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Exponential stability for Volterra–Stieltjes–type integral equations via generalized ODEs

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Abstract. The aim of this paper is to provide results on the exponential stability of the trivial solution of certain Volterra-Stieltjes integral equations involving Perron–Stieltjes integrable functions. By means of Lyapunov–type functionals, we obtain conditions for the trivial solution of a generalized ODE to be weakly exponentially stable and exponentially stable. Then, because Volterra-Stieltjes integral equations can be identified with a certain class of generalized ODEs, we translate the results we obtained to the former. In order to illustrate our results, we provide an application to impulsive differential equations.

Keywords: Volterra-Stieltjes integral equations; generalized ordinary differential equations; exponential stability; Perron–Stieltjes integrals.

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1 Introduction

In this work, we establish sufficient conditions to obtain exponential stability results for the trivial solution of the following integral form

$$x(t) = x(s_0) + \int_{s_0}^{t} f(x(s), s)ds + \int_{s_0}^{t} g(x(s), s)du(s), \quad t \geq s_0, \qquad (1.1)$$

where $f, g\colon X \times [t_0, \infty) \to X$ and $u\colon [t_0, \infty) \to \mathbb{R}$ are functions, $t_0 \in \mathbb{R}$, X is a Banach space and $f(0, t) = g(0, t) = 0$ for all $t \geq t_0$ and $s_0 \geq t_0$. The integrals on the right–hand side of (1.1) are considered in the senses of Perron and Perron–Stieltjes respectively.

In order to obtain the main results, we first present the theory of exponential stability for generalized ordinary differential equations (we write generalized ODEs, for short). The theory of exponential stability for generalized ODEs was first presented in 2014 (see [1]). Later, this theory also appeared in the book [6] and in the papers [17, 18]. In the present paper, we recover a different notion of exponential stability together with a concept of weak exponential stability employed to impulsive infinite delay differential systems in [16] and we adapt such concepts to our generalized ODEs, establishing criteria not only for this new concept of exponential stability, but also for weak exponential stability of solutions.

It is well–known that there exists a one-to-one relation between the solutions of certain generalized ODEs and the solutions of equations (1.1) (see [8, 10], for instance). In fact, generalized ODEs encompass several kinds of equations, among which we mention ordinary and impulsive differential equations [29], functional differential equations (FDEs) including those of neutral type [7, 9, 11, 12, 14, 21, 28], impulsive FDEs [1–4, 13, 15], among others. It is clear that any combination of these equations can also be described by generalized ODEs. Using the correspondence theorem between the solutions of generalized ODEs and those of equation (1.1), we obtain the desired results.

We organize this paper in the following way. In Section 2, we recall some basic concepts and results from the theory of generalized ODEs.

In Section 3, we present new concepts of weak exponential stability and exponential stability for the trivial solution of a generalized ODE. Using Lyapunov–type functionals, we establish sufficient conditions for the trivial solution of a generalized ODE to be weakly exponentially stable (see Theorem 3.4) and exponentially stable (see Theorem 3.6). Section 4 deals with the exponentially stability for equation (1.1). In Subsection 4.1, we present the general conditions for (1.1) to have a unique maximal solution and then, in Subsection 4.2, we introduce adequate concepts of weak exponential stability and exponential stability for the trivial solution of the Volterra–Stieltjes integral equation (1.1) so that, when we apply the relation between equation (1.1) and a certain generalized ODE, the intended results come naturally. Finally, in Section 5, we deal with exponential stability for impulsive differential equations.

2 Preliminaries

In this section, for the reader's convenience, we present basic concepts and some results from the theory of generalized ODEs. In order to exhibit the concept of a generalized ODE, we need to present the definition of the integral due to Jaroslav Kurzweil [6, 22, 29].

A *division* of an interval $[a, b]$ is a finite set $d = \{s_0, s_1, \ldots, s_{|d|}\}$ such that $a = s_0 \leq s_1 \leq \ldots \leq s_{|d|} = b$, where $|d|$ denotes the number of subintervals of the form $[s_{i-1}, s_i]$ of the division d.

A *tagged division* of a compact interval $[a, b] \subset \mathbb{R}$ is a finite collection of point–interval pairs $\tilde{d} = \{(\tau_i, [s_{i-1}, s_i])\}_{i=1}^{|d|}$, where $d = \{s_0, s_1, \ldots, s_{|d|}\}$ is a division of $[a, b]$ and $\tau_i \in [s_{i-1}, s_i]$ is a tag for each i.

A *gauge* on a set $J \subset [a, b]$ is any function $\delta\colon J \to (0, \infty)$. Given a gauge δ on $[a, b]$, we say that a tagged division $\tilde{d} = \{(\tau_i, [s_{i-1}, s_i])\}_{i=1}^{|d|}$ is δ–fine, whenever

$$\tau_i \in [s_{i-1}, s_i] \subset (\tau_i - \delta(\tau_i), \tau_i + \delta(\tau_i)) \quad \text{for all} \quad i = 1, 2, \ldots, |d|.$$

Let X be a Banach space endowed with a norm $\|\cdot\|$.

Definition 2.1. A function $U\colon [a,b]\times[a,b]\to X$ is said to be Kurzweil integrable on $[a,b]$, if there is an element $I\in X$ such that for each $\epsilon>0$ there exists a gauge δ on $[a,b]$ so that

$$\left\|\sum_{i=1}^{|d|}[U(\tau_i,s_i)-U(\tau_i,s_{i-1})]-I\right\|<\epsilon$$

for every δ–fine tagged division $\tilde{d}=\{(\tau_i,[s_{i-1},s_i])\}_{i=1}^{|d|}$ of $[a,b]$. In this case, I is the Kurzweil integral of U over $[a,b]$ and it is denoted by $\int_a^b DU(\tau,t)$.

The Kurzweil integral has the usual properties of uniqueness, linearity, additivity with respect to adjacent intervals, integrability on subintervals, among other properties. Moreover, it encompasses the well–known Perron–Stieltjes integral as well as its improper integrals. The reader may consult [6, 29] for more properties of this type of integration.

Remark 2.2. The Kurzweil integral can be extended to unbounded intervals, see [5] and [19].

