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Hölder regularity for fully nonlinear nonlocal equations with gradient terms

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Abstract. In this survey we prove Hölder regularity results for viscosity solutions of fully nonlinear nonlocal uniformly elliptic second order differential equations with local gradient terms. This extends the nonlocal counterpart of the work of G. Barles, E. Chasseigne and C. Imbert in JEMS, 2011, to fully nonlinear extremal nonlocal operators.

Keywords: hölder regularity, integro-differential equations, viscosity solutions.

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1 Introduction

In this survey, we study interior Hölder regularity of viscosity solutions for nonlocal equations of the form,

$$\mathcal{F}u(x) + b(x)|Du|^m = f(x) \quad \text{in} \quad \Omega, \tag{1.1}$$

where u is a real valued function, Du stands for the gradient of u, and $\mathcal{F}$ is an integral extremal fully nonlinear operator of Pucci or Isaacs type. Also,

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$b \in C^{0,\tau}$ with $\tau \in (0,1)$, $m \in (0,2]$, and $\Omega \subset \mathbb{R}^n$ is a bounded domain with $n \geq 1$.

We are interested in studying the regularity of viscosity solutions to equations of the form (1.1), under minor hypotheses about b and f.

Over the years, significant attention has been devoted to the study of integro-differential operators and fully nonlinear operators characterized by nonlinear growth in the gradient, largely attributed to their applications.

In general, two approaches are commonly used to study Hölder estimates for fully nonlinear equations. The first approach, based on the Harnack inequality, can be seen in the work of Caffarelli and Silvestre [4] and considers equations arising from stochastic control problems with Levy processes. The second approach is a method related to the theory of viscosity solutions introduced by Ishii & Lions [5], initially designed to get comparison principles, that yields results for second-order nonlinear equations, possibly degenerate, but with continuous coefficients. Furthermore, Hölder estimates were established in [3] via probability techniques, and Silvestre in [8] proved Hölder continuity for a class of differential equations associated to jump processes, via growth lemmas.

As far as local drift terms are concerned, in the work [2] by G. Barles, E. Chasseigne, C. Imbert, mixed-type operators are studied, where the local part can be fully nonlinear, but the nonlocal part is linear. In this work, we will focus on exploring the nonlocal part of the operator, i.e., focusing on the nonlocal character of the equation. We will show that the methods used in [3] can be extended by combining the techniques of [1] to produce regularity results for Pucci operators.

We note that Hölder regularity provide a lot of applications, including regularity $C^{1,\alpha}$ obtained by Caffareli & Silvestre in [4] and existence of eigenvalues for fully nonlinear integro-differential elliptic equations with a drift term [7], just to mention a few. See also the parabolic counterparts with gradient growth equal to one in [9].

Our main result reads as follows

Theorem 1.1. *Let $u \in C\left(\overline{\Omega}\right)$ be a viscosity solution of the equation (1.1), assuming that $s \in (0,1)$, $b \in C^{0,\tau}\left(\overline{\Omega}\right)$ with $\tau \in [0,1)$ and $f \in L^{\infty}(\Omega)$. If $m \in (0,1)$, then u is $\alpha-$Hölder continuous with exponent*

$$\alpha \leq \frac{2s-m+\tau}{1-m}.$$

If $m \in (1,2)$, then u is $\alpha-$Hölder continuous with exponent

$$\alpha \leq \frac{2s-m+1}{2-m}.$$

If $m = 2$, then u is $\alpha-$Hölder continuos for any $\alpha \in (0,1)$.

Remark 1.2. If $m \in (0,1]$ the conclusion of the theorem holds for any $s \geq 1/2$ with the only assumption of continuity over b; whereas it holds for any s if we assume that b is $C^{0,\tau}$ with $\tau \geq 1-2s$. If $m \in (1,2)$ the conclusion of the theorem holds for any $s \geq 1/2$ with merely continuity over b; whereas it holds for any s if we assume that b is $C^{0,\tau}$ with $\tau \geq 1-2s$. Note that if $m = 2$, the conclusion holds for any $s \in (0,1)$ with the only assumption of the continuity over b.

The rest of the paper follows this structure: the second section presents assumptions about the operators under study and definitions. The third section is dedicated to proving the main theorem in which we make use of the Ishii-Lion's method.

2 Preliminaries

In this section, we will provide the basic concepts and assumptions used in this paper. First, we present the definition of the operators we will work in this article, followed by the definition of viscosity solution.

A linear nonlocal uniformly elliptic operator is given by

$$\begin{aligned} L_K u(x) &= \text{P.V.} \int_{\mathbb{R}^n} (u(x+z)-u(x))K(z)\,dz \\ &= \lim_{\epsilon \to 0^+} \int_{B_\epsilon^c(x)} (u(x+z)-u(x))K(z)\,dz \end{aligned} \tag{2.1}$$

where P.V. denotes the principal value, and the kernel $K : \mathbb{R}^n \to \mathbb{R}$ satisfies:

(K.1) is nonnegative, symmetric and

$$\int_{\mathbb{R}^n} \min\{|z|^2, 1\} K(z)\, dz < \infty;$$

(K.2) for some $s \in (0,1)$ and ellipticity constants $0 < \lambda \leq \Lambda$ we have

$$\frac{\lambda}{|z|^{n+2s}} \leq K(z) \leq \frac{\Lambda}{|z|^{n+2s}}.$$

The most classical example of a nonlocal linear uniformly elliptic operator is provided by the fractional Laplacian by considering a kernel of the form $|z|^{-n-2s}$, that is,

$$(-\Delta)^s u(x) := C_{n,s} \text{ P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}}\, dy \quad , \forall x \in \mathbb{R}^n,$$

where $C_{n,s} > 0$ is a normalization constant. We observe that, via

$$\int_{\mathbb{R}^n} \frac{u(x+z) - u(x)}{|z|^{n+2s}}\, dz = \int_{\mathbb{R}^n} \frac{u(x-z) - u(x)}{|z|^{n+2s}}\, dz,$$

so the fractional Laplacian can be also written as

$$-(-\Delta)^s u(x) = \frac{1}{2} C_{n,s} \int_{\mathbb{R}^n} \frac{u(x+z) + u(x-z) - 2u(x)}{|z|^{n+2s}}\, dz.$$

