consider the embedding constant $\sigma > 0$ from $X \hookrightarrow \mathcal{H}$ and the numbers $t_1 = 1$ and $r_0 = \Big[(2\sigma^{p^-} C_8)^{2/p^-} + (\sigma^{-p^-}(\frac{p^-}{2} - 1)^{\frac{-2}{p^- - 2}}\Big]^{1/2}$. Let $t > t_1$ and $u(\cdot) = T(\cdot)u_0$.

If $\|u(t)\|_X < 1$, then $\|u(t)\|_{\mathcal{H}} \leq \sigma \|u(t)\|_X < \sigma$.

If $\|u(t)\|_X \geq 1$, then Multiplying the equation in (3.1) by u and using Lemma 3.1, we have that

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{\mathcal{H}}^2 \leq -\|u(t)\|_X^{p^-} + C_1\|u(t)\|_X^2 + C_2\|u(t)\|_X, \tag{3.8}$$

where $C_1 = C_1(L, \sigma) > 0$ and $C_2 = C_2(\sigma) \geq 0$ are constants ($C_2 = 0$ if, and only if, $B(0) = 0$).

Now, consider $\epsilon > 0$ arbitrarily small, $\alpha := \frac{p^-}{2}$. Then, using Young's Inequalities, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{\mathcal{H}}^2 \leq (-1 + \frac{1}{\alpha}\epsilon^\alpha + \frac{1}{p^-}\epsilon^{p^-})\|u(t)\|_X^{p^-} + C_3 + C_4, \tag{3.9}$$

where $C_3 = C_3(\epsilon, p^-) > 0$ and $C_4 = C_4(\epsilon, p^-) > 0$ are constants.

Now, as $\epsilon > 0$ was arbitrary, choose $\epsilon_0 > 0$ sufficiently small such that $\frac{1}{\alpha}\epsilon_0^\alpha + \frac{1}{p^-}\epsilon_0^{p^-} < \frac{1}{2}$ in the case $B(0) \neq 0$. For the case $B(0) = 0$, choose $\epsilon_0 > 0$ sufficiently small such that $\frac{1}{\alpha}\epsilon_0^\alpha < \frac{1}{2}$. So, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{\mathcal{H}}^2 \leq -\frac{1}{2}\|u(t)\|_X^{p^-} + C_5 \leq -\frac{\sigma^{-p^-}}{2}\|u(t)\|_{\mathcal{H}}^{p^-} + C_5, \tag{3.10}$$

where $C_5 = C_5(\epsilon_0, p^-) > 0$ is a constant.

Hence the function $y(t) := \|u(t)\|_{\mathcal{H}}^2$ satisfies the differential inequality

$$y'(t) \leq -\sigma^{-p^-} y(t)^{\frac{p^-}{2}} + 2C_5. \tag{3.11}$$

Therefore, from Lemma 5.1, p. 163 in [36], we get

$$y(t) = \|u(t)\|_{\mathcal{H}}^2 \leq \Big(2\sigma^{p^-} C_5\Big)^{2/p^-} + \Big(\sigma^{-p^-}(\frac{p^-}{2} - 1)t\Big)^{\frac{-2}{p^- - 2}}. \tag{3.12}$$

So, taking $r_1 > \max\{r_0, \sigma\}$, we have

$$\|u(t)\|_{\mathcal{H}} \leq r_1, \ \forall\, t \geq t_1. \tag{3.13}$$

Note that r_1 not depend on initial data. So, the set $\{w_0 \in \mathcal{H}; \|w_0\|_{\mathcal{H}} \leq r_1\}$ attracts bounded subsets of $\mathcal{H}$ in the $\mathcal{H}$-norm. The proof is completed. □

As a consequence of Theorem 2.10, Theorem 3.5 and Theorem 3.6 we obtain the following result

Theorem 3.7. *The semigroup $\{T(t)\}$ associated with problem (3.1) has a minimal closed global B-attractor $\mathcal{M}$, which is compact and invariant.*

4 A PDE with a non globally Lipschitz perturbation

This section is based on the papers [22, 34, 35]. In this section we consider problems of the form

$$\begin{cases} \frac{\partial u}{\partial t}(t) - \operatorname{div}\left(|\nabla u(t)|^{p(x)-2}\nabla u(t)\right) + f(x,u(t)) = g(x), \quad t > 0 \\ u(0) = u_0 \end{cases} \tag{4.1}$$

under homogeneous Dirichlet boundary conditions, where $u_0 \in \mathcal{H} := L^2(\Omega)$, Ω a bounded smooth domain in $\mathbb{R}^n$, $n \geq 1$, $g \in L^2(\Omega)$, $p(\cdot) \in C(\overline{\Omega})$ $2 < p(x) \leq a$, for all $x \in \Omega$.

We assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a non globally Lispchitz Carathéodory mapping satisfying the following conditions: there exist positive constants ℓ, k, c_1 and $c_2 \geq 1$ such that

$$(f(x,s_1) - f(x,s_2))(s_1 - s_2) \geq -\ell(s_1 - s_2)^2, \tag{4.2}$$

$\forall\, x \in \Omega$ and $s_1, s_2 \in \mathbb{R}$,

$$c_2|s|^{q(x)} - k \leq f(x,s)s \leq c_1|s|^{q(x)} + k, \tag{4.3}$$

$\forall\, x \in \Omega$ and $s \in \mathbb{R}$, where $q \in C(\overline{\Omega})$ with $2 < q^- := \inf_{x\in\Omega} q(x) \le q^+ := \sup_{x\in\Omega} q(x)$.

We denote

$$\tilde{X} := \Big\{ u : u \in L^{p^-}\left(0,T; W_0^{1,p(x)}(\Omega)\right) \cap L^{\infty}\left(0,T; L^2(\Omega)\right) \cap L^{q(x)}(\Omega \times (0,T)) \text{ with } \nabla u \in L^{p(x)}(\Omega \times (0,T)) \Big\},$$

where $L^{q(x)}(\Omega \times (0,T)) = \left\{ u : \int_0^T \int_\Omega |u(t,x)|^{p(x)}\, dx\, dt < \infty \right\}$.

Definition 4.1 ([22, Definition 2.1]). A solution of problem (4.1) is a function $u \in \tilde{X}$ such that

$$\int_0^t \int_\Omega \left(-u \frac{d\varphi}{dt} + |\nabla u|^{p(x)-2} \nabla u \nabla \varphi + f(x,u)\varphi \right) dx d\tau = \int_0^t \int_\Omega g\varphi dx d\tau - \int_\Omega u\varphi dx|_0^t$$

holds for any $t \le T$ and all $\varphi \in \tilde{X}$ with $\dfrac{d\varphi}{dt} \in \tilde{X}^*$, where $\tilde{X}^*$ is the dual space of $\tilde{X}$.

Theorem 4.2 ([22, Theorem 2.1]). *If $q^+ < \infty$ then the problem (4.1) admits a unique solution $u \in C([0,T]; \mathcal{H})$. Moreover, the mapping $u_0 \mapsto u(t)$ is continuous in $\mathcal{H}$.*

The solutions to the problem (4.1) have different nature than the solutions to problem (3.1). The solutions to the problem (4.1) are weak solutions obtained from the use of Faedo-Galerkin method.