In what follows, we present basic concepts and an important preliminary result of the theory of generalized ODEs. This theory is very well structured in the books [6], [27] and [29]. Other important references on generalized ODEs are [23], [24], [25] and [26].

Let $F\colon\Omega\to X$ be a function defined for each $(x,t)\in\Omega$, where $\Omega=X\times[t_0,\infty)$ and $t_0\in\mathbb{R}$. As introduced in [29], the equation

$$\frac{dx}{d\tau}=DF(x,t) \tag{2.1}$$

is known as a generalized ODE and it is defined via its solutions, that is, a function $x\colon J\to X$ is said to be a solution of (2.1) on the interval $J\subset[t_0,\infty)$, whenever for all $s_1,s_2\in J$, we have

$$x(s_2)-x(s_1)=\int_{s_1}^{s_2}DF(x(\tau),t), \tag{2.2}$$

where the integral on the right–hand side of (2.2) is in the sense of the Kurzweil integral.

Given an initial condition $(x_0, s_0) \in X \times J$, a solution of the initial value problem

$$\begin{cases} \dfrac{dx}{d\tau} = DF(x,t) \\ x(s_0) = x_0, \end{cases}$$

on the interval $J \subset [t_0, \infty)$ is any function $x\colon J \to X$ satisfying

$$x(s) = x_0 + \int_{s_0}^{s} DF(x(\tau), t), \quad s \in J.$$

Now, we present a special class of functions $F\colon \Omega \to X$ for which we can derive interesting properties of the solutions of (2.1).

Definition 2.3. We say that a function $F\colon \Omega \to X$ belongs to $\mathcal{F}(\Omega, h)$, if there is a nondecreasing left–continuous function $h\colon [0,\infty) \to \mathbb{R}$ such that for all (x, s_2), (x, s_1), (y, s_2), $(y, s_1) \in \Omega$,

$$\|F(x,s_2) - F(x,s_1)\| \leq |h(s_2) - h(s_1)| \quad \text{and}$$
$$\|F(x,s_2) - F(x,s_1) - F(y,s_2) + F(y,s_1)\| \leq \|x - y\| |h(s_2) - h(s_1)|$$

The next result shows interesting properties concerning the solutions of the generalized ODE (2.1), provided F belongs to the class $\mathcal{F}(\Omega, h)$.

Proposition 2.4. *Let $[\alpha, \beta] \subset [t_0, \infty)$ and $x\colon [\alpha, \beta] \to X$ be a solution of (2.1). If $F\colon \Omega \to X$ belongs to the class $\mathcal{F}(\Omega, h)$, then the following properties hold:*

(i) $\|x(t) - x(s)\| \leq |h(t) - h(s)|$ *for all t, $s \in [\alpha, \beta]$. Moreover, every point of $[\alpha, \beta]$ at which h is continuous, is a continuity point of x.*

(ii) *x is a function of bounded variation on $[\alpha, \beta]$ and*

$$\mathrm{var}_{[\alpha,\beta]} x \leq h(\beta) - h(\alpha) < \infty.$$

(iii) $x(t^+) - x(t) = F(x(t), t^+) - F(x(t), t)$ *for all $t \in [\alpha, \beta)$ and* $x(t) - x(t^-) = F(x(t), t) - F(x(t), t^-)$ *for all $t \in (\alpha, \beta]$, where* $F(x, t^+) = \lim_{s \to t^+} F(x, s)$ *for $t \in [\alpha, \beta)$ and* $F(x, t^-) = \lim_{s \to t^-} F(x, s)$ *for $t \in (\alpha, \beta]$.*

(iv) *For each* $(x_0, s_0) \in \Omega$, *there exists a unique maximal solution of the generalized ODE* (2.1), *defined in* $[s_0, \infty)$, *such that* $x(s_0) = x_0$.

The proofs of items (i), (ii) and (iii) can be found in [29]. The statement in (iv) was proved in [10].

3 Exponential stability for Generalized ODEs

Let us consider a function $F\colon \Omega \to X$ which belongs to the class $\mathcal{F}(\Omega, h)$, where $h\colon [t_0, \infty) \to \mathbb{R}$ is a left–continuous and nondecreasing function. With this function, we consider the following generalized ODE

$$\frac{dx}{d\tau} = DF(x,t). \tag{3.1}$$

Our aim is to establish results on the exponential stability and the weak exponential stability of the trivial solution of (3.1). In order to ensure that $x \equiv 0$ is a solution of (3.1), we shall assume that

$$F(0, t_2) - F(0, t_1) = 0, \quad t_1, t_2 \in [t_0, \infty).$$

In what follows, we denote by $x(\cdot) = x(\cdot, s_0, x_0)$ the maximal solution $x\colon [s_0, \infty) \to X$ of (3.1), with initial condition $x(s_0) = x_0 \in X$, $s_0 \geq t_0$. Such a solution exists and is unique due to Proposition 2.4–(iv).

Definition 3.1. The trivial solution of the generalized ODE (3.1) is

(i) **Weakly exponentially stable**, if there exist a continuous strictly increasing function $\alpha\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\alpha(0) = 0$ and a constant $\lambda > 0$, such that for $\epsilon > 0$ and $s_0 \geq t_0$, there exists $\delta = \delta(\epsilon) > 0$ such that $\alpha(\|x(t)\|) < \epsilon e^{-\lambda(t-s_0)}$ for every $t \geq s_0$, provided $\|x_0\| < \delta$.

(ii) **Exponentially stable**, if there exists a constant $\lambda > 0$, such that for every $\epsilon > 0$ and $s_0 \geq t_0$, there exists $\delta = \delta(\epsilon) > 0$ such that $\|x(t)\| < \epsilon e^{-\lambda(t-s_0)}$ for every $t \geq s_0$, provided $\|x_0\| < \delta$.

Remark 3.2. It is worth reiterating that the concepts of stability and weak stability for generalized ODEs described in Definition 3.1 are new in the literature, but they were adapted from those presented by Fu and Li in [16], in Definition 2.3, for impulsive differential equations with infinite delay.