Let $\mathcal{L}_0$ be the class of all linear uniformly elliptic operators. In other words, we say that $L \in \mathcal{L}_0$ if the operator L has the form (2.1), under the same above assumptions about (K.1) and (K.2) over K. We denote the nonlocal Pucci operators as

$$\mathcal{M}^+ u(x) = \sup_{L \in \mathcal{L}_0} Lu(x) \quad , \quad \mathcal{M}^- u(x) = \inf_{L \in \mathcal{L}_0} Lu(x)$$

i.e., the maximum and minimum operators for the class $\mathcal{L}_0$, respectively. It is noteworthy that, $\mathcal{L}_0$, $\mathcal{M}^+$ and $\mathcal{M}^-$ depend on the parameters Λ, λ

and s, in particular

$$\mathcal{M}^+u(x) = \int_{\mathbb{R}^n} \frac{S_+(u(x+z)+u(x-z)-2u(x))}{|z|^{n+2s}}\,dz,$$
$$\mathcal{M}^-u(x) = \int_{\mathbb{R}^n} \frac{S_-(u(x+z)+u(x-z)-2u(x))}{|z|^{n+2s}}\,dz,$$

where $S_+(t) = \Lambda t^+ - \lambda t^-$ and $S_-(t) = \lambda t^+ - \Lambda t^-$. Here, t^+ and t^- are defined as $t^+ = \max\{t, 0\}$ and $t^- = \max\{-t, 0\}$, respectively.

Throughout this paper, we will fix the maximal Pucci extremal operator of the form

$$\mathcal{M}^+u(x) = \sup_{L\in\mathcal{L}_0} Lu(x)$$

where the operator L is of the form,

$$\int_{\mathbb{R}^n} (u(x+z)-u(x))K(z)\,dz,$$

and the kernel $K : \mathbb{R}^n\backslash\{0\} \to \mathbb{R}$ represents the frequency of jumps in every direction $z \in \mathbb{R}^n$ and satisfies some conditions.

In proving Hölder regularity for the main problem, a suitable notion of a viscosity solution has to be considered. We recall the definition of viscosity solution for (1.1) as in [2, Definition 2.1].

Definition 2.1. We say that $u : \mathbb{R}^n \to \mathbb{R}$ is a viscosity subsolution to (1.1) if, for any test function $\varphi \in C^2(B_\delta(x_0))$ such that $u - \varphi$ attains its maximum at $x_0 \in \Omega$ for some $\delta > 0$, we have

$$\mathcal{M}^+(u, \varphi, x_0, \delta) + b(x_0)|D\varphi(x_0)|^m \geq f(x_0) \text{ in } \Omega$$

where the operator $\mathcal{M}^+$ is given by

$$\mathcal{M}^+(u,\varphi,x,\delta) = \sup_K \left\{ \int_{\mathbb{R}^n\backslash B_\delta} (u(x+z)-u(z))K(z)\,dz \right.$$
$$\left. + \int_{B_\delta} (\varphi(x+z)-\varphi(z))K(z)\,dz \right\}.$$

In a similar manner, we say that $u : \mathbb{R}^n \to \mathbb{R}$ is a viscosity supersolution to (1.1) if, for any test function $\varphi \in C^2(B_\delta(x_0))$ such that $u - \varphi$ attains its minimum at $x_0 \in \Omega$ for some $\delta > 0$, we have

$$\mathcal{M}^+(u, \varphi, x_0, \delta) + b(x_0)|D\varphi(x_0)|^m \leq f(x_0) \text{ in } \Omega$$

A continous function $u : \mathbb{R}^n \to \mathbb{R}$ is a viscosity solution to (1.1) if it is both a viscosity subsolution and supersolution.

3 Proof of Theorem 1.1

This section is devoted to the proof of the main theorem of this paper. For the proof, we will use the method introduced by Ishii-Lions [5]. Following this method, we fix $x_0 \in \Omega$ and introduce the function $\Phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ given by

$$\Phi(x, y) := u(x) - u(y) - \phi(x - y) - \Gamma(x),$$

where ϕ is a radial function and $\Gamma(x) = L_2|x - x_0|^2$ is a localized function around $x_0 \in \Omega$. Since our goal is to attain Hölder regularity, we will choose the following function $\phi(t) = L_1|t|^\alpha$, $L_1 > 0$ and $\alpha \in (0, 1)$.

Our aim is to show that

$$\Phi(x, y) \leq 0 \tag{3.1}$$

The first observation to note is that if Φ is nonpositive, then under suitable control conditions over L_1, L_2, and α, we obtain the desired regularity result

$$u(x) - u(y) \leq L_1|x - y|^\alpha.$$

To prove (3.1), we will demonstrate that the maximum is only achieved when $x = y = x_0$. To establish this, from the continuity of the function Φ, we observe that it attains a maximum point $(\overline{x}, \overline{y}) \in \overline{\Omega} \times \overline{\Omega}$.

Suppose by contradiction that $\Phi(\overline{x}, \overline{y}) > 0$. As a consequence of this, it follows that,

1. $\overline{x} \neq \overline{y}$, since otherwise $\Phi(\overline{x}, \overline{y}) = -\Gamma(\overline{x}) < 0$;

2. $L_1|\overline{x}-\overline{y}|^\alpha, L_2|\overline{x}-x_0|^2 \le u(\overline{x})-u(\overline{y}) \le 2\|u\|_\infty$;

3. $0 < u(\overline{x})-u(\overline{y})$, this implies that $u(\overline{y}) < u(\overline{x})$;

In particular, if $L_2 \ge 8\|u\|_\infty[d(x_0,\partial\Omega)]^{-2}$, we obtain

$$|\overline{x}-x_0|^2 \le 2\frac{\|u\|_\infty}{L_2} \le \frac{1}{4}[d(x_0,\partial\Omega)]^2.$$

From this it follows that $|\overline{x}-x_0| \le d(x_0,\partial\Omega)/2$. Therefore, if L_1 is sufficiently large, such that

$$\left(2\frac{\|u\|_\infty}{L_1}\right)^{1/\alpha} < \frac{1}{2}d(x_0,\partial\Omega),$$

we can see that $\overline{x},\overline{y} \in \Omega$.