The author of [22] proved the existence of a global attractor and the authors of [34, 35] studied the continuity of the flows and upper semicontinuity of global attractors with respect to variation of parameters for the problem (4.1). It is also worth to mention that the authors of [35] proved that $C_0^\infty(\Omega) \subset \mathcal{D}(-\Delta_{p(x)})$ when $p(\cdot) \in C^1(\overline{\Omega})$.

The main difference to prove the existence of global attractor to problem (4.1) in comparison with problem (3.1) is the estimate in the Sobolev space, so let us present here the technique to obtain this estimate.

Lemma 4.3. *[35] Let u be a solution of (4.1) with $u(0) = u_0 \in \mathcal{H}$.*
a) Given $T_0 > 0$, there exists a positive number r_0 such that $\|u(t)\|_{\mathcal{H}} \le r_0$,

for each $t \geq T_0$.
b) Given a bounded set $B \subset H$, there exists $D_1 > 0$ such that $\|u(t)\|_{\mathcal{H}} \leq D_1$ for all $t \geq 0$ such that $u_0 \in B$.
The constants r_0 and D_1 in Lemma 4.3 do not depend on the initial data.

Theorem 4.4. *[35] Let u be a solution of (4.1). Given $T_1 > 1$, there exists a positive constant $r_1 > 0$ such that $\|u(t)\|_X < r_1$, for every $t \geq T_1$.*

Proof. Multiplying the equation in (4.1) by u and using (4.3) we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 dx + \min\{1, c_2\}\left[\int_\Omega |\nabla u|^{p(x)}dx + \int_\Omega |u|^{q(x)}dx\right] \\
&\leq \frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 dx + \int_\Omega |\nabla u|^{p(x)}dx + c_2\int_\Omega |u|^{q(x)}dx \\
&\leq \int_\Omega \frac{g\epsilon^{\frac{1}{2}}u}{\epsilon^{\frac{1}{2}}}dx + k|\Omega| \\
&\leq \frac{1}{2\epsilon}\int_\Omega |g|^2 dx + \frac{1}{2}\epsilon\int_\Omega |u|^2 dx + k|\Omega|.
\end{aligned}$$

Then

$$\begin{aligned}
&\frac{d}{dt}\|u\|^2_{\mathcal{H}} + 2\min\{1, c_2\}\int_\Omega \left(|\nabla u|^{p(x)} + |u|^{q(x)}\right)dx \\
&\leq \frac{1}{\epsilon}\int_\Omega |g|^2 dx + \epsilon\int_\Omega |u|^2 dx + 2k|\Omega|.
\end{aligned} \tag{4.4}$$

Now, if $\theta(x) := \dfrac{q(x)}{2}$ we have

$$\begin{aligned}
\int_\Omega |u|^2 dx &\leq \int_\Omega \left(\frac{1}{\theta(x)}|u|^{q(x)} + \frac{1}{\theta'(x)}\right)dx \\
&= \int_\Omega \frac{2}{q(x)}|u|^{q(x)}dx + \int_\Omega \frac{q(x)-2}{q(x)}dx \\
&\leq \int_\Omega |u|^{q(x)}dx + C_4|\Omega|.
\end{aligned} \tag{4.5}$$

Using (4.5) in (4.4) we obtain

$$\begin{aligned}
&\frac{d}{dt}\|u\|^2_{\mathcal{H}} + 2\min\{1, c_2\}\int_\Omega \left(|\nabla u|^{p(x)} + |u|^{q(x)}\right)dx \\
\leq\ &\frac{1}{\epsilon}\|g\|^2_{\mathcal{H}} + \epsilon\int_\Omega |u|^{q(x)}dx + (2k + \epsilon C_4)|\Omega|.
\end{aligned}$$

Since $c_2 \geq 1$ we have

$$\frac{d}{dt}\|u\|_{\mathcal{H}}^2 + 2\int_\Omega |\nabla u|^{p(x)}dx + (2-\epsilon)\int_\Omega |u|^{q(x)}dx \leq \frac{1}{\epsilon}\|g\|_{\mathcal{H}}^2 + (2k+\epsilon C_4)|\Omega|,$$

and taking $\epsilon_0 > 0$ such that $2-\epsilon_0 \geq 1$ we obtain

$$\begin{aligned}\frac{d}{dt}\|u\|_{\mathcal{H}}^2 + \int_\Omega \left(|\nabla u|^{p(x)} + |u|^{q(x)}\right) dx &\leq \frac{1}{\epsilon_0}\|g\|_{\mathcal{H}}^2 + (2k+\epsilon_0 C_4)|\Omega| \\ &\leq C_5\|g\|_{\mathcal{H}}^2 + C_5|\Omega|,\end{aligned} \tag{4.6}$$

where C_4 and C_5 are positive constants. Integrating (4.6) over $[t, t+1]$, $t \geq T_0$, using Lemma 4.3 we obtain

$$\begin{aligned}\int_t^{t+1}\int_\Omega \left(|\nabla u(\tau)|^{p(x)} + |u(\tau)|^{q(x)}\right) dx d\tau &\leq C_5\|g\|_{\mathcal{H}}^2 + C_5|\Omega| + \|u(t)\|_{\mathcal{H}}^2 \\ &\leq C_5\|g\|_{\mathcal{H}}^2 + C_5|\Omega| + r_0^2 =: C_6\end{aligned} \tag{4.7}$$

for all $t \geq T_0$.

The function $[0,T] \ni t \mapsto \tilde{f}(t)(\cdot) := g(\cdot) - f(\cdot, u(t,\cdot))$ is in $L^2(0,T;H)$, so it follows from [5, Theorem 3.6] that for all $\tau \geq T_0$

$$\begin{aligned}&\int_\Omega \left|\frac{du}{dt}(\tau)\right|^2 dx + \frac{d}{dt}\int_\Omega \frac{1}{p(x)}|\nabla u(\tau)|^{p(x)}dx + \int_\Omega \frac{d}{dt}F(x,u(\tau))dx \\ &= \int_\Omega \left|\frac{du}{dt}(\tau)\right|^2 dx + \langle \partial\varphi_{p(x)}(u(\tau)), \frac{du}{dt}(\tau)\rangle + \int_\Omega f(x,u(\tau))\frac{du}{dt}(\tau)dx \\ &= \int_\Omega g\frac{du}{dt}(\tau)dx \\ &\leq \frac{1}{2}\|g\|_{\mathcal{H}}^2 + \frac{1}{2}\left\|\frac{du}{dt}(\tau)\right\|_{\mathcal{H}}^2,\end{aligned}$$

where $F(x,s) = \int_0^s f(x,\tau)d\tau$. Thus,

$$\frac{1}{2}\int_\Omega \left|\frac{du}{dt}(\tau)\right|^2 dx + \frac{d}{dt}\int_\Omega \frac{1}{p(x)}|\nabla u(\tau)|^{p(x)}dx + \frac{d}{dt}\int_\Omega F(x,u(\tau))dx \leq \frac{1}{2}\|g\|_{\mathcal{H}}^2. \tag{4.8}$$

From assumption (4.3), there exist positive constants $\tilde{c}_1, \tilde{c}_2, C_7$ such that

$$\tilde{c}_2|s|^{q(x)} - C_7 \leq F(x,s) \leq \tilde{c}_1|s|^{q(x)} + C_7. \tag{4.9}$$