Remark 3.3. The Definition 3.1 - (ii) is related with the notion of uniform stability from [9, Definition 3.2], because

$$\|x(t)\| < \epsilon e^{-\lambda(t-s_0)} \quad \Rightarrow \quad \|x(t)\| < \epsilon,$$

for every $t \geqslant s_0$, since $e^{-\lambda(t-s_0)} < 1$.

Using Lyapunov–type functionals, we now establish conditions for both the weak exponential stability and the exponential stability of the trivial solution of equation (3.1).

Theorem 3.4. *Let $V\colon [t_0,\infty)\times X \to \mathbb{R}$ be a functional for which*

(i) *there exist continuous strictly increasing functions $a\colon \mathbb{R}_+ \to \mathbb{R}_+$ and $b\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $a(0) = 0 = b(0)$ such that*

$$b(\|x\|) \leq V(t,x) \leq a(\|x\|)$$

for all $t \in [t_0,\infty)$ and all $x \in X$;

(ii) *there exist $T > 0$ and $\theta \in (0,1)$ such that if $x\colon [s_0,\infty) \to X$, $s_0 \geq t_0$, is solution of (3.1), then $V(t,x(t)) \leq \theta V(s,x(s))$, for $t,s \in [s_0,\infty)$ and $t-s \geq T$;*

(iii) *if $x\colon [s_0,\infty) \to X$, $s_0 \geq t_0$, is a solution of (3.1), then $V(t_2,x(t_2)) \leq V(t_1,x(t_1))$ for all $t_1,t_2 \in [s_0,\infty)$ such that $s_0 \leq t_1 \leq t_2 < \infty$.*

Then, the trivial solution of (3.1) is weakly exponentially stable.

Proof. Consider $\lambda = -\frac{1}{T}\ln\theta > 0$. Let $\epsilon > 0$ and $s_0 \geq t_0$. By the properties of the function a, there is a $\delta > 0$ depending on ϵ and such that $a(\delta) < \epsilon\theta$.

Let $x(t) = x(t, s_0, x_0)$ be the solution of (3.1) on $[s_0, \infty)$ such that $x(s_0) = x_0 \in X$, with $\|x_0\| < \delta$. Given $t \geq s_0$, there exists an integer $n \geq 0$ such that $nT \leq t - s_0 < nT + T$. Condition (ii) implies that

$$\begin{aligned} V(t, x(t)) &\leq \theta V(t-T, x(t-T)) \leq \theta^2 V(t-2T, x(t-2T)) \\ &\leq \ldots \leq \theta^n V(t-nT, x(t-nT)). \end{aligned} \tag{3.2}$$

Since V satisfies condition (iii), we obtain

$$V(t-nT, x(t-nT)) \leq V(s_0, x_0). \tag{3.3}$$

Then, condition (i) yields

$$V(s_0, x(s_0)) \leq a(\|x_0\|) < a(\delta). \tag{3.4}$$

By (3.2), (3.3) and (3.4), for all $t \geq s_0$, we have

$$\begin{aligned} V(t, x(t)) &\leq \theta^n V(s_0, x(s_0)) < \theta^{n+1}\theta^{-1}a(\delta) \leq \theta^{\frac{t-s_0}{T}}\theta^{-1}a(\delta) \\ &= e^{-\lambda(t-s_0)}\theta^{-1}a(\delta) < \theta^{-1}\epsilon\theta e^{-\lambda(t-s_0)} \\ &= \epsilon e^{-\lambda(t-s_0)}. \end{aligned} \tag{3.5}$$

On the other hand, by condition (i) and (3.5), for all $t \geq s_0$,

$$b(\|x(t)\|) \leq V(t, x(t)) < \epsilon e^{-\lambda(t-s_0)},$$

whence the result follows. □

Given a solution $x\colon [s_0, \infty) \to X$ of the generalized ODE (3.1), where $s_0 \geq t_0$, we consider the right derivative of V along to x given by

$$D^+V(t, x(t)) = \limsup_{\eta \to 0+} \frac{V(t+\eta, x(t+\eta)) - V(t, x(t))}{\eta},$$

for all $t \geq s_0$. Thus, we have the following straightforward result.

Corollary 3.5. *Let $V\colon [t_0, \infty) \times X \to \mathbb{R}$ be a functional satisfying:*

(i) *there are continuous strictly increasing functions* $a\colon \mathbb{R}_+ \to \mathbb{R}_+$ *and* $b\colon \mathbb{R}_+ \to \mathbb{R}_+$, *with* $a(0) = 0 = b(0)$, *such that for every* $t \in [t_0, \infty)$ *and* $x \in X$,

$$b(\|x\|) \leq V(t,x) \leq a(\|x\|);$$

(ii) *there exist* $T > 0$ *and* $\theta \in (0,1)$ *such that* $a(\|x(t)\|) \leq \theta b(\|x(s)\|)$ *for every solution* $x\colon [s_0,\infty) \to X$ *of* (3.1), *with* $t, s \in [s_0, \infty)$ *and* $t - s \geq T$;

(iii) $D^+V(t, x(t)) \leq 0$, *for every solution* $x\colon [s_0,\infty) \to X$ *of* (3.1), *with* $s_0 \geq t_0$.

Then, the trivial solution of (3.1) *is weakly exponentially stable.*

Theorem 3.6. *Let* $V\colon [t_0,\infty) \times X \to \mathbb{R}$ *be a functional satisfying conditions* (ii) *and* (iii) *of Theorem 3.4. Moreover, assume that* V *satisfies:*

(i^*) *there exist* $k > 0$, $m \in \mathbb{N}$, *and a continuous strictly increasing function* $a\colon \mathbb{R}_+ \to \mathbb{R}_+$, *with* $a(0) = 0$, *and such that* $k\|x\|^m \leq V(t,x) \leq a(\|x\|)$ *for every* $t \in [t_0, \infty)$ *and* $x \in X$.

Then, the trivial solution $x \equiv 0$ *of* (3.1) *is exponentially stable.*

Proof. Let $\widehat{\lambda} = -\frac{1}{T}\ln\theta > 0$ and $\lambda = \frac{\widehat{\lambda}}{m}$, where m is given by condition (i^*). Let $\epsilon > 0$ and $s_0 \geq t_0$. Since $a(0) = 0$, the continuity of a at 0 implies the existence of a $\delta > 0$ depending on ϵ and such that $a(\delta) < \epsilon^m \theta k$.

