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A system of two σ-evolution equations with non-effective damping and coupling derivative-type semilinear terms

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Abstract. In this work, we consider a two by two system of evolution equations with a non-effective damping and semilinear coupling of derivative type. We provide sufficient conditions on the power nonlinearities in the coupling term which guarantee the existence of global-in-time small data solutions.

Keywords: Weakly coupled system, σ-evolution equation, structural non-effective damping, semilinear terms of derivative type.

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1 Introduction

In this paper, we consider the weakly coupled system of σ-evolution equations with non-effective damping and derivative-type semilinear cou-

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pling

$$\begin{cases} \partial_t^2 u_1 + (-\Delta)^\sigma u_1 + \partial_t (-\Delta)^{\theta_1/2} u_1 = |\partial_t u_2|^{p_2}, & t \geq 0,\ x \in \mathbb{R}^n, \\ \partial_t^2 u_2 + (-\Delta)^\sigma u_2 + \partial_t (-\Delta)^{\theta_2/2} u_2 = |\partial_t u_1|^{p_1}, & t \geq 0,\ x \in \mathbb{R}^n, \\ (u_1, \partial_t u_1, u_2, \partial_t u_2)(0,x) = (0, \varphi_1, 0, \varphi_2)(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $\sigma > 1$, $\sigma \leq \theta_1, \theta_2 \leq 2\sigma$ and $p_1, p_2 > 1$. The term $(-\Delta)^s$ for $s > 0$, denotes the s-th power of the Laplace operator, that we may define as $(-\Delta)^s f = \mathcal{F}^{-1}(|\xi|^{2s}\widehat{f})$ where $f \in H^{2s}$ and $\mathcal{F}$ denotes the Fourier transform with respect to the x variable.

The study of weakly coupled systems gained increasing interest in recent years, particularly the research of the so-called critical curve for the existence of solutions of the system, that is the threshold curve in the plane $p_1 - p_2$ between the global existence of small data solutions in the supercritical case and the nonexistence of global solutions in the subcritical case, under suitable sign assumption on the initial data. The concept of critical curve for systems is analogous to that of critical exponent for semilinear equations. We mention that the definition of critical exponent may be refined to the definition of critical nonlinearity, as discussed in [11] (see also [8]), but a possible extension to systems is not yet clarified.

The main purpose of this work is to find conditions to prove the global-in-time existence of small data solutions to (1.1). In the case of the semilinear scalar equation

$$\begin{cases} v_{tt} + (-\Delta)^\sigma v + (-\Delta)^{\theta/2} v_t = |v_t|^p, & t \geq 0,\ x \in \mathbb{R}^n, \\ v(0,x) = 0,\ v_t(0,x) = \varphi(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.2}$$

with $1 < \sigma < \theta \leq 2\sigma$, the critical exponent is (see [3, 5, 6]):

$$p_c = 1 + \frac{\sigma}{n}, \tag{1.3}$$

at least for $\sigma \geq 3$ and $n \leq \sigma - 2$ (see also [7] for the critical exponent in presence of a nonlinear memory term). For the weakly coupled system of damped wave equations

$$\begin{cases} \partial_t^2 u_1 - \Delta u_1 + \partial_t u_1 = |u_2|^{p_2}, & t \geq 0, \ x \in \mathbb{R}^n, \\ \partial_t^2 u_2 - \Delta u_2 + \partial_t u_2 = |u_1|^{p_1}, & t \geq 0, \ x \in \mathbb{R}^n, \\ (u_1, \partial_t u_1, u_2, \partial_t u_2)(0,x) = (0, \varphi_1, 0, \varphi_2)(x), & x \in \mathbb{R}^n, \end{cases}$$

the critical curve is (see [22], see also [14, 15, 16, 17])

$$\alpha(p_1,p_2) := \frac{p_1 p_2 - 1}{\max\{p_1,p_2\}+1} = \frac{2}{n}. \tag{1.4}$$

Condition (1.4) is equivalent to

$$\max\{p_1,p_2\}\left(\min\{p_1,p_2\}+1-p_{\mathrm{F}}(n)\right) = p_{\mathrm{F}}(n),$$

where $p_{\mathrm{F}}(n) = 1 + 2/n$ is the Fujita critical exponent [12] for the damped wave equation [13, 23]. The system of structurally damped waves (1.1), $\sigma = \theta_1 = \theta_2 = 1$, was studied in [2]. Recently, the system of two σ-evolution equations with non-effective damping and a general semilinear coupling $N(u_1,u_2)$ has been studied in [10]. However, the case of derivative-type nonlinearities was not considered. Weakly coupled systems with derivative-type coupling were considered in [1, 18, 19, 20].