Integrating (4.8) over $[\eta, t+1]$, $T_0 \le t < \eta < t+1$, yields

$$\begin{aligned}
&\int_\Omega \frac{1}{p(x)}|\nabla u(t+1)|^{p(x)}dx + \int_\Omega F(x,u(t+1))dx \\
&\le \frac{1}{2}\int_\eta^{t+1}\int_\Omega \left|\frac{du}{dt}(\tau)\right|^2 dx\ d\tau + \int_\Omega \frac{1}{p(x)}|\nabla u(t+1)|^{p(x)}dx + \int_\Omega F(x,u(t+1))dx \\
&\le \frac{1}{2}\|g\|_H^2 + \int_\Omega \frac{1}{p(x)}|\nabla u(\eta)|^{p(x)}dx + \int_\Omega F(x,u(\eta))dx.
\end{aligned}$$

Integrating the above inequality with respect to η between t and $t+1$, we obtain

$$\begin{aligned}
&\int_\Omega \frac{1}{p(x)}|\nabla u(t+1)|^{p(x)}dx + \int_\Omega F(x,u(t+1))dx \\
&\le \frac{1}{2}\|g\|_{\mathcal{H}}^2 + \int_t^{t+1}\int_\Omega \frac{1}{p(x)}|\nabla u(\eta)|^{p(x)}dxd\eta + \int_t^{t+1}\int_\Omega F(x,u(\eta))dxd\eta.
\end{aligned}$$

Using the above inequality, the assumptions on $p(x)$, (4.9) and (4.7), we get

$$\begin{aligned}
&\frac{1}{a}\int_\Omega |\nabla u(t+1)|^{p(x)}dx + \tilde{c}_2\int_\Omega |u(t+1)|^{q(x)}dx - C_7|\Omega| \\
&\le \frac{1}{2}\|g\|_{\mathcal{H}}^2 + \int_t^{t+1}\int_\Omega \frac{1}{p(x)}|\nabla u(\eta)|^{p(x)}dxd\eta + \int_t^{t+1}\int_\Omega F(x,u(\eta))dxd\eta \\
&\le \frac{1}{2}\|g\|_{\mathcal{H}}^2 + \frac{1}{p}\int_t^{t+1}\int_\Omega |\nabla u(\eta)|^{p(x)}dxd\eta + \tilde{c}_1\int_t^{t+1}\int_\Omega |u(\eta)|^{q(x)}dxd\eta + C_7|\Omega| \\
&\le \frac{1}{2}\|g\|_{\mathcal{H}}^2 + \max\left\{\frac{1}{p}, \tilde{c}_1\right\}\int_t^{t+1}\int_\Omega \left(|\nabla u(\eta)|^{p(x)} + |u(\eta)|^{q(x)}\right)dxd\eta + C_7|\Omega| \\
&\le \frac{1}{2}\|g\|_{\mathcal{H}}^2 + \max\left\{\frac{1}{p}, \tilde{c}_1\right\}C_6 + C_7|\Omega|.
\end{aligned}$$

Then we conclude that

$$\begin{aligned}
&\int_\Omega \left(|\nabla u(t)|^{p(x)} + |u(t)|^{q(x)}\right)dx \\
&\le \frac{1}{\min\left\{\frac{1}{a}, \tilde{c}_2\right\}}\left[\frac{1}{2}\|g\|_{\mathcal{H}}^2 + \max\left\{\frac{1}{p}, \tilde{c}_1\right\}C_6 + 2C_7|\Omega|\right] =: C_8
\end{aligned} \tag{4.10}$$

for all $t \ge T_0 + 1$.

Now, if $t \geq T_0 + 1$ and $\|\nabla u(t)\|_{p(x)} \geq 1$, we obtain

$$\|u(t)\|_X^{p^-} \leq \int_\Omega |\nabla u(t)|^{p(x)} dx \leq C_8$$

and so

$$\|u(t)\|_X \leq \max\left\{C_8^{\frac{1}{p^-}}, 1\right\} =: r_1 \quad \forall\, t \geq T_0 + 1.$$

The proof is completed. □

Proposition 4.5. *Let $\{S(t)\}$ be the semigroup associated with problem (4.1) on $\mathcal{H}$. Then, for each $t \geq 1$, $S(t) : \mathcal{H} \to \mathcal{H}$ is a compact operator.*

Proof. Once we have the estimates in the Sobolev Space, i.e., estimates for the norm $\|\cdot\|_X$, we can prove the results by using the average technique as in Theorem 3.5. Let $t \geq 1$ and $\{u_{0k}\}$ be a sequence of initial data, $\|u_{0k}\|_{\mathcal{H}} \leq r$, and $(u)_k(t) = S(t)u_{0k}$ the corresponding solution of problem (4.1). Take an arbitrary interval $(\alpha, \beta) \subset (1, T)$ with $2\alpha < \beta$. Applying Theorem 4.4 we may estimate, for a positive constant K,

$$\frac{1}{\beta - \alpha} \int_\alpha^\beta \|(u)_k(t)\|_X^{p^-}\, dt \quad \leq \quad \frac{1}{\beta - \alpha} \int_\alpha^\beta R^{p^-} dx \leq K. \tag{4.11}$$

There exists a sequence $\{t_k\} \subset [\alpha, \beta]$ such that

$$I_k(t_k) := \|(u)_k(t_k)\|_X^{p^-} \leq K + 1.$$

Indeed: if we assume that $I_k(t) > K+1$ for some $k \in \mathbb{N}$ and all $t \in (\alpha, \beta)$, then

$$K \geq \frac{1}{\beta - \alpha} \int_\alpha^\beta \|(u)_k(t)\|_X^{p^-}\, dt = \frac{1}{\beta - \alpha} \int_\alpha^\beta I_k(t)\, dt \geq K + 1,$$

which is impossible. Since $\|(u)_k(t_k)\|_X$ are uniformly bounded with respect to k and the embedding $X \subset \mathcal{H}$ is compact, there is a subsequence $\{t_{k_j}\}$ such that $t_{k_j} \to t_0$ and $S(t_{k_j})u_{0k_j} = (u)_{k_j}(t_{k_j}) \to v_0$ in $\mathcal{H}$ as $k_j \to \infty$.

Using the joint continuity of the map S (see [34, Theorem 5]), we have

$$S(t)u_{0k_j} = S(t - t_{k_j})S(t_{k_j})u_{0k_j} \to S(t - t_0)v_0$$

as $k_j \to \infty$. □

Note that Lemma 4.3 provide us $B-$dissipativity. Therefore, the existence of a global attractor can be obtained by applying Theorem 2.10.

5 A partial differential inclusion

This section is based on the paper [27]. Let us consider the following problem

$$(P1) \begin{cases} \dfrac{\partial u}{\partial t}(t) - div(|\nabla u(t)|^{p(x)-2}\nabla u(t)) \in F(u(t)) + h, \quad t > 0 \\ u(0) = u_0 \end{cases}$$

under homogeneous Dirichlet boundary conditions, where $p(\cdot) \in C(\bar{\Omega})$, $p^- := \inf p(x) > 2$, $\Omega \subset \mathbb{R}^n$, $n \geq 1$, is a bounded smooth domain, $h, u_0 \in \mathcal{H} := L^2(\Omega)$, $F : \mathcal{D}(F) \subset \mathcal{H} \to \mathcal{P}(\mathcal{H})$, given by $F(y(\cdot)) = \{\xi(\cdot) \in \mathcal{H} : \xi(x) \in f(y(x))$ x-a.e. in $\Omega\}$ with $f : \mathbb{R} \to \mathcal{C}_v(\mathbb{R})$ ($\mathcal{C}_v(\mathbb{R})$ is the set of all nonempty, bounded, closed, convex subsets of $\mathbb{R}$) be a multivalued map. Assume that f is Lipschitz, i.e., $\exists\ C \geq 0$ such that $\text{dist}_H(f(x), f(z)) \leq C|x - z|\ \forall\ x, z \in \mathbb{R}$. Consequently, the map $F(u) + h$ has values in $\mathcal{C}_v(\mathcal{H})$ and it is also Lipschitz.