Let $x(t) = x(t, s_0, x_0)$ be the solution of the generalized ODE (3.1) on $[s_0, \infty)$ satisfying $x(s_0) = x_0 \in X$, with $\|x_0\| < \delta$. By the same procedure used in the proof of Theorem 3.4, $k\|x(t)\|^m < e^{-\widehat{\lambda}(t-s_0)}\theta^{-1}a(\delta)$. Hence,

$$\|x(t)\| < \left(\frac{\theta^{-1}a(\delta)}{k}\right)^{\frac{1}{m}} e^{-\frac{\widehat{\lambda}}{m}(t-s_0)} < \epsilon e^{-\frac{\widehat{\lambda}}{m}(t-s_0)} = \epsilon e^{-\lambda(t-s_0)},$$

for all $t \geq s_0$, which completes the proof. □

Corollary 3.7. *Let* $V\colon [t_0,\infty) \times X \to \mathbb{R}$ *be a functional satisfying condition* (iii) *of Corollary 3.5 and condition* (i^*) *of Theorem 3.6. Assume, in addition, that* (*ii**) *there exist* $T > 0$ *and* $\theta \in (0, 1)$ *such that* $a(\|x(t)\|) \le \theta k \|x(s)\|^m$ *for every solution* $x\colon [s_0, \infty) \to X$ *of* (4.1), *with* $t, s \in [s_0, \infty)$ *and* $t - s \ge T$.

Then, the trivial solution $x \equiv 0$ *of* (3.1) *is exponentially stable.*

4 Exponential stability for a Volterra–Stieltjes–type integral equation

This section is divided into two parts. In Subsection 4.1, we describe the setting of our equations and we bring up the aforementioned result that links our Volterra–Stieltjes–type integral equation with a generalized ODE. The main results are grouped Subsection 4.2.

4.1 Basic framework

Let X be a Banach space with norm $\|\cdot\|$ and $t_0 \in \mathbb{R}$. For $s_0 \ge t_0$, consider the following Volterra–Stieltjes–type integral equation

$$x(t) = x(s_0) + \int_{s_0}^{t} f(x(s), s)ds + \int_{s_0}^{t} g(x(s), s)du(s), \quad t \ge s_0, \qquad (4.1)$$

where $f, g\colon X \times [t_0, \infty) \to X$ and $u\colon [t_0, \infty) \to \mathbb{R}$ are functions. The first integral on the right–hand side of (4.1) is understood in the sense of Definition 2.1, with $U(t, \tau) = t f(\tau, x(\tau))$, in which case we refer to as the Perron integral. The second integral can also be understood in the sense of Definition 2.1, with $U(t, \tau) = u(t) g(\tau, x(\tau))$, and we refer to it as the Perron–Stieltjes integral.

Let $a, b \in \mathbb{R}$, with $a < b$. The space of all regulated functions defined on the interval $[a, b]$ and taking values in X is denoted by $G([a, b], X)$. This space, equipped with the usual supremum norm given by

$$\|x\|_\infty = \sup_{t \in [a,b]} \|x(t)\|, \quad x \in G([a, b], X),$$

is a Banach space, see [20, Theorem 3.6]. By $G([t_0, \infty), X)$, we mean the space of all functions $x\colon [t_0, \infty) \to X$ such that $x|_{[\alpha,\beta]}$ belongs to the space

$G([\alpha,\beta],X)$ for all $[\alpha,\beta]\subset[t_0,\infty)$. Then, by $G_0([t_0,\infty),X)$, we denote the space of all functions $x\in G([t_0,\infty),X)$ fulfilling

$$\sup_{s\in[t_0,\infty)} e^{-(s-t_0)}\,\|x(s)\|<\infty.$$

The space $G_0([t_0,\infty),X)$, endowed with the norm

$$\|x\|_{[t_0,\infty)}=\sup_{s\in[t_0,\infty)} e^{-(s-t_0)}\,\|x(s)\|\,,\quad x\in G_0([t_0,\infty),X),$$

is a Banach space. A proof of this fact was presented in [6, Proposition 1.9].

In order to obtain existence and uniqueness of solutions of (4.1), we assume that the functions $f,g\colon X\times[t_0,\infty)\to X$ and $u\colon[t_0,\infty)\to\mathbb{R}$ satisfy the conditions below.

(A1) $u\colon[t_0,\infty)\to\mathbb{R}$ is nondecreasing and left–continuous on (t_0,∞).

(A2) The Perron integral $\int_{s_1}^{s_2} f(x(s),s)ds$ exists for all $x\in G([t_0,\infty),X)$ and all $s_1,\ s_2\in[t_0,\infty)$.

(A3) The Perron–Stieltjes integral $\int_{s_1}^{s_2} g(x(s),s)du(s)$ exists for all $x\in G([t_0,\infty),X)$ and all $s_1,\ s_2\in[t_0,\infty)$.

(A4) There exist a locally Perron integrable function $M_1\colon[t_0,\infty)\to\mathbb{R}_+$ and a locally Perron–Stieltjes integrable function $M_2\colon[t_0,\infty)\to\mathbb{R}_+$ with respect to u such that, for all $x\in G([t_0,\infty),X)$ and all $s_1,\ s_2\in[t_0,\infty)$, with $s_1\le s_2$, we have

$$\left\|\int_{s_1}^{s_2} f(x(s),s)ds\right\|\le\int_{s_1}^{s_2} M_1(s)ds\quad\text{and}$$
$$\left\|\int_{s_1}^{s_2} g(x(s),s)du(s)\right\|\le\int_{s_1}^{s_2} M_2(s)du(s).$$

(A5) There exist a locally Perron integrable function $L_1\colon[t_0,\infty)\to\mathbb{R}_+$ and locally Perron–Stieltjes integrable function $L_2\colon[t_0,\infty)\to\mathbb{R}_+$ with respect to u such that, for all $x,\ z\in G_0([t_0,\infty),X)$ and all

$s_1,\ s_2 \in [t_0, \infty)$, with $s_1 \leq s_2$, we have

$$\left\| \int_{s_1}^{s_2} [f(x(s), s) - f(z(s), s)]ds \right\| \leq \|x - z\|_{[t_0,\infty)} \int_{s_1}^{s_2} L_1(s)ds \quad \text{and}$$

$$\left\| \int_{s_1}^{s_2} [g(x(s), s) - g(z(s), s)]du(s) \right\| \leq \|x - z\|_{[t_0,\infty)} \int_{s_1}^{s_2} L_2(s)du(s).$$

A proof of the next result follows as in [10, Theorems 4.2 and 4.7].