Consistently with the previous results for other systems with semilinear coupling and with the result for the scalar equation, the critical curve for system (1.1) is expected to be

$$\alpha(p_1,p_2) = \frac{\sigma}{n}\,.$$

The presence of a nonlinearity of derivative type, however, makes the proof of a nonexistence result more complicated. In particular, the techniques used in [10] do not work to prove blow up in finite time in the critical case. The proof of blow up in the critical and subcritical case will be studied by the authors in a subsequent work. In this paper, we will provide a global existence result of small data solutions, in low dimension space.

Without any loss of generality, we assume in the following that $p_1 \leq p_2$.

Theorem 1.1. *Assume that*

$$\sigma > 2+\sqrt{2}, \quad n \leq \frac{\sigma(\sigma-2)}{2(\sigma-1)}. \tag{1.5}$$

Let $p_2 \geq p_1 > p_c - 1$ *be such that the condition*

$$\alpha(p_1,p_2) > \frac{\sigma}{n}, \tag{1.6}$$

holds. Then, there exists $\varepsilon > 0$ *such that, for any* $(\varphi_1,\varphi_2) \in \mathcal{A} = L^1 \cap (L^{p_1} \times L^{p_2})$, *with*

$$\|(\varphi_1,\varphi_2)\|_{\mathcal{A}} = \|(\varphi_1,\varphi_2)\|_{L^1} + \|\varphi_1\|_{L^{p_1}} + \|\varphi_2\|_{L^{p_2}} \leq \varepsilon,$$

there exists a solution

$$(u_1,u_2) \in \big(\mathcal{C}([0,\infty),H^\sigma)\big)^2 \cap \mathcal{C}^1([0,\infty),L^2 \cap (L^{p_1}\times L^{p_2}))$$

to (1.1). *Moreover,*

$$\begin{aligned} \|(\partial_t u_1,(-\Delta)^{\frac{\sigma}{2}}u_1)(t,\cdot)\|_{L^2} &\leq (1+t)^{-\frac{n}{2\theta_1}}\|(\varphi_1,\varphi_2)\|_{\mathcal{A}},\\ \|(\partial_t u_2,(-\Delta)^{\frac{\sigma}{2}}u_2)(t,\cdot)\|_{L^2} &\leq (1+t)^{-\frac{n}{2\theta_2}+\gamma(p_1)}\|(\varphi_1,\varphi_2)\|_{\mathcal{A}}, \end{aligned} \tag{1.7}$$

and

$$\begin{aligned} \|\partial_t u_1(t,\cdot)\|_{L^{p_1}} &\leq C\,(1+t)^{-\frac{n}{\sigma}\left(1-\frac{1}{p_1}\right)}\|(\varphi_1,\varphi_2)\|_{\mathcal{A}},\\ \|\partial_t u_2(t,\cdot)\|_{L^{p_2}} &\leq C\,(1+t)^{-\frac{n}{\sigma}\left(1-\frac{1}{p_2}\right)+\gamma(p_1)}\|(\varphi_1,\varphi_2)\|_{\mathcal{A}}, \end{aligned} \tag{1.8}$$

where

$$\gamma(p_1) = \frac{n}{\sigma}(p_c - p_1)_+, \tag{1.9}$$

provided that $p_1 \neq p_c(n/\sigma)$. *If* $p_1 = p_c$, *then the quantity* $(1+t)^{\gamma(p_1)}$ *in* (1.7) *and* (1.8) *is replaced by* $\log(e+t)$.

In (1.9) and through this paper, $(a)_+ := \max\{a,0\}$. The quantity $(1+t)^{\gamma(p_1)}$ represents the possible loss of decay with respect to the corresponding linear estimates for u_2.

We stress that $p_2 > p_c$, due to assumption (1.6), and $p_2 \geq p_1 > p_c - 1 > 2$, since $n < \sigma/2$ as a consequence of (1.5).