On the other hand, we have that $\Phi_{\overline{y}}(x) = u(x)-\varphi(x,\overline{y})$ reaches a maximum in $B_\delta(\overline{x})$ at the point $\overline{x}$, where $\varphi(x,y) = \phi(x-y)+\Gamma(x)$. Then $\varphi \in C^2(\Omega)$ is a test function in the definition of u as a viscosity subsolution of (1.1), so it follows

$$\mathcal{M}^+(u,\varphi,\overline{x},\delta) + b(\overline{x})|D_x\varphi(\overline{x},\overline{y})|^m \ge f(\overline{x}).$$

Similarly, $\Phi_{\overline{x}}(y) = u(y)+\varphi(\overline{x},y)$ reaches a minimum in $B_\delta(\overline{y})$ at the point $\overline{y}$; we obtain the viscosity supersolution inequality,

$$\mathcal{M}^+(u,-\varphi,\overline{y},\delta) + b(\overline{y})|-D_y\varphi(\overline{x},\overline{y})|^m \le f(\overline{y}).$$

Combining the previous viscosity inequalities, we obtain the following expression that brings together local and nonlocal terms

$$\begin{aligned}\mathcal{M}^+(u,\varphi,\overline{x},\delta) &- \mathcal{M}^+(u,-\varphi,\overline{y},\delta)\\ &+ b(\overline{x})|D_x\varphi(\overline{x},\overline{y})|^m - b(\overline{y})|D_y\varphi(\overline{x},\overline{y})|^m - f(\overline{x}) + f(\overline{y}) \ge 0. \quad (3.2)\end{aligned}$$

We must show that the right-hand side of the above inequality is negative. Our aim now is to acquire estimations for both local and non-local terms.

3.1 Estimates for the local terms.

Let us denote the final terms of the inequality (3.2) as

$$\begin{aligned} T_l := (b(\overline{x}) - b(\overline{y}))|D_x\varphi(\overline{x},\overline{y})|^m \\ + b(\overline{y})(|D_x\varphi(\overline{x},\overline{y})|^m - |D_y\varphi(\overline{x},\overline{y})|^m) \\ - (f(\overline{x}) + f(\overline{y})), \end{aligned} \tag{3.3}$$

where the derivatives for the function $\varphi(x,y) = \phi(x-y) + \Gamma(x)$ are given by

$$\begin{aligned} D_x\varphi(\overline{x},\overline{y}) &= \alpha L_1|\overline{x}-\overline{y}|^{\alpha-2}(\overline{x}-\overline{y}) + 2L_2(\overline{x}-x_0), \\ D_y\varphi(\overline{x},\overline{y}) &= -\alpha L_1|\overline{x}-\overline{y}|^{\alpha-2}(\overline{x}-\overline{y}). \end{aligned}$$

To estimate the local terms in the inequality (3.2), we must distinguish between two cases for the exponent m, in addition to making use of the fact that b belongs to the Hölder space $C^{0,\tau}$, so we can find a constant $C_b > 0$ such that $|b(\overline{x}) - b(\overline{y})| \leq C_b|\overline{x}-\overline{y}|^\tau$. When $\tau = 0$, we only need to use that b is bounded.

First, let us note that if $m \in (0,1]$, then $|p+q|^m - |p|^m \leq |q|^m$. In this case, it follows from (3.3)

$$\begin{aligned} T_l \leq C_b|\overline{x}-\overline{y}|^\tau \left[(\alpha L_1|\overline{x}-\overline{y}|^{\alpha-1})^m + (2L_2|\overline{x}-x_0|)^m\right] \\ + (2L_2|\overline{x}-x_0|)^m\|b\|_\infty + 2\|f\|_\infty. \end{aligned}$$

On the other hand, if $m \geq 1$ we know that for $|p|,|q| \geq 0$,

$$(|p|+|q|)^m \leqslant 2^{m-1}\max\{|p|^m,|q|^m\} \leqslant 2^{m-1}(|p|^m+|q|^m).$$

Furthermore, the function $t \to |t|^m$ is convex for $t \in (0,+\infty)$, which implies $|p+q|^m \leq |p|^m + m|p+q|^{m-2}\langle p+q, q\rangle$. Using the above inequality, it follows that

$$|p+q|^m - |p|^m \leqslant m2^{m-1}(|p|^{m-1} + |q|^{m-1})|q|.$$

In this case, of (3.3), we have

$$
\begin{aligned}
T_l \leq C_b 2^{m-1}|\overline{x}-\overline{y}|^{\tau}\Big[\Big(\alpha L_1|\overline{x}-\overline{y}|^{\alpha-1}\Big)^m + \Big(2L_2|\overline{x}-x_0|\Big)^m\Big] \\
+ m2^m L_2\Big[\big(\alpha L_1|\overline{x}-\overline{y}|^{\alpha-1}\big)^{m-1} \\
+ \big(2L_2|\overline{x}-x_0|\big)^{m-1}\Big]|\overline{x}-x_0|\|b\|_\infty + 2\|f\|_\infty.
\end{aligned}
$$

Given these inequalities, obtained for the local term when $m \in (0,1]$ and $m > 1$, note that $\Phi(\overline{x},\overline{y}) > 0$ implies that $|\overline{x}-x_0| < 2\|u\|_\infty$, therefore, we can choose L_1 to be sufficiently large in order to control the terms that depend on α, L_2, $\|b\|_\infty$, $\|f\|_\infty$.

3.2 Estimates for nonlocal terms

Now we need to estimate the remaining terms of (3.2), which involve the nonlocal terms.