5.1 A preliminary abstract theory for inclusions

This subsection is based on the paper [20], therefore we strongly suggest that the reader consult the original sources when using the results for further research. We include them to make this text self-contained.

Consider the following evolution inclusion

$$\frac{dy(t)}{dt} \in A(y(t)) + F(y(t)),\ t \in [0, T], \tag{5.1}$$

with the initial condition

$$y(0) = y_0 \in H. \tag{5.2}$$

Let us consider the next conditions:

(A) The operator A is maximal monotone in the Hilbert space H.

(F_1) $F : H \to \mathcal{C}_v(H)$, where $\mathcal{C}_v(H)$ is the set of all nonempty, bounded, closed and convex subsets of H.

(F_2) The map F is Lipschitz on $\overline{\mathcal{D}(A)}$, i.e., there exists $c \geq 0$ such that

$$\text{dist}_H(F(y_1), F(y_2)) \leq c \parallel y_1 - y_2 \parallel_H, \text{ for all } y_1, y_2 \in \overline{\mathcal{D}(A)},$$

where $\text{dist}_H(\cdot, \cdot)$ denotes the Hausdorff metric of bounded sets.

Consider also the next inclusion

$$\frac{dy(t)}{dt} \in A(y(t)) + f(t), \ t \in [0, T], \tag{5.3}$$

with the initial condition

$$y(0) = y_0 \in H, \tag{5.4}$$

where $f(\cdot) \in L^1([0, T]; H)$ and $L^1([0, T]; H)$ is the space of Bochner integrable functions.

Definition 5.1. [20] The continuous function $y : [0, T] \to H$ is called an integral solution of the problem (5.3), (5.4) if:

i) $y(0) = y_0$;

ii) $\forall\, u \in \mathcal{D}(A),\ \forall\, v \in A(u)$,

$$\|y(t) - u\|_H^2 \leq \|y(s) - u\|_H^2 + 2\int_s^t \langle f(\tau) + v, y(\tau) - u\rangle d\tau, \ t \geq s. \tag{5.5}$$

Definition 5.2. [17] The continuous function $y : [0, T] \to H$ is called a strong solution of the problem (5.3), (5.4) if $y(0) = y_0$ and $y(\cdot)$ is absolutely continuous on every compact subsets of $(0, T)$ and satisfies (5.3) almost everywhere on $(0, T)$.

Definition 5.3. [20] The continuous function $y : [0,T] \to H$ is called an integral solution of the problem (5.1), (5.2) if:

i) $y(0) = y_0$;

ii) For some selection $f \in L^1([0,T],H)$, $f(t) \in F(y(t))$ a.e. on $[0,T]$ and the inequality (5.5) holds.

Definition 5.4. [17,31] The continuous function $y : [0,T] \to H$ is called a strong solution of the problem (5.1), (5.2) if there exists a selection $f \in L^1([0,T],H)$, $f(t) \in F(y(t))$ a.e. on $[0,T]$ such that $y : [0,T] \to H$ is a strong solution of the problem (5.3), (5.4).

Remark 5.5. [4,20] If the condition (A) holds and $f \in L^1([0,T];H)$, then for every $y_0 \in \overline{\mathcal{D}(A)}$, there exists a unique integral solution $y(\cdot)$ of the problem (5.3), (5.4) for each $T > 0$. We shall denote $y(\cdot) = I(y_0)f(\cdot)$. Moreover, for any integral solutions $y_i(\cdot) = I(y_{i0})f_i(\cdot)$, $i = 1,2$, the next inequality holds:

$$\|y_1(t) - y_2(t)\| \le \|y_1(s) - y_2(s)\| + \int_s^t \|f_1(\tau) - f_2(\tau)\| d\tau, \; t \ge s. \quad (5.6)$$

Let us denote by $D(y_0)$ the set of all integral solutions of (5.1) such that $y(0) = y_0$.

Lemma 5.6. *[20] The multivalued map $G : \mathbb{R}_+ \times \overline{\mathcal{D}(A)} \to \mathcal{P}(\overline{\mathcal{D}(A)})$ defined by $G(t,y_0) := \{y(t) : y(\cdot) \in D(y_0)\}$ is a multivalued semigroup.*

Let us consider the following condition:

$(\mathcal{C})$ The sets $M_K := \{u \in D(\varphi) : \|u\|_H \le K, \; \varphi(u) \le K\}$ are compact in H for any $K > 0$.

Theorem 5.7. *[20] Let $(\mathcal{C})$ be satisfied. Suppose that there exist $\delta > 0$, $M > 0$ such that $\forall\, u \in \mathcal{D}(\partial\varphi)$, $\|u\| \ge M$, $\forall\, y \in -\partial\varphi(u) + F(u) + h$,*

$$(y,u) \le -\delta. \quad (5.7)$$

Then the multivalued semigroup G has a global attractor R. It is the minimal closed set attracting each bounded set. It is compact, invariant and maximal among all negatively semi-invariant bounded subsets in H.

5.2 Existence of the global attractor for the inclusion $(P1)$

From Lemma 5.6, we have

Proposition 5.8. *The inclusion in $(P1)$ defines a strict multivalued semigroup (or strict m-semiflow) $G_1(t,\cdot) : \mathcal{H} \to P(\mathcal{H})$ where $G_1(t,u_0)$ is the set of all integral solutions of $(P1)$ beginning at $u_0 \in \mathcal{H}$ valuated at time t.*

Now, we establish our result:

Theorem 5.9. *[27] The multivalued semigroup associated with problem $(P1)$ has a global attractor R_1. It is the minimal closed set attracting each bounded set. It is compact, invariant and maximal among all negatively semi-invariant bounded subsets in H.*

Proof. First, we will to prove that the condition $(\mathcal{C})$ is satisfied. Indeed, since $X \subset\subset \mathcal{H}$ and

$$M_K := \{u \in \mathcal{D}(\varphi); \|u\|_{\mathcal{H}} \leq K,\ \varphi(u) \leq K\} = \overline{M_K},$$

