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A note on broken waveguides

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Abstract. We study the spectrum of the Dirichlet Laplacian in a broken waveguide. It is found out information about its essential and discrete spectrum. In particular, it is shown that the number of discrete eigenvalues depends on a local deformation.

Keywords: Rectangular waveguides, Dirichlet Laplacian, Essential spectrum, Discrete spectrum.

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1 Introduction

Let Γ be a strip in the plane $\mathbb{R}^2$ or a waveguide in the space $\mathbb{R}^3$. Denote by $-\Delta_\Gamma^D$ the Dirichlet Laplacian in Γ. The spectrum of the operator $-\Delta_\Gamma^D$ has been extensively studied in the last years. In fact, the subject is non-trivial since the results depend on the geometry of Γ. For example, in the particular case where Γ is a straight strip, or a straight waveguide, it is known that the spectrum of $-\Delta_\Gamma^D$ is purely essential, i.e, there no exist discrete eigenvalues. However, several works show that deformations in a

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straight waveguide can create discrete eigenvalues for the operator. See, for example, [3, 4, 6, 8, 9, 10, 11, 12, 13, 7] for these discussions. In the next paragraphs we present the model of a "broken strip" in $\mathbb{R}^2$.

For each $\theta \in (0, \pi/2)$, consider the region

$$V_\theta := \left\{(x,y) : x\tan\theta < |y| < \left(x + \frac{\pi}{\sin\theta}\right)\tan\theta\right\};$$

V_θ is called broken strip. In the pioneer work [10] the authors investigated the case $\theta = \pi/4$ and proved that the operator $-\Delta^D_{V_\theta}$ has a unique discrete eigenvalue. In [1] it was found that $-\Delta^D_{V_\theta}$ has at least one discrete eigenvalue and more than one for any sufficiently small angle. In [6] the authors proved that, for each $\theta \in (0, \pi/2)$, $-\Delta^D_{V_\theta}$ has at least one discrete eigenvalue; the number of discrete eigenvalues is always finite; this number tends to infinity as θ approaches to zero. In [14] the authors proved the existence of a critical angle α^* so that, for all $\theta \in (\alpha^*, \pi/2)$, the total multiplicity of the discrete spectrum is one; an asymptotic lower bound for the multiplicity as θ approaches to zero was also found.

Inspired by the previous works, we study the following model. Fix $\beta \in (0, \infty)$, and consider the region

$$\Gamma_\beta := \left\{(s,t,z) : t \in (0, 2\pi), \left(|s| - \pi\frac{\sqrt{1+\beta^2}}{\beta}\right)\beta < z < \beta|s|\right\}. \quad (1.1)$$

We call Γ_β of a broken waveguide; see Figure 1.1. Note that, Γ_β is symmetric with respect to the tz-plane, it has a corner at the tz-plane, and, unless a neighborhood of this corner, it is the union of two half straight waveguides; which one with a constant cross section $S := (0, 2\pi) \times (0, \pi)$. Finally, the parameter β is related to the "opening" of the corner.

Denote by $-\Delta^D_{\Gamma_\beta}$ the Dirichlet Laplacian in Γ_β, i.e., the self-adjoint operator associated with the quadratic form

$$Q^D_{\Gamma_\beta}(\psi) = \int_{\Gamma_\beta} |\nabla\psi|^2 \mathrm{d}\mathbf{s}, \quad \mathrm{dom}\, Q^D_{\Gamma_\beta} = H^1_0(\Gamma_\beta); \quad (1.2)$$

$\mathbf{s} = (s,t,z)$ denotes a point of Γ_β, and $\nabla\psi$ denotes the gradient vector of ψ. The goal of this paper is to know how the geometry of Γ_β affects the

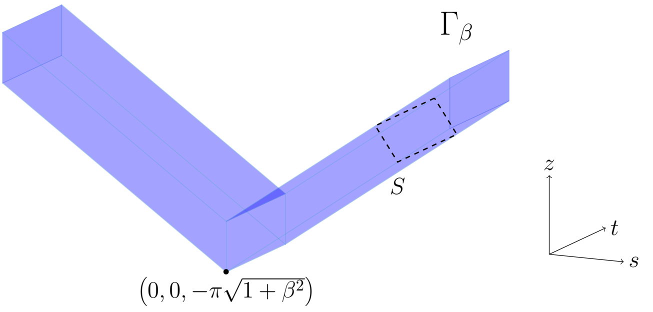


Figura 1.1: The broken waveguide Γ_β with $\beta = 0.7$.

spectrum of the operator $-\Delta^D_{\Gamma_\beta}$. In next paragraphs we present our main results.