As in [1], for given $\epsilon > 0$ and $\bar{x} \in \Omega$ fixed, we have

$$
\mathcal{M}^+u(\overline{x}) \leq L_K u(\overline{x}) + \epsilon
$$

for some $L_K \in \mathcal{L}_0$. Indeed, this follows from the definition of supremum, since $\mathcal{M}^+u(\overline{x}) - \epsilon$ is not an upper bound for the set $\{L_K u(\overline{x})\}$, since it is less than $\mathcal{M}^+u(\overline{x}) = \sup L_K u(\overline{x})$. Therefore, we can assert that there exists a kernel K such that

$$
\mathcal{M}^+u(\overline{x}) \leq L_K u(\overline{x}) + \epsilon \qquad \text{and} \qquad \mathcal{M}^+u(\overline{y}) \geq L_K u(\overline{y}).
$$

In other words, we only need to consider such a linear operator in the rest of the argument,

$$
L_K u(\overline{x}) - L_K u(\overline{y}) + \epsilon \geq \mathcal{M}^+u(\overline{x}) - \mathcal{M}^+u(\overline{y}),
$$

where the operators L_K are of the form (2.1).

Given that these integrals may be singular at the origin, we will partitionate the domain of integration into three parts: $A_1 := \mathbb{R}^n \backslash B_1$, which relates to the nonlocal term T^1_{nl}; $A_2 := \{z \in B_\delta : (1-\eta)|z||\overline{x}-\overline{y}| \leq |\langle z, \overline{x}-\overline{y}\rangle|\}$, with $\eta \in (0,1)$ small enough, is related to the nonlocal term T^2_{nl}; and $A_3 := B_1 \backslash A_2$, which relates to the nonlocal term T^3_{nl}, so that

$$L_K u(\overline{x}) - L_K u(\overline{y}) = T^1_{nl} + T^2_{nl} + T^3_{nl},$$

where the nonlocal terms for $i \in \{1,2,3\}$ are of the form,

$$\begin{aligned} T^i_{nl}(\overline{x},\overline{y}) = \int_{A_i} (u(\overline{x}+z) - u(\overline{x}))K(z)\,dz \\ - \int_{A_i} (u(\overline{y}+z) - u(\overline{y}))K(z)\,dz, \end{aligned}$$

where A_i with $i \in \{1,2,3\}$ describe each of the preceding domains. To avoid overloading the notation, we will denote $a := \overline{x} - \overline{y}$.

For the initial term T^1_{nl}, there is no indeterminacy away from the origin. Consequently, we deduce that

$$T^1_{nl} \leq 4\|u\|_\infty \int_{\mathbb{R}^n \backslash B_1} K(z)\,dz \leq C_1. \tag{3.4}$$

where

$$C_1 := \Lambda \frac{2\sigma(S_{n-1})}{s} \|u\|_\infty$$

with $\Lambda > 0$ the ellipticity constant and $\sigma(S_{n-1})$ the surface area of the sphere of dimension $n-1$.

Before proceeding with the estimation to evaluate T^2_{nl}, let us note that since $(\overline{x},\overline{y})$ is the maximum point for Φ, it follows that for any $d, d' \in \mathbb{R}^n$,

$$\Phi(\overline{x}+d, \overline{y}+d') \leq \Phi(\overline{x},\overline{y}),$$

More precisely, we have

$$\begin{aligned} (u(\overline{x}+d) - u(\overline{x})) - \big(u(\overline{y}+d') - u(\overline{y})\big) \\ \leq \phi(a + (d-d')) - \phi(a) \\ + \Gamma(\overline{x}+d) - \Gamma(\overline{x}). \end{aligned} \tag{3.5}$$

Applying this inequality, first with $d = z, d' = 0$ and then with $d = 0, d' = z$, we obtain

$$\begin{aligned} T^2_{nl} \leq & \int_{A_2} (\phi(a+z) - \phi(a))\, K(z)\, dz \\ & + \int_{A_2} (\Gamma(\overline{x}+z) - \Gamma(\overline{x}))\, K(z)\, dz \\ & + \int_{A_2} (\phi(a-z) - \phi(a))\, K(z)\, dz. \end{aligned}$$

Now, applying Taylor's formula with an integral remainder (for instance Theorem 9.9 in [6]) to the function $\phi(a+z)$

$$\phi(a+z) = \phi(a) + D\phi(a)\cdot z + \int_0^1 (1-s)\langle D^2\phi(a+sz)\cdot z, z\rangle\, ds.$$

Analogously, we can derive a similar expression for $\phi(a-z)$,

$$\phi(a-z) = \phi(a) - D\phi(a)\cdot z + \int_{-1}^0 (1+s)\langle D^2\phi(a+sz)\cdot z, z\rangle\, ds,$$

adding these two equations, we observe that the linear terms in z cancel, leaving us with the following result

$$\phi(a+z) + \phi(a-z) - 2\phi(a) = \int_{-1}^1 (1-|s|)\langle D^2\phi(a+sz)z, z\rangle\, ds.$$

Observe that the derivative of the function $\phi(t) = L_1|t|^\alpha$ is given by

$$D\phi(a) = \alpha L_1 |a|^{\alpha-2} a.$$

For the second derivative, denoting by $\widehat{a} = a/|a|$,

$$\begin{aligned} D^2\phi(a) &= \alpha L_1 |a|^{\alpha-2} \left((\alpha-2)\frac{a}{|a|} \otimes \frac{a}{|a|} + I_n \right) \\ &= \alpha L_1 |a|^{\alpha-2} \left((\alpha-2)\widehat{a} \otimes \widehat{a} + I_n \right). \end{aligned}$$

In particular,

$$D^2\phi(a+sz)\cdot z = \alpha L_1 |a+sz|^{\alpha-2} \left((\alpha-2)(\widehat{a+sz}) \otimes (\widehat{a+sz}) \cdot z + z I_n \right).$$

As we seek to estimate the Taylor expansion, we observe that

$$\begin{aligned}\langle D^2\phi(a+sz)\cdot z, z\rangle &= \alpha L_1|a+sz|^{\alpha-2}\left((\alpha-2)(\widehat{a+sz})\otimes(\widehat{a+sz})\cdot z + zI_n\right)\cdot z\\ &= \alpha L_1|a+sz|^{\alpha-2}\left(|z|^2+(\alpha-2)|\langle\widehat{a+sz}, z\rangle|^2\right)\end{aligned} \tag{3.6}$$