it is sufficient to show that for each $K > 0$, M_K is a bounded set in X. Let be $K > 0$ and $u \in M_K$. Then, $\langle A_1 u, u\rangle_{X^*,X} \leq Kp^+$. From Lemma 3.1, $\|u\|_X \leq \max\{[Kp^+]^{\frac{1}{p^-}},\ [Kp^+]^{\frac{1}{p^+}}\}$. So, the condition $(\mathcal{C})$ is satisfied. Now, we intend to show that the condition (5.7) in Theorem 5.7 is satisfied. Let $u \in \mathcal{D}(A)$, $\xi \in F(u)$. Since the map f is Lipschitz and has values in $\mathcal{C}_v(\mathbb{R})$ it's easy to see that there exist $D_1,\ D_2 \geq 0$ such that $\sup_{y\in f(s)} |y| \leq D_1 + D_2|s|$, $\forall\ s \in \mathbb{R}$. Consequently, there are constants $k_1, k_2 > 0$ such that $\|\xi + h\|_{\mathcal{H}} \leq k_1\|u\|_{\mathcal{H}} + k_2$, $\forall\ \xi \in F(u)$. Using the immersion $X \subset \mathcal{H}$ we have that $\|u\|_{\mathcal{H}} \leq \sigma\|u\|_X$ for some $\sigma > 1$. Using Lemma 3.1 we obtain $\langle A_1 u, u\rangle_{X^*,X} \geq (\frac{1}{\sigma})^{p^-}||u||_{\mathcal{H}}^{p^-}$ for $||u||_{\mathcal{H}} \geq \sigma$. Then, using the Cauchy-Schwarz and the Young inequalities we get $\langle -Au + \xi + h, u\rangle_{X^*,X} \leq -(\frac{1}{\sigma})^{p^-}||u||_{\mathcal{H}}^{p^-} + k_1\|u\|_{\mathcal{H}}^2 + k_2\|u\|_{\mathcal{H}} \leq$

$-\frac{1}{2\sigma^{p^-}}||u||_{\mathcal{H}}^{p^-} + k_3$ for $||u||_{\mathcal{H}} \geq \sigma$, with $k_3 := \frac{k_1^\alpha}{\alpha\epsilon_0^\alpha} + \frac{k_2^{q^-}}{q^-\epsilon_0^{q^-}}$, where $\frac{2}{p^-} + \frac{1}{\alpha} = 1$, $\frac{1}{p^-} + \frac{1}{q^-} = 1$ and $\epsilon_0 > 0$ is such that $\frac{2}{p^-}\epsilon_0^{p^-/2} + \frac{1}{p^-}\epsilon_0^{p^-} < \frac{1}{2\sigma^{p^-}}$. Considering $M := \max\left\{\left[2\sigma^{p^-}(1+k_3)\right]^{1/p^-}, \sigma\right\} > 0$ and $\delta := 1$ we have that $\langle -A_1u + \xi + h, u\rangle_{X^*,X} \leq -\delta$ for all $u \in \mathcal{D}(A_{\mathcal{H}})$ with $\|u\|_{\mathcal{H}} > M$. So, condition (5.7) is satisfied and the result follows from Theorem 5.7. □

6 A coupled system of partial differential inclusions

This section is based on paper [32]. This work concerns the study of asymptotic behavior of coupled systems of $p(x)$-Laplacian differential inclusions. We obtain that the generalized semiflow generated by the coupled system has a global attractor.

Let us consider the following system

$$\begin{cases} u_t - \operatorname{div}(D_1|\nabla u|^{p_1(x)-2}\nabla u) \in F(u,v) & \text{in } (0,\infty)\times\Omega \\ v_t - \operatorname{div}(D_2|\nabla v|^{p_2(x)-2}\nabla v) \in G(u,v) & \text{in } (0,\infty)\times\Omega \\ u(0,x) = u_0(x),\ v(0,x) = v_0(x) \text{ in } \Omega \end{cases} \tag{6.1}$$

under homogeneous Dirichlet boundary conditions, where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $N \geq 1$, $p_i(x)$ continuous in $\overline{\Omega}$ and $p_i^- := \operatorname{ess\,inf} p_i > 2$, for $i = 1,2$, $u_0, v_0 \in \mathcal{H} := L^2(\Omega)$, $D_1, D_2 \in L^\infty(\Omega)$ with $0 < \sigma \leq D_i(x) \leq M < \infty$ a.e. in Ω, $i = 1,2$, and $F, G : \mathcal{H}\times\mathcal{H} \to P(\mathcal{H})$ are bounded, upper semicontinuous and positively sublinear multivalued operators. In this section, we will denote $X_1 := W_0^{1,p_1(x)}(\Omega)$ and $X_2 := W_0^{1,p_2(x)}(\Omega)$. By [28] we know that $-\operatorname{div}(|\nabla u|^{p_i(x)-2}\nabla u)$, $i = 1,2$, generate a compact semigroup and the result remains true for $-\operatorname{div}(D_i|\nabla u|^{p_i(x)-2}\nabla u)$ with D_i as above.

Definition 6.1. Let U be a topological space. A mapping $G : U \to P(H)$ is called upper semicontinuous at $u \in U$, if

(i) $G(u)$ is nonempty, bounded, closed and convex.

(ii) For each open subset D in H satisfying $G(u) \subset D$, there exists a neighborhood V of u, such that $G(v) \subset D$, for each $v \in V$.

If G is upper semicontinuous at each $u \in U$, then it is called upper semicontinuous on U.

Definition 6.2. Let M be a Lebesgue measurable subset in $\mathbb{R}^q$, $q \geq 1$. By a *selection* of $E : M \to P(H)$ we mean a function $f : M \to H$ such that $f(y) \in E(y)$ a.e. $y \in M$, and we denote by Sel E the set

$$\text{Sel } E \doteq \{f,\ f : M \to H \text{ is a measurable selection of } E\}.$$

We obtain existence of local solutions to our system (6.1) by applying the following result:

Theorem 6.3 ([31]). *Let A and B be univalued operators which are subdifferentials of convex, proper and lower semicontinuous (l.s.c.) non negative maps, ψ_A, ψ_B, respectively, defined in a real Hilbert space H, with $\psi_A(0) = \psi_B(0) = 0$. Also suppose each one A and B generates a compact semigroup, and let $F, G : H \times H \to P(H)$ be upper semicontinuous and bounded multivalued maps. Then given a bounded subset $B_0 \subset H \times H$, there exists $T_0 > 0$ such that for each $(u_0, v_0) \in B_0$ there exists at least one strong solution (u, v) of*

$$(P) \begin{cases} u_t + Au \in F(u,v) \\ v_t + Bv \in G(u,v) \\ u(0) = u_0,\ v(0) = v_0 \end{cases}$$

defined on $[0, T_0]$.

Remark 6.4. A strong solution [weak solution] of (P) is a pair (u, v) satisfying: $u, v \in C([0,T]; H)$ for which there exists $f, g \in L^1(0,T;H)$, $f(t) \in F(u(t), v(t))$, $g(t) \in G(u(t), v(t))$ *a.e.* in $(0,T)$, and such that (u, v) is a strong solution [weak solution] (see Definition 3.1 and Theorem 3.4 in [5]) over $(0,T)$ to the system below:

$$\begin{cases} u_t + Au = f \\ v_t + Bv = g \\ u(0) = u_0, v(0) = v_0 \end{cases}$$

In order to get global solutions, as in [31], we use the suitable conditions on terms F and G which appear in the following

Definition 6.5. The pair (F, G) of maps $F, G : \mathcal{H} \times \mathcal{H} \to P(\mathcal{H})$, which take bounded subsets of $\mathcal{H} \times \mathcal{H}$ into bounded subsets of $\mathcal{H}$, is called **positively sublinear** if there exist $a > 0$, $b > 0$, $c > 0$ and $m_0 > 0$ such that for each $(u, v) \in \mathcal{H} \times \mathcal{H}$ with $\|u\| > m_0$ or $\|v\| > m_0$ for which either there exists $f_0 \in F(u, v)$ satisfying $\langle u, f_0 \rangle > 0$ or there exists $g_0 \in G(u, v)$ with $\langle v, g_0 \rangle > 0$, we have both

$$\|f\| \leq a\|u\| + b\|v\| + c$$

and

$$\|g\| \leq a\|u\| + b\|v\| + c$$

for each $f \in F(u, v)$ and each $g \in G(u, v)$.