Theorem 4.1. *Assume that conditions (A1)–(A5) hold. Choose an arbitrary $\tau_0 \in [t_0, \infty)$ and define $F\colon X \times [t_0, \infty) \to X$ by*

$$F(x, t) = \int_{\tau_0}^{t} f(x, s)ds + \int_{\tau_0}^{t} g(x, s)du(s), \quad (x, t) \in X \times [t_0, \infty). \tag{4.2}$$

Then, the following conditions are true:

(i) *$F \in \mathcal{F}(\Omega, h)$, where $\Omega = X \times [t_0, \infty)$, and $h : [t_0, \infty) \to \mathbb{R}$ given by*

$$h(t) = \int_{\tau_0}^{t} (M_1(s) + L_1(s))ds + \int_{\tau_0}^{t} (M_2(s) + L_2(s))dg(s),$$

for $t \in [t_0, \infty)$, is a nondecreasing and left–continuous function.

(ii) *If $x \in G([a, b], X)$, with $[a, b] \subset [t_0, \infty)$, then both the Kurzweil integral $\int_a^b DF(x(\tau), t)$ and the Perron–Stieltjes integral $\int_a^b f(x(s), s)dg(s)$ exist and have the same value.*

In the sequel, we present the correspondence result which links the solutions of (4.1) to the solutions of a certain class of generalized ODE given by

$$\frac{dx}{d\tau} = DF(x, t), \tag{4.3}$$

where F is defined by

$$F(x, t) = \int_{t_0}^{t} f(x, s)ds + \int_{t_0}^{t} g(x, s)du(s),$$

with $(x, t) \in X \times [t_0, \infty)$. This relation is essential to our main results.

The proof of Theorem 4.2 below follows as in [10, Theorem 4.8].

Theorem 4.2. *Assume that conditions (A1)–(A5) hold.*

(i) *A function $x\colon J \to X$ is solution of (4.1) on $J \subset [t_0, \infty)$ if and only if it is solution of the generalized ODE (4.3) on J.*

(ii) *For all $(x_0, s_0) \in X \times [t_0, \infty)$, there exists a unique maximal solution of (4.1) defined in $[s_0, \infty)$ for which $x(s_0) = x_0$.*

4.2 Main results

Recall the Volterra–Stieltjes–type integral equation (4.1) such that $f, g\colon X \times [t_0, \infty) \to X$ satisfy conditions (A2) to (A5) and $u\colon [t_0, \infty) \to \mathbb{R}$ satisfies condition (A1). Assume, in addition, that

$$f(0, t) = g(0, t) = 0, \quad t \geq t_0,$$

which implies that $x \equiv 0$ is a solution of the MDE (4.1) in every subinterval of $[t_0, \infty)$. By $x(t) = x(t, s_0, x_0)$, we denote the unique maximal solution $x\colon [s_0, \infty) \to X$ of (4.1), with initial condition $x(s_0) = x_0$. Such a solution exists by Theorem 4.2.

Next, we present the new concepts of weak exponential stability and exponential stability for the trivial solution of (4.1), which were adapted from [16], Definition 2.3.

Definition 4.3. The trivial solution of (4.1) is

(i) **weakly exponentially stable**, if there exist a continuous strictly increasing function $\alpha\colon \mathbb{R}_+ \to \mathbb{R}_+$, with $\alpha(0) = 0$, and a constant $\lambda > 0$, such that for $\epsilon > 0$ and $s_0 \geq t_0$, there exists $\delta = \delta(\epsilon) > 0$ such that $\alpha(\|x(t)\|) < \epsilon e^{-\lambda(t-s_0)}$ for every $t \geq s_0$, provided $\|x_0\| < \delta$;

(ii) **exponentially stable**, if there exists a constant $\lambda > 0$, such that for $\epsilon > 0$ and $s_0 \geq t_0$, there exists $\delta = \delta(\epsilon) > 0$ such that $\|x(t)\| < \epsilon e^{-\lambda(t-s_0)}$ for every $t \geq s_0$, provided $\|x_0\| < \delta$.

In the next results, we provide sufficient conditions to ensure that the trivial solution of (4.1) is weak exponentially stable and exponentially stable.

Theorem 4.4. *Let $V\colon [t_0,\infty)\times X\to\mathbb{R}$ be a functional for which*

(i) *there exist continuous strictly increasing functions $a\colon \mathbb{R}_+\to\mathbb{R}_+$ and $b\colon \mathbb{R}_+\to\mathbb{R}_+$ satisfying $a(0)=0=b(0)$ such that*

$$b(\|x\|)\le V(t,x)\le a(\|x\|)$$

for all $t\in[t_0,\infty)$ and all $x\in X$;

(ii) *there exist $T>0$ and $\theta\in(0,1)$ such that if $x\colon [s_0,\infty)\to X$ is solution of the MDE (4.1), with $s_0\ge t_0$, then $V(t,x(t))\le\theta V(s,x(s))$ whenever $t,s\in[s_0,\infty)$ and $t-s\ge T$;*

(iii) *if $x\colon [s_0,\infty)\to X$ is a solution of (4.1), with $s_0\ge t_0$, then $V(t_2,x(t_2))\le V(t_1,x(t_1))$ for all $t_1,t_2\in[s_0,\infty)$, such that $s_0\le t_1\le t_2<\infty$.*

Then, the trivial solution of (4.1) is weakly exponentially stable.