Note that within the set A_2, which includes $z \in B_\delta$, the following bounds are established with $\delta = |a|\rho_0$ and $\rho_0 \in (0,1)$

$$\begin{aligned}|a+sz| &\le |a| + |s||z| \le |a| + \delta = |a|\,(1+\rho_0) \qquad (3.7)\\ |a+sz| &\ge |a| - |s||z| \ge |a| - \delta = |a|\,(1-\rho_0)\\ |(a+sz)\cdot z| &= |a\cdot z + s|z|^2| \ge |a\cdot z| - \delta|z|\\ &\ge (1-\eta)|a||z| - \rho_0|a||z|\\ &= (1-\eta-\rho_0)\,|a||z|.\end{aligned}$$

Through the preceding inequalities and the Taylor expansion (3.6), we have

$$\begin{aligned}&\left\langle D^2\phi(a+sz)\cdot z, z\right\rangle\\ &\quad\le \alpha L_1|a+sz|^{\alpha-4}\left(|a+sz|^2|z|^2+(\alpha-2)|(a+sz)\cdot z|^2\right)\\ &\quad\le \alpha L_1|a+sz|^{\alpha-4}\left(|a+sz|^2+(\alpha-2)\,(1-\eta-\rho_0)^2\,|a|^2\right)|z|^2\\ &\quad\le \alpha L_1|a|^{\alpha-2}(1+\rho_0)^{\alpha-4}\left((1+\rho_0)^2-(2-\alpha)(1-\eta-\rho_0)^2\right)|z|^2.\end{aligned}$$

From this estimate, we define

$$C_2 := (1+\rho_0)^{\alpha-4}\left((2-\alpha)(1-\eta-\rho_0)^2-(1+\rho_0)^2\right)$$

then,

$$\int_{A_2}\left(\phi(a+z)+\phi(a-z)-2\phi(a)\right)K(z)\,dz \le -C_2\alpha L_1|a|^{\alpha-2}\int_{A_2}|z|^2K(z)\,dz.$$

Given $A_2 \subset B_1$, we can observe that this set is indeed a cone, so if $z \in A_2$ then $-z \in A_2$. Following the definition of Γ,

$$\int_{A_2}\left(\Gamma(\overline{x}+z)-\Gamma(\overline{x})\right)K(z)\,dz = L_2\int_{A_2}(|z|^2+\langle\overline{x}-x_0, z\rangle)K(z)\,dz,$$

by using the symmetry of the kernel, we obtain

$$L_2 \int_{A_2} (2\langle \overline{x} - x_0, z\rangle) K(z)\, dz = 0,$$

that is,

$$\int_{A_2} \left(\Gamma(\overline{x}+z) - \Gamma(\overline{x})\right) K(z)\, dz = L_2 \int_{A_2} |z|^2 K(z)\, dz.$$

Given that we are in the cone, we need to estimate the integral of the kernel. For this, let us note that

$$\int_{A_2} |z|^2 K(z)\, dz \le \sigma(S_{n-1})\Lambda \int_0^\delta r^{1-2s}\, dr = \Lambda \frac{\sigma(S_{n-1})}{2(1-s)} \delta^{2-2s}.$$

Hence, by defining this constant

$$C_2' := \Lambda \frac{\sigma(S_{n-1})}{2-2s} \rho_0^{2-2s},$$

we can see that the second nonlocal term T_{nl}^2 is bounded by

$$T_{nl}^2 \le -C_2 C_2' \alpha L_1 |a|^{\alpha-2s} + L_2 C_2' |a|^{2-2s}. \tag{3.8}$$

In order to estimate the final nonlocal term T_{nl}^3 in $B_1 \backslash A_2$, that is,

$$\begin{aligned} T_{nl}^3 \le \int_{B_1\backslash A_2} &\left(\phi(a+z) - \phi(a)\right) K(z)\, dz \\ &+ \int_{B_1\backslash A_2} \left(\Gamma(\overline{x}+z) - \Gamma(\overline{x})\right) K(z)\, dz \\ &\quad + \int_{B_1\backslash A_2} \left(\phi(a-z) - \phi(a)\right) K(z)\, dz \end{aligned}$$

we will write $B_1\backslash A_2 = (B_1\backslash B_\delta) \cup (B_\delta\backslash A_2)$. In this way, in the annulus $B_1\backslash B_\delta$, we can use a Taylor expansion for ϕ, while in the domain $B_\delta\backslash A_2$, we utilize the concavity of the function t^α for $t \in (0, \infty)$.

Let us consider $d \in \mathbb{R}^n$, then in $B_1\backslash B_\delta$

$$\begin{aligned} \phi(a+d) - \phi(a) &\le L_1(|a| + |d|)^\alpha - L_1(|a|)^\alpha \\ &\le \alpha L_1 |a|^{\alpha-1} |d|. \end{aligned}$$

In this way, taking first $d = z$ and then $d = -z$, we can conclude that in $B_1 \backslash B_\delta$,

$$\int_{B_1 \backslash B_\delta} (\phi(a+z) + \phi(a-z) - 2\phi(a)) K(z)\, dz \leq 2\alpha L_1 |a|^{\alpha-1} \int_{B_1 \backslash B_\delta} |z| K(z)\, dz.$$

Given the last integral with $s \in (0,1)$, we must separate the analysis in some cases. If $s \neq 1/2$, then

$$\int_{B_1 \backslash B_\delta} |z| K(z)\, dz \leq \Lambda \sigma(S_{n-1}) \int_\delta^1 r^{-2s}\, dr = \Lambda \frac{\sigma(S_{n-1})}{1-2s} \left(1 - \delta^{1-2s}\right),$$

and we must again distinguish into two cases, as if $s < 1/2$

$$\int_{B_1 \backslash B_\delta} |z| K(z)\, dz \leq \Lambda \frac{\sigma(S_{n-1})}{1-2s} = \overline{C}_3,$$

whereas if $s > 1/2$ with $\delta = |a|\rho_0$

$$\int_{B_1 \backslash B_\delta} |z| K(z)\, dz \leq \Lambda \frac{\sigma(S_{n-1})}{2s-1} (|a|\rho_0)^{1-2s} = |a|^{1-2s} \widetilde{C}_3.$$