2 $L^1 - L^p$ estimates for the scalar equation

For several nonlinear equations, global-in-time existence critical exponents are known to be related to asymptotic in time decay of some space norms of the solution. This is also the case for the system (1.1). So, in order to prove Theorem 1.1, we take advantage of the decay estimates, for the linear scalar equation obtained in [4] (see also [9])

$$\begin{cases} v_{tt} + (-\Delta)^{\sigma} v + (-\Delta)^{\theta/2} v_t = 0, & t \geq 0,\ x \in \mathbb{R}^n, \\ v(0,x) = 0,\ v_t(0,x) = \varphi(x), & x \in \mathbb{R}^n, \end{cases} \tag{2.1}$$

where $1 < \sigma < \theta \leq 2\sigma$. Then, for $\sigma > 2$, the solution to (2.1) satisfies the $L^1 \cap L^p - L^p$ estimate

$$\|\partial_t v(t,\cdot)\|_{L^p} \leq C(1+t)^{-\frac{n}{\sigma}\left(1-\frac{1}{p}\right)}(\|\varphi\|_{L^1} + \|\varphi\|_{L^p}), \tag{2.2}$$

for $t \geq 0$, for any $p \in (2,\infty]$ such that

$$1 - \frac{1}{p} < \sigma\left(\frac{1}{2} - \frac{1}{p}\right).$$

In particular, assumption (1.5) is equivalent to ask that the above condition is satisfied for any $p > p_c - 1 = \sigma/n$.

Moreover, if

$$n\left(1 - \frac{1}{p}\right) < \theta,$$

then the solution to (2.1) satisfies the $L^1 - L^p$ estimate (singular at $t = 0$):

$$\|\partial_t v(t,\cdot)\|_{L^p} \leq C\big((1+t)^{-\frac{n}{\sigma}(1-\frac{1}{p})} + t^{-\delta}e^{-ct}\big)\|\varphi\|_{L^1}, \tag{2.3}$$

for some $\delta \in (0,1)$, for any $t > 0$.

The following energy estimate for the solution to (2.1) also holds:

$$\|(\partial_t, (-\Delta)^{\frac{\sigma}{2}})v(t,\cdot)\|_{L^2} \leq C\,(1+t)^{-\frac{n}{2\theta}}\,(\|\varphi\|_{L^1} + \|\varphi\|_{L^2})\,, \tag{2.4}$$

for any $t \geq 0$. Moreover, if $n < 2\theta$, the singular energy estimate also holds

$$\|(\partial_t, (-\Delta)^{\frac{\sigma}{2}})v(t,\cdot)\|_{L^2} \leq C\,\left((1+t)^{-\frac{n}{2\theta}} + t^{-\delta}e^{-ct}\right)\|\varphi\|_{L^1}, \tag{2.5}$$

for some $\delta \in (0,1)$, for any $t > 0$.

3 Proof of Theorem 1.1

Let us introduce some useful notation. Let $E_i(t,x)$ be the fundamental solution to the linear scalar equations associated with (1.1) for $i=1,2$, namely

$$u(t,x)=E_i(t,x)\ast_{(x)}\varphi_i(x)$$

is the solution to

$$\begin{cases}\partial_t^2 u_i+(-\Delta)^\sigma u_i+\partial_t(-\Delta)^{\theta_i/2}u_i=0, & t\geq 0,\ x\in\mathbb{R}^n,\\ u_i(0,x)=0,\ \partial_t u_i(0,x)=\varphi_i(x), & x\in\mathbb{R}^n.\end{cases}$$

Having in mind Duhamel's principle, for a suitable Banach space X, let us consider the integral operator $N:X\to X$ defined by $N[u_1,u_2]=A+(Fu_2,Gu_1)$, where

$$\begin{aligned}A(t,x)&=\big(E_1(t,x)\ast_{(x)}\varphi_1(x),E_2(t,x)\ast_{(x)}\varphi_2(x)\big),\\ Fu_2(t,x)&=\int_0^t E_1(t-s,x)\ast_{(x)}|\partial_t u_2(s,x)|^{p_2}ds,\\ Gu_1(t,x)&=\int_0^t E_2(t-s,x)\ast_{(x)}|\partial_t u_1(s,x)|^{p_1}ds.\end{aligned}$$