Proof. Let $F\colon X\times[t_0,\infty)\to X$ be a function defined by

$$F(x,t)=\int_{t_0}^{t}f(x,s)ds+\int_{t_0}^{t}g(x,s)du(s),$$

with $(x,t)\in X\times[t_0,\infty)$. Since $f,g\colon X\times[t_0,\infty)\to X$ satisfy conditions (A2) to (A5), and $u\colon[t_0,\infty)\to\mathbb{R}$ satisfies condition (A1), Theorem 4.1 implies $F\in\mathcal{F}(\Omega,h)$, with $\Omega=X\times[t_0,\infty)$ and $h\colon[t_0,\infty)\to\mathbb{R}$ given by

$$h(t)=\int_{\tau_0}^{t}(M_1(s)+L_1(s))ds+\int_{\tau_0}^{t}(M_2(s)+L_2(s))dg(s).$$

Note that the function h is left–continuous on (t_0,∞), since u is left–continuous on (t_0,∞). In addition, the assumption $f(0,t)=g(0,t)=0$, for all $t\in[t_0,\infty)$, implies that $F(0,t_2)-F(0,t_1)=0$ for all $t_2,t_1\ge t_0$, which implies that $x\equiv 0$ is a solution of the generalized ODE

$$\frac{dx}{d\tau}=DF(x,t), \tag{4.4}$$

on the interval $[t_0,\infty)$.

The relation between the solutions of our class of generalized ODE (4.4) and the solutions of (4.1), which is described by Theorem 4.2, allows us to verify that V satisfies the conditions (i), (ii) and (iii) of Theorem 3.4. Consequently, the trivial solution of the generalized ODE (4.4) is weakly exponentially stable. Using again the relation between solutions, the trivial solution of (4.1) is weakly exponentially stable. $\square$

Corollary 4.5. *Let $V\colon [t_0,\infty)\times X\to\mathbb{R}$ be a functional satisfying*

(i) *there exist continuous strictly increasing functions $a\colon \mathbb{R}_+\to\mathbb{R}_+$ and $b\colon \mathbb{R}_+\to\mathbb{R}_+$ satisfying $a(0)=0=b(0)$ such that*

$$b(\|x\|)\le V(t,x)\le a(\|x\|)$$

for all $t\in[t_0,\infty)$ and all $x\in X$;

(ii) *there exist $T>0$ and $\theta\in(0,1)$ such that $a(\|x(t)\|)\le\theta b(\|x(s)\|)$ for every solution $x\colon[s_0,\infty)\to X$ of (4.1), whenever $t,s\in[s_0,\infty)$ and $t-s\ge T$;*

(iii) *$D^+V(t,x(t))\le 0$ for every solution $x\colon[s_0,\infty)\to X$ of (4.1), with $s_0\ge t_0$.*

Then, the trivial solution of (4.1) is weakly exponentially stable.

Theorem 4.6. *Let $V\colon [t_0,\infty)\times X\to\mathbb{R}$ be a Lyapunov functional satisfying conditions (ii) and (iii) of Theorem 4.4. If, in addition,*

(i^*) *there exist $k>0$, $m\in\mathbb{N}$, and a continuous strictly increasing function $a\colon\mathbb{R}_+\to\mathbb{R}_+$, with $a(0)=0$, such that $k\|x\|^m\le V(t,x)\le a(\|x\|)$ for all $t\in[t_0,\infty)$ and $x\in X$.*

Then, the trivial solution of (4.1) is exponentially stable.

Proof. As in the proof of Theorem 4.4, one may check that all the hypotheses of Theorem 3.6 are satisfied. Hence, the trivial solution of the generalized ODE (4.4) is exponentially stable. Using the relation between the solutions of our class of generalized ODE (4.4) and the solutions of (4.1) yields that the trivial solution of (4.1) is also exponentially stable. $\square$

Corollary 4.7. *Let $V\colon [t_0,\infty)\times X\to\mathbb{R}$ be a functional satisfying condition (iii) of Corollary 4.5 and condition (i^*) of Theorem 4.6. If, moreover,*

(ii^*) *there exist $T>0$ and $\theta\in(0,1)$ such that $a(\|x(t)\|)\leq\theta k\|x(s)\|^m$ for every solution $x\colon[s_0,\infty)\to X$ of (4.1), whenever $t,s\in[s_0,\infty)$ and $t-s\geq T$.*

Then, the trivial solution of (4.1) is exponentially stable.

5 An application

Let $\mathbb{R}^n$ be endowed with norm $\|x\|_1=\sum_{i=1}^n|x_i|$, $x\in\mathbb{R}^n$. Consider the impulsive differential equation (IDE) subject to pre-assigned moments of impulse effects

$$\begin{cases}\dot{x}=-b(t)q(x), \quad t\neq t_i,\ t\geq 0,\\ \Delta(x(t_i))=x(t_i+)-x(t_i)=B_ix(t_i), \quad i\in\mathbb{N}.\end{cases} \tag{5.1}$$

Suppose the following conditions hold:

(B1) $b\colon\mathbb{R}_+\to[0,\infty)$ is a non–negative function, $q\colon O\subset\mathbb{R}^n\to\mathbb{R}^n$ is a function defined on a bounded subset O of $\mathbb{R}^n$, with $q(0)=0$ and $x_iq_i(x)>0$ whenever $x_i\neq 0$ and $i\in\{1,2,\ldots,n\}$, and the Perron integral $\int_a^b b(s)q(x(s))ds$ exists for $x\in G(\mathbb{R}_+,O)$ and $0\leq a<b$;

(B2) there exists a locally Perron integrable function $m\colon\mathbb{R}_+\to\mathbb{R}_+$ such that for all $a,b\in\mathbb{R}_+$, with $a<b$, we have

$$\left\|\int_a^b b(s)q(x(s))ds\right\|_1\leq\int_a^b m(s)ds,$$

for every $x\in G(\mathbb{R}_+,O)$;

(B3) there exists a locally Perron integrable function $\ell\colon\mathbb{R}_+\to\mathbb{R}_+$ such that for $a,b\in\mathbb{R}_+$, with $a<b$, we have

$$\left\|\int_a^b[b(s)q(x(s))-b(s)q(z(s))]ds\right\|_1\leq\|x-z\|_{[0,\infty)}\int_a^b\ell(s)ds,$$

for every $x,\ z\in G_0(\mathbb{R}_+,O)$;

(B4) $0 < t_1 < t_2 < \ldots < t_k < \ldots$ and $\lim_{k\to\infty} t_k = \infty$;

(B5) $-1 < B_i < -1 + \theta_0$ for some $\theta_0 \in (0, 1)$ and for all $i \in \mathbb{N}$;

(B6) there exists $T > 0$ such that on each interval $(kT, (k+1)T)$, $k \in \mathbb{N}$, there exists at least one discontinuity point t_j.