If we define the following constant $C_3 = \max\{\overline{C}_3, \widetilde{C}_3\}$ we obtain the following upper bound for ϕ in $B_1 \backslash B_\delta$, when $s < 1/2$,

$$\int_{B_1 \backslash B_\delta} (\phi(a+z) + \phi(a-z) - 2\phi(a)) K(z)\, dz \leq 2\alpha L_1 C_3 |a|^{\alpha-1},$$

and if $s > 1/2$,

$$\int_{B_1 \backslash B_\delta} (\phi(a+z) + \phi(a-z) - 2\phi(a)) K(z)\, dz \leq 2\alpha L_1 C_3 |a|^{\alpha-2s}.$$

In the case when $s = 1/2$ we get

$$\int_{B_1 \backslash B_\delta} |z| K(z)\, dz \leq -\Lambda \sigma(S_{n-1}) \ln(|a|\rho_0),$$

and if we define the constant

$$C_3' := \Lambda \sigma(S_{n-1})$$

we obtain the following upper bound

$$\int_{B_1\setminus B_\delta} \phi(a+z)+\phi(a-z)-2\phi(a))K(z)\,dz$$
$$\leq -2\alpha L_1 C_3'|a|^{\alpha-1}\ln(|a|\rho_0).$$

In $B_\delta\setminus A_2$ we use a Taylor expansion, since $\alpha\in(0,1)$ we see that

$$\phi(a+z)+\phi(a-z)-2\phi(a)=\int_{-1}^{1}(1-|s|)\langle D^2\phi(a+sz)z,z\rangle\,ds.$$

In particular,

$$\langle D^2\phi(a+sz)\cdot z,z\rangle=\alpha L_1|a+sz|^{\alpha-2}\left(|z|^2-(2-\alpha)|\langle\widehat{a+sz},z\rangle|^2\right)$$
$$\leq \alpha L_1|a+sz|^{\alpha-2}|z|^2.$$

From (3.7), we can conclude that

$$\langle D^2\phi(a+sz)\cdot z,z\rangle\leq \alpha L_1|a|^{\alpha-2}(1-\rho_0)^{\alpha-2}|z|^2,$$

since the kernel K is bounded by constant $\Lambda>0$, the estimate of the integral over $B_\delta\setminus A_2$ is given by

$$\int_{B_\delta\setminus A_2}\frac{1}{|z|^{n+2s-2}}\,dz\leq \Lambda\sigma(S_{n-1})\int_0^\delta\frac{1}{r^{2s-1}}\,dr=\Lambda\frac{\sigma(S_{n-1})}{2(1-s)}\delta^{2-2s}.$$

Now, if we define

$$C_4:=\Lambda\frac{\sigma(S_{n-1})}{2(1-s)}\rho_0^{2-2s},$$

we obtain the following bound for ϕ,

$$\int_{B_\delta\setminus A_2}(\phi(a+z)+\phi(a-z)-2\phi(a))\,K(z)\,dz\leq \alpha(1-\rho_0)^{\alpha-2}L_1C_4|a|^{\alpha-2s}.$$

Finally, for the integral involving Γ, we can observe that

$$\int_{B_1\setminus A_2}(\Gamma(\overline{x}+z)-\Gamma(\overline{x}))\,K(z)\,dz\leq\int_{B_1}(\Gamma(\overline{x}+z)-\Gamma(\overline{x}))\,K(z)\,dz$$
$$\leq L_2\int_{B_1}|z|^2K(z)\,dz.$$

In a similar manner to the previous kernel estimation,

$$\int_{B_1} \frac{1}{|z|^{n+2s-2}}\, dz \leq \Lambda \frac{\sigma(S_{n-1})}{2(1-s)} =: C_4',$$

we can conclude that the integral of Γ over $B_1 \backslash A_2$ is bounded by

$$\int_{B_1 \backslash A_2} (\Gamma(\overline{x}+z) - \Gamma(\overline{x}))\, K(z)\, dz \leq L_2 C_4'.$$

In conclusion, regrouping all terms related to the nonlocal term T^3_{nl}, we must take into consideration the values that $s \in (0,1)$ can take. If $s < 1/2$,

$$T^3_{nl} \leq 2\alpha L_1 C_3 |a|^{\alpha-1} + \alpha(1-\rho_0)^{\alpha-2} L_1 C_4 |a|^{\alpha-2s} + L_2 C_4', \tag{3.9}$$

if $s = 1/2$,

$$T^3_{nl} \leq -2\alpha L_1 \ln(|a|\rho_0) C_3' |a|^{\alpha-1} + \alpha(1-\rho_0)^{\alpha-2} L_1 C_4 |a|^{\alpha-1} + L_2 C_4'.$$

and if $s > 1/2$

$$T^3_{nl} \leq 2\alpha L_1 C_3 |a|^{\alpha-2s} + \alpha(1-\rho_0)^{\alpha-2} L_1 C_4 |a|^{\alpha-2s} + L_2 C_4', \tag{3.10}$$

In the case when $s = 1/2$, we observe that $\lim_{r\to 0} \ln(|r|) r^{\alpha} \to 0$. Therefore, we also obtain powers of $|a|$ of the form $\alpha - 2s$. Thus, (3.10) remains valid in this case as well.

3.3 Estimates for local and nonlocal terms

We star our analysis considering $s \geq 1/2$. Let us see what happens when putting together the local terms (3.3) and the nonlocal terms (3.4), (3.8), (3.10), in both cases $m \in (0,1]$ and $m \in (1,2]$.