Global existence of small data solutions to (1.1) follows by a standard contraction argument. Namely, we want to prove that the operator N is a contraction on the functional space X. Then the unique fixed point of N will be the unique solution to (1.1). We will prove the following a priori estimates

$$\|N[u_1,u_2]\|_X\leq C_1\mathcal{A}, \tag{3.1}$$

and

$$\|N[u_1,u_2]-N[\tilde{u}_1,\tilde{u}_2]\|_X\leq C\|(u_1,u_2)-(\tilde{u}_1,\tilde{u}_2)\|_X \Big(\|(u_1,u_2)\|_X^{p_2-1}+\|(\tilde{u}_1,\tilde{u}_2)\|_X^{p_2-1}+\|(u_1,u_2)\|_X^{p_1-1}+\|(\tilde{u}_1,\tilde{u}_2)\|_X^{p_1-1}\Big). \tag{3.2}$$

Estimates (3.1), (3.2) also imply the desired estimates for the solution of the problem.

One of the tools we will use is the following classical lemma.

Lemma 3.1. *Let $\nu > -1$ and $\mu \in \mathbb{R}$. Then it holds*

$$\int_0^t (t-s)^\nu\,(1+s)^\mu\,ds \lesssim \begin{cases} (1+t)^\nu & \text{if } \mu < -1,\\ (1+t)^\nu \log(e+t), & \text{if } \mu = -1,\\ (1+t)^{1+\nu+\mu} & \text{if } \mu > -1. \end{cases} \tag{3.3}$$

and

$$\int_0^t (t-s)^\nu\,e^{-c(t-s)}(1+s)^\mu\,ds \lesssim (1+t)^\mu.$$

Moreover, the estimate is also valid if $(t-s)^\nu$ is replaced by $(1+t-s)^\nu$ in the integral.

Lemma 3.1 has been proved in many different versions by many authors. One earlier version of this lemma goes back to [21].

Proof of Theorem 1.1. Let us define the space

$$X := \big(\mathcal{C}([0,\infty), H^\sigma)\big)^2 \cap \mathcal{C}^1([0,\infty), L^2 \cap (L^{p_1}\times L^{p_2}))$$

with norm:

$$\|(u_1,u_2)\|_X := \sup_{t\geq 0}\Big(M_1(u_1) + (1+t)^{-\gamma(p_1)}M_2(u_2)\Big),$$

where

$$M_1(u_1) = (1+t)^{\frac{n}{\sigma}\left(1-\frac{1}{p_1}\right)}\|\partial_t u_1(t,\cdot)\|_{L^{p_1}} + (1+t)^{\frac{n}{2\theta_1}}\|(\partial_t u_1, (-\Delta)^{\frac{\sigma}{2}}u_1)(t,\cdot)\|_{L^2},$$

and

$$M_2(u_2) = (1+t)^{\frac{n}{\sigma}\left(1-\frac{1}{p_2}\right)}\|\partial_t u_2(t,\cdot)\|_{L^{p_2}} + (1+t)^{\frac{n}{2\theta_2}}\|(\partial_t u_1, (-\Delta)^{\frac{\sigma}{2}}u_2)(t,\cdot)\|_{L^2}.$$

Moreover, if $p_1 = p_c$, then we replace $(1+t)^{-\gamma(p_1)}$ by $(\log(e+t))^{-1}$.

For any $(u_1, u_2) \in X$ and for any $s \geq 0$, it holds

$$\|\partial_t u_1(s,\cdot)\|_{L^{p_1}} \leq (1+s)^{-\frac{n}{\sigma}\left(1-\frac{1}{p_1}\right)} \|(u_1,u_2)\|_X, \tag{3.4}$$

$$\|\partial_t u_2(s,\cdot)\|_{L^{p_2}} \leq (1+s)^{-\frac{n}{\sigma}\left(1-\frac{1}{p_2}\right)+\gamma(p_1)} \|(u_1,u_2)\|_X, \tag{3.5}$$

Using the linear estimates from Section 2, we can prove that $\|A\|_X \leq C\|(\varphi_1,\varphi_2)\|_{\mathcal{A}}$. Indeed, recalling that $p_2 \geq p_1 > p_c - 1$, the estimate (2.2) with $p = p_1$ gives us

$$\|\partial_t u_1(t,\cdot)\|_{L^{p_1}} \lesssim (1+t)^{-\frac{n}{\sigma}\left(1-\frac{1}{p_1}\right)} (\|u_1\|_{L^1} + \|u_1\|_{L^{p_1}}).$$

Similarly for u_2, choosing $p = p_2$. Similarly, we can use the energy estimate (2.4), so that we get (3.1). It remains to show (3.2). We first estimate the terms $\|\partial_t(Fu_2(t,\cdot) - F\tilde{u}_2(t,\cdot))\|_{L^{p_1}}$ and $\|\partial_t(Gu_1(t,\cdot) - G\tilde{u}_1(t,\cdot))\|_{L^{p_2}}$.