Proposition 6.6. *[31] If $D(u_0, v_0)$ is the set of all solutions of (6.1) with initial data (u_0, v_0) then*

$$\mathbb{G} := \bigcup_{(u_0, v_0) \in \mathcal{H} \times \mathcal{H}} D(u_0, v_0)$$

is a generalized semiflow in $\mathcal{H} \times \mathcal{H}$ which is called the generalized semiflow associated with (6.1).

Therefore, according to Theorem 9 in [30] (see also Remark 6 in [30]), in order to assure the existence of a compact invariant global B-attractor for (6.1), it is enough to guarantee that the generalized semiflow $\mathbb{G}$ defined by (6.1) is B-dissipative.

Theorem 6.7. *[32] Let $F, G : \mathcal{H} \times \mathcal{H} \to P(\mathcal{H})$ bounded, upper semicontinuous and positively sublinear operators. There exists a bounded set B_0 in $\mathcal{H} \times \mathcal{H}$ and $t_0 > 0$ such that for any $\varphi \in \mathbb{G}$, $\varphi(t) \in B_0$, $\forall\, t \geq t_0$. Then, in particular, the generalized semiflow $\mathbb{G}$ defined by (6.1) is B-dissipative.*

Proof. Let $\varphi = (u,v) \in \mathbb{G}$ a solution of (6.1). Then there exists a pair $(f,g) \in \text{Sel } F(u,v) \times \text{Sel } G(u,v)$, $f,g \in L^1(0,T;\mathcal{H})$ for each $T>0$ such that u,v satisfy the problem

$$\begin{cases} \frac{\partial u}{\partial t} - \text{div}(D_1|\nabla u|^{p_1(x)-2}\nabla u) = f & \text{in} \quad (0,T)\times\Omega \\ \frac{\partial v}{\partial t} - \text{div}(D_2|\nabla v|^{p_2(x)-2}\nabla v) = g & \text{in} \quad (0,T)\times\Omega \\ u(0,x) = u_0(x),\ v(0,x) = v_0(x) & \text{in} \quad \Omega \end{cases}. \tag{6.2}$$

Multiplying the first equation by u we obtain

$$\Big\langle \frac{\partial u(t)}{\partial t}, u(t)\Big\rangle_{\mathcal{H}} + \Big\langle -\text{div}(D_1|\nabla u|^{p_1(x)-2}\nabla u(t)), u(t)\Big\rangle_{\mathcal{H}} = \langle f(t), u(t)\rangle_{\mathcal{H}}.$$

Let $I := (0,T)$, $I_1 := \{t \in I : \|u(t)\|_{X_1} < 1\}$ and $I_2 := \{t \in I : \|u(t)\|_{X_1} \geq 1\}$. Then by Lemma 3.1

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{\mathcal{H}}^2 + \sigma\|u(t)\|_X^{p_1^+} \leq \langle f(t), u(t)\rangle_{\mathcal{H}} \quad \text{if} \quad t \in I_1,$$

and

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{\mathcal{H}}^2 + \sigma\|u(t)\|_X^{p_1^-} \leq \langle f(t), u(t)\rangle_{\mathcal{H}} \quad \text{if} \quad t \in I_2.$$

Thus, if α_1 is the constant of the continuous imbedding $X_1 \subset \mathcal{H}$, it follows that

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{\mathcal{H}}^2 \leq \begin{cases} -\frac{\sigma}{\alpha_1^{p_1^+}}\|u(t)\|_{\mathcal{H}}^{p_1^+} + \langle f(t), u(t)\rangle_{\mathcal{H}} & \text{if} \quad t \in I_1 \\ -\frac{\sigma}{\alpha_1^{p_1^-}}\|u(t)\|_{\mathcal{H}}^{p_1^-} + \langle f(t), u(t)\rangle_{\mathcal{H}} & \text{if} \quad t \in I_2 \end{cases}. \tag{6.3}$$

In an analogous way, multiplying the second equation in (6.2) by v we obtain

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|_{\mathcal{H}}^2 \leq \begin{cases} -\frac{\sigma}{\alpha_2^{p_2^+}}\|v(t)\|_{\mathcal{H}}^{p_2^+} + \langle g(t), v(t)\rangle_{\mathcal{H}} & \text{if} \quad t \in \tilde{I}_1 \\ -\frac{\sigma}{\alpha_2^{p_2^-}}\|v(t)\|_{\mathcal{H}}^{p_2^-} + \langle g(t), v(t)\rangle_{\mathcal{H}} & \text{if} \quad t \in \tilde{I}_2 \end{cases},$$

where $\tilde{I}_1 := \{t \in I : \|v(t)\|_{X_2} < 1\}$, $\tilde{I}_2 := \{t \in I : \|v(t)\|_{X_2} \geq 1\}$ and α_2 is the constant of the continuous imbedding $X_2 \subset \mathcal{H}$.

Now, let $r := \frac{p_1^+}{p_1^-} > 1$ and r' such that $\frac{1}{r} + \frac{1}{r'} = 1$ then by Young's inequality

$$\|u(t)\|_{\mathcal{H}}^{p_1^-} \leq \frac{1}{r'} + \frac{1}{r}\|u(t)\|_{\mathcal{H}}^{p_1^+},$$

and so

$$-\frac{\sigma}{\alpha_1^{p_1^+}}\|u(t)\|_{\mathcal{H}}^{p_1^+} \leq r\Big(-\frac{\sigma}{\alpha_1^{p_1^+}}\|u(t)\|_{\mathcal{H}}^{p_1^-} + \frac{\sigma}{\alpha_1^{p_1^+}r'}\Big). \tag{6.4}$$

Using (6.4) in (6.3) we get

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{\mathcal{H}}^2 \leq -C_2\|u(t)\|_{\mathcal{H}}^{p_1^-} + \langle f(t), u(t)\rangle_{\mathcal{H}} + C_1 \quad \forall\, t \in I = (0,T),$$

where $C_2 := \min\left\{\frac{\sigma}{\alpha_1^{p_1^+}}, \frac{\sigma}{\alpha_1^{p_1^-}}\right\}$ and $C_1 := \frac{r\sigma}{\alpha_1^{p_1^+}r'}$.

In an analogous way, taking $\tilde{r} := \frac{p_2^+}{p_2^-} > 1$ and $\tilde{r}'$ such that $\frac{1}{\tilde{r}} + \frac{1}{\tilde{r}'} = 1$ we get

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|_{\mathcal{H}}^2 \leq -\tilde{C_2}\|v(t)\|_{\mathcal{H}}^{p_2^-} + \langle g(t), v(t)\rangle_{\mathcal{H}} + \tilde{C_1} \quad \forall\, t \in I = (0,T),$$

where $\tilde{C_2} := \min\left\{\frac{\sigma}{\alpha_2^{p_2^+}}, \frac{\sigma}{\alpha_2^{p_2^-}}\right\}$ and $\tilde{C_1} := \frac{\tilde{r}\sigma}{\alpha_2^{p_2^+}\tilde{r}'}$.