Given $s_0 \geq 0$, a function $x\colon [s_0, \infty) \to \mathbb{R}^n$ is a solution of (5.1), if $x(t) \in O$ for all $t \in [s_0, \infty)$, x is continuous on every interval $[0, t_1] \cap [s_0, \infty)$ and $(t_i, t_{i+1}] \cap [s_0, \infty)$ for $i \in \mathbb{N}$, $x'(t) = -b(t)q(x(t))$ for almost all $t \in [s_0, \infty)$ and $x(t_i+) = x(t_i) + B_i x(t_i)$ whenever $t_i \in [s_0, \infty)$.

If $x\colon [s_0, \infty) \to \mathbb{R}^n$ is solution of (5.1), with $s_0 \geq 0$, then x satisfies the Volterra integral equation

$$x(t) = x(s_0) - \int_{s_0}^{t} b(s)q(x(s))\,\mathrm{d}s + \sum_{i=1}^{\infty} B_i x(t_i) H_{t_i}(t), \quad t \geq s_0,$$

where H_{t_i} is the left–continuous Heaviside function concentrated at t_i, i.e.,

$$H_{t_i}(t) = \begin{cases} 0, & \text{for } \ 0 \leq t \leq t_i, \\ 1, & \text{for } \ t > t_i. \end{cases}$$

Define $g\colon \mathbb{R}_+ \times O \to \mathbb{R}^n$ by $g(t, x) = -b(t)q(x)$, and consider the auxiliary functions:

$$\widetilde{g}(t, x) = \begin{cases} g(t, x), & \text{if } t \in \mathbb{R}_+ \setminus \{t_1, t_2, \ldots\}, \\ B_k x(t_k), & \text{if } t = t_k, \quad k \in \mathbb{N}, \end{cases}$$

and

$$\widetilde{u}(t) = \begin{cases} t, & \text{if } t \in [0, t_1], \\ t + k, & \text{if } t \in (t_k, t_{k+1}], \ k \in \mathbb{N}. \end{cases}$$

According to [12, Theorem 3.1], the IDE (5.1) can be transformed into the following Volterra–Stieltjes–type integral equation

$$x(t) = x(s_0) + \int_{s_0}^{t} \widetilde{g}(s, x(s))d\widetilde{u}(s), \quad t \geq s_0. \tag{5.2}$$

This means that x is a solution of the IDE (5.1) if and only if x is a solution of 5.2. Since conditions (B1) and (B1) hold and O is bounded, one can prove that conditions (A1)–(A5) are satisfied with f, g and u replaced by 0, $\widetilde{g}$ and $\widetilde{u}$ respectively, see [12, Lemma 3.3]. Note that the boundedness of O is used to conclude condition (A4), since the proof of [12, Lemma 3.3] uses the boundedness of the impulse operator.

Define $V\colon \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ by $V(t,x) = \|x\|_1$. We show next that the function V satisfies the hypotheses of Theorem 4.6.

At first, consider $a\colon \mathbb{R}_+ \to \mathbb{R}_+$ defined by $a(t) = t$, $k = \frac{1}{2}$ and $m = 1$. Then,

$$k\|x\|_1^m \leq V(t,x) \leq a(\|x\|_1),$$

which yields condition (i^*).

Now, let us consider an auxiliary function $U\colon \mathbb{R}^n \to \mathbb{R}_+$ given by $U(x) = \|x\|_1$, $x \in \mathbb{R}^n$. Let $x\colon [0,\infty) \to O$ be a solution of (5.2). We claim that, for all $t \geq 0$,

$$D^+U(x(t)) = \limsup_{\eta \to 0^+} \frac{U(x(t+\eta)) - U(x(t))}{\eta} \leq 0.$$

In fact, we consider two cases: when $t \neq t_i$ for all $i \in \mathbb{N}$ and when $t = t_i$ for some $i \in \mathbb{N}$.

Case 1: $t \neq t_i$ for all $i \in \mathbb{N}$. Since $x \equiv 0$ is the unique solution of (5.1) such that $x(0) = 0$, we may consider:

(i) If $x(t) \neq 0$, then

$$D^+U(x(t)) = \nabla U(x(t))x'(t) = -\sum_{i=1}^{n} b(t)\,\mathrm{sgn}(x_i(t))q_i(x(t)) \leq 0,$$

since b is non-negative and $x_i h_i(x) > 0$ for $x_i \neq 0$. Here, by $\mathrm{sgn}(z)$ we mean the sign of z.

(ii) If $x(t) = 0$, then $D^+U(x(t)) = 0$.

Case 2: $t = t_i$ for some $i \in \mathbb{N}$. In this case, by (B5), we get

$$\begin{aligned} U(x(t_i^+)) &= U(x(t_i) + B_i x(t_i)) = \|(1 + B_i)(x(t_i))\|_1 \\ &< \theta_0 \|x(t_i)\|_1 = \theta_0 U(x(t_i)). \end{aligned}$$

Thus, for $\eta > 0$ sufficiently small, we have $U(x(t_i+\eta)) \leq U(x(t_i))$. Then, for $t = t_i$, we obtain

$$D^+U(x(t)) = \limsup_{\eta\to 0^+} \frac{U(x(t+\eta)) - U(x(t))}{\eta} \leq 0.$$

Analyzing the two previous cases, we conclude that $D^+U(x(t)) \leq 0$ for all $t \geq 0$. Consequently, $U(x(t)) \leq U(x(s))$ whenever $t \geq s \geq 0$, which implies that

$$V(t, x(t)) = U(x(t)) \leq U(x(s)) = V(s, x(s))$$

for all $t \geq s \geq 0$ and condition (iii) is also satisfied.