Due to the fact that $n < \sigma/2 \leq \theta_i$, for $i = 1, 2$, we can use the $L^1 - L^{p_1}$ singular estimate (2.3) for some $\delta \in (0,1)$. Using that $u_2, \tilde{u}_2 \in X$, we have that (3.5) holds for $u_2 - \tilde{u}_2$, so we may estimate

$$\begin{aligned}
&\|\partial_t(Fu_2(t,\cdot) - F\tilde{u}_2(t,\cdot))\|_{L^{p_1}} \\
&\qquad \lesssim \int_0^t (1+t-s)^{-\frac{n}{\sigma}\left(1-\frac{1}{p_1}\right)} \||\partial_t u_2(s,\cdot)|^{p_2} - |\partial_t \tilde{u}_2(s,\cdot)|^{p_2}\|_{L^1}\, ds \\
&\qquad\qquad + \int_0^t (t-s)^{-\delta} e^{-c(t-s)} \||\partial_t u_2(s,\cdot)|^{p_2} - |\partial_t \tilde{u}_2(s,\cdot)|^{p_2}\|_{L^1}\, ds \\
&\qquad \lesssim \int_0^t (1+t-s)^{-\frac{n}{\sigma}\left(1-\frac{1}{p_1}\right)} \|\partial_t(u_2(s,\cdot) - \tilde{u}_2(s,\cdot))\|_{L^{p_2}}^{p_2}\, ds \\
&\qquad\qquad + \int_0^t (t-s)^{-\delta} e^{-c(t-s)} \|\partial_t(u_2(s,\cdot) - \tilde{u}_2(s,\cdot))\|_{L^{p_2}}^{p_2}\, ds \\
&\quad \lesssim \|(u_1,u_2)\|_X^{p_2} \left(I_1(t) + \tilde{I}_1(t)\right),
\end{aligned}$$

where

$$\begin{aligned}
I_1(t) &= \int_0^t (1+t-s)^{-\frac{n}{\sigma}\left(1-\frac{1}{p_1}\right)} (1+s)^{\left(-\frac{n}{\sigma}\left(1-\frac{1}{p_2}\right)+\gamma(p_1)\right)p_2}\, ds, \\
\tilde{I}_1(t) &= \int_0^t (t-s)^{-\delta} e^{-c(t-s)} (1+s)^{\left(-\frac{n}{\sigma}\left(1-\frac{1}{p_2}\right)+\gamma(p_1)\right)p_2}\, ds,
\end{aligned}$$

Similarly, using (3.4) for $u_1 - \tilde{u}_1$, we estimate

$$\begin{aligned}
&\|\partial_t(Gu_1(t,\cdot) - Gu_1(t,\cdot))\|_{L^{p_2}} \\
&\qquad \lesssim \int_0^t (1+t-s)^{-\frac{n}{\sigma}\left(1-\frac{1}{p_2}\right)} \||\partial_t u_1(s,\cdot)|^{p_1} - |\partial_t \tilde{u}_1(s,\cdot)|^{p_1}\|_{L^1}\, ds \\
&\qquad\qquad + \int_0^t (t-s)^{-\delta} e^{-c(t-s)} \||\partial_t u_1(s,\cdot)|^{p_1} - |\partial_t \tilde{u}_1(s,\cdot)|^{p_1}\|_{L^1}\, ds \\
&\qquad \lesssim \int_0^t (1+t-s)^{-\frac{n}{\sigma}\left(1-\frac{1}{p_2}\right)} \|\partial_t(u_1(s,\cdot) - \tilde{u}_1(s,\cdot))\|_{L^{p_1}}^{p_1}\, ds \\
&\qquad\qquad + \int_0^t (t-s)^{-\delta} e^{-c(t-s)} \|\partial_t(u_1(s,\cdot) - \tilde{u}_1(s,\cdot))\|_{L^{p_1}}^{p_1}\, ds \\
&\quad \lesssim \|(u_1,u_2)\|_X^{p_1} \left(I_2(t) + \tilde{I}_2(t)\right),
\end{aligned}$$