Thus, we obtain

$$\begin{cases} \frac{1}{2}\frac{d}{dt}\|u(t)\|_{\mathcal{H}}^2 \leq -C_2\|u(t)\|_{\mathcal{H}}^{p_1^-} + \langle f(t), u(t)\rangle_{\mathcal{H}} + C_1 \\ \frac{1}{2}\frac{d}{dt}\|v(t)\|_{\mathcal{H}}^2 \leq -\tilde{C_2}\|v(t)\|_{\mathcal{H}}^{p_2^-} + \langle g(t), v(t)\rangle_{\mathcal{H}} + \tilde{C_1} \end{cases} \tag{6.5}$$

where $C_2, \tilde{C_2}, C_1, \tilde{C_1}$ are positive real numbers depending on $p_1^-, p_2^-, p_1^+, p_2^+, \Omega, \sigma$.

We can suppose, without losing generality that $p_1^- \geq p_2^-$. If $p_1^- = p_2^-$ we obtain a similar expression as (6.5) with p_2^- on the place of p_1^-. If $p_1^- > p_2^-$, taking $s := \frac{p_1^-}{p_2^-} > 1$, s' such that $\frac{1}{s} + \frac{1}{s'} = 1$ and $\epsilon > 0$ we have

$$\|u(t)\|_{\mathcal{H}}^{p_2^-} = \frac{\epsilon}{\epsilon}\|u(t)\|_{\mathcal{H}}^{p_2^-} \leq \frac{1}{s'\epsilon^{s'}} + \frac{1}{s}\epsilon^s\|u(t)\|_{\mathcal{H}}^{p_1^-}$$

and then

$$-C_2\|u(t)\|_{\mathcal{H}}^{p_1^-} \leq \frac{s}{\epsilon^s}\Big[\frac{C_2}{s'\epsilon^{s'}} - C_2\|u(t)\|_{\mathcal{H}}^{p_2^-}\Big].$$

So, we have

$$\begin{cases} \frac{1}{2}\frac{d}{dt}\|u(t)\|_{\mathcal{H}}^2 \leq -\frac{C_2 s}{\epsilon^s}\|u(t)\|_{\mathcal{H}}^{p_2^-} + \langle f(t), u(t)\rangle_{\mathcal{H}} + C_1 + \frac{sC_2}{s'\epsilon^s\epsilon^{s'}} \\ \frac{1}{2}\frac{d}{dt}\|v(t)\|_{\mathcal{H}}^2 \leq -\tilde{C}_2\|v(t)\|_{\mathcal{H}}^{p_2^-} + \langle g(t), v(t)\rangle_{\mathcal{H}} + \tilde{C}_1 \end{cases}. \tag{6.6}$$

Now, we use that (F, G) is positively sublinear (see Definition 6.5) to estimate $\langle f(t), u(t)\rangle_{\mathcal{H}}$ and $\langle g(t), v(t)\rangle_{\mathcal{H}}$. To do this, we have to consider the following three steps:

1. If $\|u(t)\| \leq m_0$ and $\|v(t)\| \leq m_0$ then as F and G take bounded subsets of $\mathcal{H} \times \mathcal{H}$ into bounded subsets of $\mathcal{H}$ there exists $C > 0$ such that

$$\langle f(t), u(t)\rangle_{\mathcal{H}} \leq \|f(t)\|\|u(t)\| \leq Cm_0$$

and

$$\langle g(t), v(t)\rangle_{\mathcal{H}} \leq \|g(t)\|\|v(t)\| \leq Cm_0.$$

2. If $\|u(t)\| > m_0$ or $\|v(t)\| > m_0$ and $\langle f_0, u(t)\rangle \leq 0$ and $\langle g_0, v(t)\rangle \leq 0$ $\forall\ f_0 \in F(u(t), v(t))$ and $\forall\ g_0 \in G(u(t), v(t))$ then $\langle f(t), u(t)\rangle_{\mathcal{H}} \leq 0$ and $\langle g(t), v(t)\rangle_{\mathcal{H}} \leq 0$.

3. If $\|u(t)\| > m_0$ or $\|v(t)\| > m_0$ and $\langle f_0, u(t)\rangle > 0$ or $\langle g_0, v(t)\rangle > 0$ for some $f_0 \in F(u(t), v(t))$ or for some $g_0 \in G(u(t), v(t))$ then, for $\epsilon > 0$, $\theta := \frac{p_2^-}{2} > 1$ and $\nu := \frac{p_2^-}{(p_2^-)'} > 1$, we get

$$\begin{aligned}
\langle f(t), u(t)\rangle &\leq \|f(t)\|\|u(t)\| \\
&\leq \frac{\epsilon}{\epsilon}a\|u(t)\|^2 + \frac{\epsilon}{\epsilon}b\|u(t)\|\|v(t)\| + \frac{\epsilon}{\epsilon}c\|u(t)\| \\
&\leq \frac{1}{\theta'}\Big(\frac{a}{\epsilon}\Big)^{\theta'} + \frac{1}{\theta}\epsilon^{\theta}\|u(t)\|_{\mathcal{H}}^{p_2^-} + \frac{1}{(p_2^-)'}\Big(\frac{b}{\epsilon}\Big)^{(p_2^-)'}\|v(t)\|_{\mathcal{H}}^{(p_2^-)'} \\
&\quad + \frac{1}{p_2^-}\epsilon^{p_2^-}\|u(t)\|_{\mathcal{H}}^{p_2^-} + \frac{1}{(p_2^-)'}\Big(\frac{c}{\epsilon}\Big)^{(p_2^-)'} + \frac{1}{p_2^-}\epsilon^{p_2^-}\|u(t)\|_{\mathcal{H}}^{p_2^-} \\
&= \Big(\frac{2}{p_2^-}\epsilon^{\frac{p_2^-}{2}} + \frac{2}{p_2^-}\epsilon^{p_2^-}\Big)\|u(t)\|_{\mathcal{H}}^{p_2^-} + \frac{\epsilon}{\epsilon}\frac{1}{(p_2^-)'}\Big(\frac{b}{\epsilon}\Big)^{(p_2^-)'}\|v(t)\|_{\mathcal{H}}^{(p_2^-)'} \\
&\quad + \Big(\frac{1}{\theta'}\Big(\frac{a}{\epsilon}\Big)^{\theta'} + \frac{1}{(p_2^-)'}\Big(\frac{c}{\epsilon}\Big)^{(p_2^-)'}\Big)
\end{aligned}$$

$$\leq \Big(\frac{2}{p_2^-}\epsilon^{\frac{p_2^-}{2}} + \frac{2}{p_2^-}\epsilon^{p_2^-}\Big)\|u(t)\|_{\mathcal{H}}^{p_2^-} + \frac{\epsilon^\nu}{\nu}\|v(t)\|_{\mathcal{H}}^{p_2^-}$$
$$+\Big[\frac{1}{\nu'}\Big(\frac{1}{\epsilon}\frac{1}{(p_2^-)'}\Big(\frac{b}{\epsilon}\Big)^{(p_2^-)'}\Big)^{\nu'} + \frac{1}{\theta'}\Big(\frac{a}{\epsilon}\Big)^{\theta'} + \frac{1}{(p_2^-)'}\Big(\frac{c}{\epsilon}\Big)^{(p_2^-)'}\Big]$$

and in an analogous way

$$\langle g(t), v(t)\rangle \leq \Big(\frac{2}{p_2^-}\epsilon^{\frac{p_2^-}{2}} + \frac{2}{p_2^-}\epsilon^{p_2^-}\Big)\|v(t)\|_{\mathcal{H}}^{p_2^-} + \frac{\epsilon^\nu}{\nu}\|u(t)\|_{\mathcal{H}}^{p_2^-}$$
$$+\Big[\frac{1}{\nu'}\Big(\frac{1}{\epsilon}\frac{1}{(p_2^-)'}\Big(\frac{a}{\epsilon}\Big)^{(p_2^-)'}\Big)^{\nu'} + \frac{1}{\theta'}\Big(\frac{b}{\epsilon}\Big)^{\theta'} + \frac{1}{(p_2^-)'}\Big(\frac{c}{\epsilon}\Big)^{(p_2^-)'}\Big].$$