When $m \in (0,1]$, we obtain the following estimation,

$$\begin{aligned}
0 \leq & \mathcal{M}^+(u,\varphi,\overline{x},\delta) - \mathcal{M}^+(u,-\varphi,\overline{y},\delta) \\
& \quad + b(\overline{x})|D_x\varphi(\overline{x},\overline{y})|^m - b(\overline{y})|D_y\varphi(\overline{x},\overline{y})|^m - f(\overline{x}) + f(\overline{y}) \\
& \leq 2\|f\| + C_b|a|^\tau \left[(\alpha L_1|a|^{\alpha-1})^m + (2L_2|\overline{x}-x_0|)^m\right] \\
& \quad + \|b\|_\infty(2L_2|\overline{x}-x_0|)^m + C_1\|u\|_\infty + L_2C_2'|a|^{2-2s} + L_2C_4' + \epsilon \\
& \quad\quad - C_2C_2'\alpha L_1|a|^{\alpha-2s} + 2\alpha L_1C_3|a|^{\alpha-1} + \alpha(1-\rho_0)^{\alpha-2}L_1C_4|a|^{\alpha-2s} \\
& = \nu_1(L_2,\|b\|_\infty,f) - \Big(C_2C_2' - (1-\rho_0)^{\alpha-2}C_4 \\
& \qquad\qquad -C_b(\alpha L_1|a|^\alpha)^{m-1}|a|^{\tau+(2s-m)}\Big)\alpha L_1|a|^{\alpha-2s},
\end{aligned} \tag{3.11}$$

where the terms depending on L_2, along with the bounded and constant terms, have been grouped into ν_1.

We know that η, ρ_0 are small enough for $1-\eta-\rho_0 > 0$ and $(2-\alpha)(1-\eta-\rho_0)^2 - (1+\rho_0)^2 > 0$, where the last term is part of C_2. Note that the first-order term $(1-\rho_0)^{\alpha-2} \approx 1-(\alpha-2)\rho_0$, so for ρ_0 small enough, we conclude that $C_2 - (1-\rho_0)^{\alpha-2} > 0$.

After completing all these calculations, the goal is to prove the result for sufficiently small values of α and sufficiently large values of L_1. Therefore, we seek to have

$$(\alpha L_1|a|^\alpha)^{m-1}|a|^{\tau+(2s-m)} = o(1),$$

which holds if $\tau + 2s - m + \alpha(m-1) \geq 0$ and $m \in (0,1)$, that is,

$$\alpha \leq \frac{\tau - m + 2s}{1-m} \quad \text{if } m \in (0,1) \quad \text{and} \quad \tau + 2s - 1 \geq 0 \quad \text{if } m = 1.$$

We choose L_1 such that $L_1 > 2^{1+\alpha}d(x_0,\Omega)^{-\alpha}\|u\|_\infty$ and

$$L_1 \geq \frac{\nu_1(L_2,\|b\|_\infty,f)}{(C_2C_2' - (1-\rho_0)^{\alpha-2}C_4)\alpha}$$

Then, based on the condition $L_1|a|^\alpha \leq 2\|u\|_\infty$, for $\alpha \in (0,2s)$, we deduce that $|a|^{\alpha-2s} > 1$ since these terms tend to 0 as $\overline{x}$ approaches $\overline{y}$. It follows that (3.11)

$$0 \leq \nu(L_2,\|b\|_\infty,f)(1-|a|^{\alpha-2s}) < 0$$

This leads to a contradiction, which allows us to conclude that

$$\Phi(x,y) \leq 0,$$

proving the regularity of the solution for sufficiently small values of α.

Furthermore, in the case where $m=1$, if $s \geq 1/2$, then the conclusion holds for any $\tau \geq 0$, that is, b it can only be a continuous function. Now, it follows for $m \in (0,1)$ that

$$\alpha \leq \frac{\tau + 2s - 1}{1-m}.$$

When considering the case of $m \in (1,2]$, we obtain the following estimation

$$\begin{aligned}
&\mathcal{M}^+(u,\varphi,\overline{x},\delta) - \mathcal{M}^+(u,-\varphi,\overline{y},\delta) \\
&\quad + b(\overline{x})|D_x\varphi(\overline{x},\overline{y})|^m - b(\overline{y})|D_y\varphi(\overline{x},\overline{y})|^m - f(\overline{x}) + f(\overline{y}) \\
&\quad \leq -L_1|a|^{\alpha-2s}\Big[\alpha C_2 C_2' - 2\alpha C_3 \\
&\qquad\qquad - \alpha(1-\rho_0)^{\alpha-2}C_4 - \alpha^m 2^{m-1} C_b |a|^\tau (L_1|a|^\alpha)^{m-1}|a|^{2s-m} \\
&\qquad\qquad - 2^{2m-1}L_1^{-1}L_2^m C_b|\overline{x}-x_0|^m|a|^{\tau-\alpha+2s} \\
&\qquad\qquad - m\alpha^{m-1}2^m L_2|a|(L_1|a|^\alpha)^{m-2}|a|^{2s-m}|\overline{x}-x_0|\|b\|_\infty\Big] \\
&\quad + C_1 + L_2C_4' + L_2C_2'|a|^{2-\alpha} + m2^{2m-1}L_2^{-m-2}|\overline{x}-x_0|^m\|b\|_\infty + 2\|f\|_\infty + \epsilon.
\end{aligned} \tag{3.12}$$

In this case, similar to the previous scenario, we require that the terms involving the powers of $|a|$ be small. Therefore, we need that

$$|a|^\tau(L_1|a|^\alpha)^{(m-1)}|a|^{2s-m} = o(1) \quad \text{and} \quad |a|(L_1|a|^\alpha)^{m-2}|a|^{2s-m} = o(1).$$

Hence, we impose that $\tau+\alpha(m-1)+2s-m \geq 0$ and $1+\alpha(m-2)+2s-m \geq 0$, from where we obtain

$$\alpha \leq \frac{1-m+2s}{2-m} \quad \text{if } m \in (1,2) \quad \text{and} \quad \tau + \alpha + 2(s-1) \geq 0 \quad \text{if } m = 2.$$

Now, the terms depending on L_2, along with the bounded and constant terms have been grouped into ν_2 in (3.12),

$$0 \leq \nu_2(L_2, \|b\|_\infty, f) - (C_2C_2' - (1-\rho_0)^{\alpha-2}C_4)\alpha L_1|a|^{\alpha-2s}$$

Hence, we can choose $L_1 > 0$ large enough to control the rest of the terms,

$$L_1 \geq \frac{\nu_2(L_2, \|b\|_\infty, f)}{(C_2C_2' - (1-\rho_0)^{\alpha-2}C_4)\alpha}.$$

We observe that $\alpha \in (0, 2s)$, then $|a|^{\alpha-2s} > 1$, it follows that

$$\nu_2(L_2, \|b\|_\infty, f)(1 - |a|^{\alpha-2s}) < 0,$$

while the size of the left-hand side is always positive, thus we get the contradiction.