where

$$\begin{aligned}
I_2(t) &= \int_0^t (1+t-s)^{-\frac{n}{\sigma}\left(1-\frac{1}{p_2}\right)} (1+s)^{-\frac{n}{\sigma}(p_1-1)}\, ds, \\
\tilde{I}_2(t) &= \int_0^t (t-s)^{-\delta} e^{-c(t-s)} (1+s)^{-\frac{n}{\sigma}(p_1-1)}\, ds,
\end{aligned}$$

Now let us distinguish two cases. If $p_1 > p_c$, then $\gamma(p_1) = 0$ and using that the condition $p_i > p_c$ is equivalent to

$$-\frac{n}{\sigma}(p_i - 1) < -1,$$

for $i = 1, 2$, and Lemma 3.1 with $\mu < -1$, we get

$$\begin{aligned}
I_1(t) &= \int_0^t (1+t-s)^{-\frac{n}{\sigma}\left(1-\frac{1}{p_1}\right)} (1+s)^{-\frac{n}{\sigma}\left(1-\frac{1}{p_2}\right)p_2}\, ds \lesssim (1+t)^{-\frac{n}{\sigma}\left(1-\frac{1}{p_1}\right)}, \\
I_2(t) &= \int_0^t (1+t-s)^{-\frac{n}{\sigma}\left(1-\frac{1}{p_2}\right)} (1+s)^{-\frac{n}{\sigma}\left(1-\frac{1}{p_1}\right)p_1}\, ds \lesssim (1+t)^{-\frac{n}{\sigma}\left(1-\frac{1}{p_2}\right)}.
\end{aligned}$$

We stress that we are allowed to use Lemma 3.1 since

$$-\frac{n}{\sigma}\left(1 - \frac{1}{p_1}\right) > -\frac{n}{\sigma} > -1,$$

due to assumption (1.5). Regarding $\tilde{I}_1(t)$, using again Lemma 3.1, we can see

$$\tilde{I}_1(t) = \int_0^t (t-s)^{-\delta} e^{-c(t-s)} (1+s)^{\left(-\frac{n}{\sigma}\left(1-\frac{1}{p_2}\right)\right)p_2}\, ds \lesssim (1+t)^{-\frac{n}{\sigma}(p_2-1)}$$
$$\leq (1+t)^{-1} \leq (1+t)^{-\frac{n}{\sigma}\left(1-\frac{1}{p_1}\right)},$$

and similarly we may see that $\tilde{I}_2(t)$ can be controlled by the decay of $I_2(t)$. Otherwise, we assume $p_1 \leq p_c < p_2$. On the one hand, for the estimate of I_2, using Lemma 3.1 with $\mu > -1$ if $p_1 < p_c$ and $\mu = -1$ if $p_1 = p_c$, we get

$$I_2(t) \lesssim \begin{cases} (1+t)^{-\frac{n}{\sigma}\left(1-\frac{1}{p_2}\right)+\gamma(p_1)} & \text{if } p_1 < p_c, \\ (1+t)^{-\frac{n}{\sigma}\left(1-\frac{1}{p_2}\right)} \log(e+t) & \text{if } p_1 = p_c. \end{cases}$$

On the other hand, for the estimate of I_1, since the condition (1.6) is equivalent to

$$\left(-\frac{n}{\sigma}\left(1-\frac{1}{p_2}\right)+\gamma(p_1)\right)p_2 < -1,$$

we can use Lemma 3.1 with $\mu < -1$ to obtain

$$I_1(t) \lesssim (1+t)^{-\frac{n}{\sigma}\left(1-\frac{1}{p_1}\right)},$$

and the same if we replace $(1+s)^{\gamma(p_1)p_2}$ by $(\log(e+s))^{p_2}$ when $p_1 = p_c$. Moreover, as before, it is easy to see that the estimates of $\tilde{I}_1(t)$ and $\tilde{I}_2(t)$ may be controlled by the estimates of I_1 and I_2, respectively.