Therefore, joining up 1.), 2.) and 3.) we get

$$\langle f(t), u(t)\rangle \leq \Big(\frac{2}{p_2^-}\epsilon^{\frac{p_2^-}{2}} + \frac{2}{p_2^-}\epsilon^{p_2^-}\Big)\|u(t)\|_{\mathcal{H}}^{p_2^-} + \frac{\epsilon^\nu}{\nu}\|v(t)\|_{\mathcal{H}}^{p_2^-} + m_0C$$
$$+\Big[\frac{1}{\nu'}\Big(\frac{1}{\epsilon}\frac{1}{(p_2^-)'}\Big(\frac{b}{\epsilon}\Big)^{(p_2^-)'}\Big)^{\nu'} + \frac{1}{\theta'}\Big(\frac{a}{\epsilon}\Big)^{\theta'} + \frac{1}{(p_2^-)'}\Big(\frac{c}{\epsilon}\Big)^{(p_2^-)'}\Big] \quad (6.7)$$

and

$$\langle g(t), v(t)\rangle \leq \Big(\frac{2}{p_2^-}\epsilon^{\frac{p_2^-}{2}} + \frac{2}{p_2^-}\epsilon^{p_2^-}\Big)\|v(t)\|_{\mathcal{H}}^{p_2^-} + \frac{\epsilon^\nu}{\nu}\|u(t)\|_{\mathcal{H}}^{p_2^-} + m_0C$$
$$+\Big[\frac{1}{\nu'}\Big(\frac{1}{\epsilon}\frac{1}{(p_2^-)'}\Big(\frac{a}{\epsilon}\Big)^{(p_2^-)'}\Big)^{\nu'} + \frac{1}{\theta'}\Big(\frac{b}{\epsilon}\Big)^{\theta'} + \frac{1}{(p_2^-)'}\Big(\frac{c}{\epsilon}\Big)^{(p_2^-)'}\Big] \quad (6.8)$$

Using (6.7) and (6.8) in (6.6) we get

$$\begin{cases} \frac{1}{2}\frac{d}{dt}\|u(t)\|_{\mathcal{H}}^2 \leq \Big(-\frac{C_2 s}{\epsilon^s} + \frac{2}{p_2^-}\epsilon^{\frac{p_2^-}{2}} + \frac{2}{p_2^-}\epsilon^{p_2^-}\Big)\|u(t)\|_{\mathcal{H}}^{p_2^-} + \frac{\epsilon^\nu}{\nu}\|v(t)\|_{\mathcal{H}}^{p_2^-} + C_3(\epsilon) \\ \frac{1}{2}\frac{d}{dt}\|v(t)\|_{\mathcal{H}}^2 \leq \Big(-\tilde{C}_2 + \frac{2}{p_2^-}\epsilon^{\frac{p_2^-}{2}} + \frac{2}{p_2^-}\epsilon^{p_2^-}\Big)\|v(t)\|_{\mathcal{H}}^{p_2^-} + \frac{\epsilon^\nu}{\nu}\|u(t)\|_{\mathcal{H}}^{p_2^-} + C_4(\epsilon) \end{cases}$$

where $C_3(\epsilon) = \tilde{C}_3(\epsilon, p_1^-, p_2^-) + C_1$ and $C_4(\epsilon) = \tilde{C}_4(\epsilon, p_1^-, p_2^-) + \tilde{C}_1$. Thus, adding the last two inequalities we obtain

$$\frac{1}{2}\frac{d}{dt}\Big(\|u(t)\|_{\mathcal{H}}^2 + \|v(t)\|_{\mathcal{H}}^2\Big) \leq \Big(-\frac{C_2 s}{\epsilon^s} + \frac{2}{p_2^-}\epsilon^{\frac{p_2^-}{2}} + \frac{2}{p_2^-}\epsilon^{p_2^-} + \frac{\epsilon^\nu}{\nu}\Big)\|u(t)\|_{\mathcal{H}}^{p_2^-}$$
$$+\Big(-\tilde{C}_2 + \frac{2}{p_2^-}\epsilon^{\frac{p_2^-}{2}} + \frac{2}{p_2^-}\epsilon^{p_2^-} + \frac{\epsilon^\nu}{\nu}\Big)\|v(t)\|_{\mathcal{H}}^{p_2^-} + C_3(\epsilon) + C_4(\epsilon).$$

As $\epsilon > 0$ is arbitrary, take ϵ_0 sufficiently small such that

$$\frac{2}{p_2^-}\epsilon_0^{\frac{p_2^-}{2}} + \frac{2}{p_2^-}\epsilon_0^{p_2^-} + \frac{\epsilon_0^{\nu}}{\nu} < \frac{\tilde{C}_2}{2} \quad \text{and} \quad \frac{C_2 s}{\epsilon_0^s} \geq \tilde{C}_2.$$

Then

$$\frac{1}{2}\frac{d}{dt}\Big(\|u(t)\|_{\mathcal{H}}^2 + \|v(t)\|_{\mathcal{H}}^2\Big) \leq -C_5\Big(\|u(t)\|_{\mathcal{H}}^{p_2^-} + \|v(t)\|_{\mathcal{H}}^{p_2^-}\Big) + C_6$$

where $C_5 := \frac{\tilde{C}_2}{2} > 0$ and $C_6 = C_3(\epsilon_0) + C_4(\epsilon_0) > 0$.

Thus,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\Big(\|u(t)\|_{\mathcal{H}}^2 + \|v(t)\|_{\mathcal{H}}^2\Big) &\leq -C_5\Big(\|u(t)\|_{\mathcal{H}}^{2\frac{p_2^-}{2}} + \|v(t)\|_{\mathcal{H}}^{2\frac{p_2^-}{2}}\Big) + C_6 \\ &\leq -\frac{C_5}{2^{\frac{p_2^-}{2}}}\Big(\|u(t)\|_{\mathcal{H}}^2 + \|v(t)\|_{\mathcal{H}}^2\Big)^{\frac{p_2^-}{2}} + C_6. \end{aligned}$$

Therefore, the function $y(t) := \|u(t)\|_{\mathcal{H}}^2 + \|v(t)\|_{\mathcal{H}}^2$ satisfies the inequality

$$y'(t) \leq -\frac{2C_5}{2^{\frac{p_2^-}{2}}} y(t)^{\frac{p_2^-}{2}} + 2C_6, \qquad t > 0$$

and we can conclude the proof appealing to Lemma 5.1 in [36]. □

7 Final remarks

About existence of global attractors for autonomous equations with variable exponents in unbounded domains we refer the reader to [1,2] and for an equation involving two different variable exponents see [26].

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