Let $t, s \geq 0$ be such that $t - s \geq T$. By condition (B6), there exists t_k such that $s < t_k < t$. Therefore,

$$U(x(t)) \leq U(x(t_k^+)) \leq \theta_0 U(x(t_k)) \leq \theta_0 U(x(s)),$$

whence it follows that $V(t, x(t)) \leq \theta_0 V(s, x(s))$. Hence, condition (ii) holds. Since all conditions of Theorem 4.6 are satisfied, the trivial solution of the Volterra–Stieltjes–type integral equation (5.2) is exponentially stable and, therefore, the trivial solution of the IDE (5.1) is also exponentially stable.

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References

[1] S. M. Afonso, E. M. Bonotto, and M. Federson. On exponential stability of functional differential equations with variable impulse perturbations. *Differential Integral Equations*, 27(7/8):721–742, 2014.

[2] S. M. Afonso, E. M. Bonotto, M. Federson, and L. Gimenes. Stability of functional differential equations with variable impulsive perturbations via generalized ordinary differential equations. *Bulletin des Sciences Mathèmatiques*, 137(2):189–214, 2013.

[3] S. M. Afonso, E. M. Bonotto, M. Federson, and L. P. Gimenes. Boundedness of solutions of retarded functional differential equations with variable impulses via generalized ordinary differential equations. *Mathematische Nachrichten*, 285(5–6):545–561, 2012.

[4] S. M. Afonso, E. M. Bonotto, M. Federson, and Š. Schwabik. Discontinuous local semiflows for Kurzweil equations leading to LaSalle's invariance principle for differential systems with impulses at variable times. *Journal of Differential Equations*, 250(7):2969–3001, 2011.

[5] R. G. Bartle. *A Modern Theory of Integration*. Crm Proceedings & Lecture Notes. American Mathematical Society, 2001.

[6] E. M. Bonotto, M. Federson, and J. G. Mesquita. *Generalized ordinary differential equations in abstract spaces and applications*. John Wiley & Sons, Hoboken, NJ, 2021.

[7] M. Federson, M. Frasson, J. Mesquita, and P. Tacuri. Measure neutral functional differential equations as generalized ODEs. *Journal of Dynamics and Differential Equations*, 31(1):207–236, 2019.

[8] M. Federson, R. Grau, J. Mesquita, and E. Toon. Boundedness of solutions of measure differential equations and dynamic equations on time scales. *Journal of Differential Equations*, 263(1):26–56, 2017.

[9] M. Federson, R. Grau, J. Mesquita, and E. Toon. Lyapunov stability for measure differential equations and dynamic equations on time scales. *Journal of Differential Equations*, 267(7):4192–4223, 2019.

[10] M. Federson, R. Grau, and J. G. Mesquita. Prolongation of solutions of measure differential equations and dynamic equations on time scales. *Mathematische Nachrichten*, 292(1):22–55, 2019.

[11] M. Federson, J. G. Mesquita, and A. Slavík. Measure functional differential equations and functional dynamic equations on time scales. *Journal of Differential Equations*, 252(6):3816–3847, 2012.

[12] M. Federson, J. G. Mesquita, and A. Slavík. Basic results for functional differential and dynamic equations involving impulses. *Mathematische Nachrichten*, 286(2–3):181–204, 2013.

[13] M. Federson and Š. Schwabik. Generalized ODE approach to impulsive retarded functional differential equations. *Differential Integral Equations*, 19(11):1201–1234, 2006.

[14] M. Federson and Š. Schwabik. Stability for retarded functional differential equations. *Ukrainian Math. J.*, 60(1):121–140, 2008.

[15] M. Federson and Š. Schwabik. A new approach to impulsive retarded differential equations: stability results. *Functional Differential Equations*, 16(4):583–607, 2009.

[16] X. Fu and X. Li. Razumikhin–type theorems on exponential stability of impulsive infinite delay differential systems. *Journal of Computational and Applied Mathematics*, 224(1):1–10, 2009.

[17] C. A. Gallegos, R. Grau, and J. G. Mesquita. Stability, asymptotic and exponential stability for various types of equations with discontinuous solutions via Lyapunov functionals. *Journal of Differential Equations*, 299:256–283, 2021.

[18] C. A. Gallegos and G. Robledo. On stability for generalized linear differential equations and applications to impulsive systems. *Proceedings of the Royal Society of Edinburgh: Section A Mathematics*, 154(2):381–407, 2024.

[19] R. Henstock. *Lectures on the Theory of Integration.* Series in real analysis. World Scientific, 1988.

[20] C. S. Hönig. *Volterra Stieltjes-integral equations.* North-Holland Publishing Co., Amsterdam, 1975. Functional analytic methods; linear constraints, Mathematics Studies, No. 16, Notas de Matemática, No. 56. [Notes on Mathematics, No. 56].

[21] C. Imaz and Z. Vorel. Generalized ordinary differential equations in Banach spaces and applications to functional equations. *Bol. Soc. Mat. Mexicana*, 11:47–59, 1966.

[22] J. Kurzweil. Generalized ordinary differential equations and continuous dependence on a parameter. *Czechoslovak Mathematical Journal*, 07(3):418–449, 1957.

[23] J. Kurzweil. Generalized ordinary differential equations. *Czechoslovak Mathematical Journal*, 08(3):360–388, 1958.

[24] J. Kurzweil. Unicity of solutions of generalized differential equations. *Czechoslovak Mathematical Journal*, 08(4):502–509, 1958.

[25] J. Kurzweil. Addition to my paper "Generalized ordinary differential equations and continuous dependence on a parameter". *Czechoslovak Mathematical Journal*, 09(4):564–573, 1959.

[26] J. Kurzweil. Problems which lead to a generalization of the concept of an ordinary nonlinear differential equation. In *Differential Equations and Their Applications. Proceedings of the Conference held in Prague in September 1962*, pages 65–76, Prague, 1963. Publishing House of the Czechoslovak Academy of Sciences.

[27] J. Kurzweil. *Generalized ordinary differential equations*, volume 11 of *Series in Real Analysis.* World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2012. Not absolutely continuous solutions.

[28] F. Oliva and Z. Vorel. Functional equations and generalized ordinary differential equations. *Bol. Soc. Mat. Mexicana*, 11:40–46, 1966.

[29] Š. Schwabik. *Generalized ordinary differential equations*, volume 5 of *Series in Real Analysis.* World Scientific Publishing Co. Inc., River Edge, NJ, 1992.