Now we proceed to the estimates related to the energy norm. Using the singular energy estimate (2.5), we have

$$\begin{aligned}
&\|(\partial_t,(-\Delta)^{\frac{\sigma}{2}})(Fu_2(t,\cdot) - F\tilde{u}_2(t,\cdot))\|_{L^2} \\
&\quad\lesssim \int_0^t (1+t-s)^{-\frac{n}{2\theta_1}} \||\partial_t u_2(s,\cdot)|^{p_2} - |\partial_t \tilde{u}_2(s,\cdot)|^{p_2}\|_{L^1}\, ds \\
&\qquad + \int_0^t (t-s)^{-\delta} e^{-c(t-s)} \||\partial_t u_2(s,\cdot)|^{p_2} - |\partial_t \tilde{u}_2(s,\cdot)|^{p_2}\|_{L^1}\, ds \\
&\quad\lesssim \|(u_1,u_2)\|_X^{p_2} \left(I_3(t) + I_4(t)\right),
\end{aligned}$$

for some $\delta \in (0,1)$, where

$$I_3(t) = \int_0^t (1+t-s)^{-\frac{n}{2\theta_1}} (1+s)^{\left(-\frac{n}{\sigma}\left(1-\frac{1}{p_2}\right)+\gamma(p_1)\right)p_2} \, ds,$$

$$I_4(t) = \int_0^t (t-s)^{-\delta} e^{-c(t-s)} (1+s)^{\left(-\frac{n}{\sigma}\left(1-\frac{1}{p_2}\right)+\gamma(p_1)\right)p_2} \, ds$$

As before, it will be enough to estimate only the integral I_3, since I_4 has a better decay. Similarly

$$\begin{aligned} &\|(\partial_t,(-\Delta)^{\frac{\sigma}{2}})(Gu_1(t,\cdot) - Gu_1(t,\cdot))\|_{L^2} \\ &\quad \lesssim \int_0^t (1+t-s)^{-\frac{n}{2\theta_2}} \, \||\partial_t u_1(s,\cdot)|^{p_1} - |\partial_t \tilde{u}_1(s,\cdot)|^{p_1}\|_{L^1} \, ds \\ &\qquad + \int_0^t (t-s)^{-\delta} e^{-c(t-s)} \, \||\partial_t u_1(s,\cdot)|^{p_1} - |\partial_t \tilde{u}_1(s,\cdot)|^{p_1}\|_{L^1} \, ds \\ &\quad \lesssim \|(u_1,u_2)\|_X^{p_1} \, I_5, \end{aligned}$$

where

$$I_5(t) = \int_0^t (1+t-s)^{-\frac{n}{2\theta_2}} (1+s)^{-\frac{n}{\sigma}(p_1-1)} \, ds.$$

As before, in the case $p_1 > p_c$, we may use Lemma 3.1 with $\mu < -1$ to get

$$I_3(t) = \int_0^t (1+t-s)^{-\frac{n}{2\theta_1}} (1+s)^{\left(-\frac{n}{\sigma}\left(1-\frac{1}{p_2}\right)\right)p_2} \, ds \lesssim (1+t)^{-\frac{n}{2\theta_1}},$$

$$I_5(t) = \int_0^t (1+t-s)^{-\frac{n}{2\theta_2}} (1+s)^{\left(-\frac{n}{\sigma}\left(1-\frac{1}{p_1}\right)\right)p_1} \, ds \lesssim (1+t)^{-\frac{n}{2\theta_2}}.$$

Otherwise, assume $p_1 \le p_c < p_2$. On the one hand, for the estimate of I_5 using Lemma 3.1 with $\mu \ge -1$, we obtain

$$I_5(t) \lesssim \begin{cases} (1+t)^{-\frac{n}{2\theta_2}+\gamma(p_1)} & \text{if } p_1 < p_c, \\ (1+t)^{-\frac{n}{2\theta_2}} \log(e+t) & \text{if } p_1 = p_c. \end{cases}$$

On the other hand, from (1.6) we obtain

$$I_3(t) = \int_0^t (1+t-s)^{-\frac{n}{2\theta_1}} (1+s)^{\left(-\frac{n}{\sigma}\left(1-\frac{1}{p_2}\right)+\gamma(p_1)\right)p_2} \, ds \lesssim (1+t)^{-\frac{n}{2\theta_1}},$$

and the same if we replace $(1+s)^{\gamma(p_1)p_2}$ by $(\log(e+s))^{p_2}$ when $p_1 = p_c$.

This concludes the proof of (3.2).

□

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