Furthermore, in the case when $m = 2$, from (3.12), we impose that $\tau - \alpha + 2s \geq 0$ and from the restriction $\tau + \alpha + 2(s-1) \geq 0$, we obtain that $\tau \geq 1 - 2s$, however we know that $1 - 2s \leq 0$, so the conclusion holds for any $\tau \geq 0$, that is, b it can be only a continuous function.

Now we see what happens when $s < 1/2$, using estimates for the local terms (3.3) and the non-local terms (3.4), (3.8), (3.9) for both cases $m \in (0, 1]$ and $m \in (1, 2]$.

In the case where $m \in (0, 1]$, using $s < 1/2$, we obtain

$$\begin{aligned}
&\mathcal{M}^+(u, \varphi, \overline{x}, \delta) - \mathcal{M}^+(u, -\varphi, \overline{y}, \delta) \\
&\quad + b(\overline{x})|D_x\varphi(\overline{x}, \overline{y})|^m - b(\overline{y})|D_y\varphi(\overline{x}, \overline{y})|^m - f(\overline{x}) + f(\overline{y}) \\
&\quad \leq -L_1|a|^{\alpha-2s}\Big[\alpha C_2C_2' - 2\alpha C_3|a|^{2s-1} \\
&\qquad\qquad - \alpha(1-\rho_0)^{\alpha-2}C_4 - C_b\alpha^m|a|^\tau(L_1|a|^\alpha)^{(m-1)}|a|^{2s-m} \\
&\qquad\qquad - C_b|a|^{\tau+2s-\alpha}(2L_2|\overline{x} - x_0|)^m\Big] \\
&\quad + C_1 + L_2C_2'|a|^{2-\alpha} + L_2C_4' + \|b\|_\infty(2L_2|\overline{x} - x_0|)^m + 2\|f\|_\infty + \epsilon.
\end{aligned} \tag{3.13}$$

In a similar manner to the previous, we need

$$|a|^{\tau}(L_1|a|^{\alpha})^{(m-1)}|a|^{2s-m} = o(1),$$

that is, we impose

$$\alpha \leq \frac{\tau - m + 2s}{1-m} \quad \text{if } m \in (0,1) \quad \text{and } s < \frac{1}{2},$$

while if $m = 1$, we obtain $\tau + 2s - 1 \geq 0$. In other words, if $s < 1/2$ and $m \in (0,1]$ then we must have b that is $C^{0,\tau}$ with $\tau \geq 1 - 2s$.

In the case where $m \in (1,2]$, using $s < 1/2$, we obtain

$$\begin{aligned}
&\mathcal{M}^+(u,\varphi,\overline{x},\delta) - \mathcal{M}^+(u,-\varphi,\overline{y},\delta) \\
&\quad + b(\overline{x})|D_x\varphi(\overline{x},\overline{y})|^m - b(\overline{y})|D_y\varphi(\overline{x},\overline{y})|^m - f(\overline{x}) + f(\overline{y}) \\
&\quad \leq -L_1|a|^{\alpha-2s}\Big[\alpha C_2 C_2' - 2\alpha C_3|a|^{2s-1} \\
&\qquad\qquad - \alpha(1-\rho_0)^{\alpha-2}C_4 - \alpha^m 2^{m-1}C_b|a|^{\tau}(L_1|a|^{\alpha})^{(m-1)}|a|^{2s-m} \\
&\qquad\qquad - 2^{2m-1}L_1^{-1}L_2^m C_b|\overline{x}-x_0|^m|a|^{\tau-\alpha+2s} \\
&\qquad\qquad - m\alpha^{m-1}2^m L_2|a|(L_1|a|^{\alpha})^{m-2}|a|^{2s-m}|\overline{x}-x_0|\|b\|_\infty\Big] \\
&\quad + C_1 + L_2C_2'|a|^{2-\alpha} + L_2C_4' + m2^{2m-1}L_2^{-m-2}|\overline{x}-x_0|^m\|b\|_\infty + 2\|f\|_\infty + \epsilon.
\end{aligned} \tag{3.14}$$

In a completely analogous manner to the previous, we need that

$$|a|^{\tau}(L_1|a|^{\alpha})^{(m-1)}|a|^{2s-m} = o(1) \quad \text{and} \quad |a|(L_1|a|^{\alpha})^{m-2}|a|^{2s-m} = o(1),$$

that is,

$$\alpha \leq \frac{1-m+2s}{2-m} \quad \text{if } m \in (1,2) \quad \text{and } \tau + \alpha + 2s - 2 \geq 0 \quad \text{if } m = 2.$$

Furthermore, in the case when $m = 2$, from (3.14), we impose the condition $\tau - \alpha + 2s \geq 0$ and from the restriction $\tau + \alpha + 2(s-1) \geq 0$, we obtain that $\tau \geq 1 - 2s$, however we know that $1 - 2s > 0$. Therefore, the conclusion holds for b in $C^{0,\tau}$ with $\tau \geq 1 - 2s$.

Remark 3.1. We mention that the case of a minimal Pucci operator of the form $\mathcal{M}^{-}u(x) = \inf_{L \in \mathcal{L}_0} Lu(x)$ will be analogous, as well as for Isaacs operators in the form

$$\mathcal{F}u(x) = \inf_a \sup_b L_{a,b}u(x) \quad , \quad \mathcal{F}u(x) = \sup_a \inf_b L_{a,b}u(x)$$

where $L_{a,b} = L$ for some $K = K_{a,b}$ for a family of indexes $a \in A$, $b \in B$, this type of operator is treated as in article [1].

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