We start studying the essential spectrum of $-\Delta^D_{\Gamma_\beta}$. At first, by a "separation of variables", it is simple to find out that the spectrum of the Dirichlet Laplacian in the straight waveguide $\mathbb{R} \times S$, denoted by $-\Delta^D_{\mathbb{R}\times S}$, is purely essential and it is the interval $[5/4, \infty)$; in particular, the value $5/4$ is the first eigenvalue of the Dirichlet Laplacian in S. Roughly speaking, since Γ_β and $\mathbb{R} \times S$ have the same geometry at the infinity, as a direct consequence of Proposition 3 in [15], one has the stability of the essential spectrum of both operators, $-\Delta^D_{\Gamma_\beta}$ and $-\Delta^D_{\mathbb{R}\times S}$. In other words,

Proposition 1.1. *For each $\beta \in (0, \infty)$, the essential spectrum of $-\Delta^D_{\Gamma_\beta}$ is the interval $[5/4, \infty)$.*

As discussed in [15], the essential spectrum is determined by the geometry of Γ_β at infinity only. Now, the problem is to know if the corner deformation in Γ_β can create discrete eigenvalues. We obtain

Proposition 1.2. *For each $\beta \in (0, \infty)$, the discrete spectrum of $-\Delta^D_{\Gamma_\beta}$ is non empty.*

The proof of this result is presented in Section 4. Roughly speaking, from the previous discussions and Proposition 1.2, we can say that a corner deformation in a straight waveguide creates discrete eigenvalues for the Dirichlet Laplacian operator.

Now, since the discrete spectrum of $-\Delta^D_{\Gamma_\beta}$ is non empty, the next question is to find out information about the finiteness or infinity of this set. In this case, we have

Proposition 1.3. *For each $\beta \in (0, \infty)$, the discrete spectrum of $-\Delta^D_{\Gamma_\beta}$ is finite.*

The proof of Proposition 1.3 follows the same arguments of Proposition 5.1 in [6]; it will be omitted in this text.

Now, an interesting question is to know if the parameter β influences the number of discrete eigenvalues. This is answered by the following results.

Proposition 1.4. *For each $\beta \in (0, \sqrt{6/5}]$, the operator $-\Delta^D_{\Gamma_\beta}$ has exactly one discrete eigenvalue.*

Furthermore,

Proposition 1.5. *The number of discrete eigenvalues of $-\Delta^D_{\Gamma_\beta}$ increases as $\beta \to \infty$.*

The proofs of these propositions are presented in Section 5. We pay attention that the assumption $\beta \in (0, \sqrt{6/5}]$ in Proposition 1.4 is only a sufficient condition to ensure the uniqueness of the discrete eigenvalue.

This paper is organized as follows. In Section 2 and 3 we discuss the strategies to study the spectrum of the operator $-\Delta^D_{\Gamma_\beta}$. The proofs of Propositions 1.2, 1.4 and 1.5 are presented in Sections 4 and 5, respectively.

2 Auxiliary problem

Define

$$\Gamma^+_\beta := \Gamma_\beta \cap \{\mathbf{s} = (s, t, z) \in \mathbb{R}^3 : s > 0\},$$

and $\partial_D \Gamma^+_\beta := \partial \Gamma_\beta \cap \partial \Gamma^+_\beta$. Consider the quadratic form

$$Q^{DN}_{\Gamma^+_\beta}(\psi) = \int_{\Gamma^+_\beta} |\nabla \psi|^2 \mathrm{d}\mathbf{s}, \qquad (2.1)$$

$$\operatorname{dom} Q^{DN}_{\Gamma^+_\beta} = \{\psi \in H^1(\Gamma^+_\beta) : \psi = 0 \text{ in } \partial_D \Gamma^+_\beta\}.$$

Denote by $-\Delta^{D}_{\Gamma^+_\beta}$ the self-adjoint operator associated with $Q^{DN}_{\Gamma^+_\beta}$. In particular,

$$\operatorname{dom}(-\Delta^{DN}_{\Gamma^+_\beta}) = \big\{\psi \in H^1(\Gamma^+_\beta) : \Delta\psi \in L^2(\Gamma^+_\beta), \\ \psi = 0 \text{ in } \partial_D\Gamma^+_\beta, \text{ and } \partial\psi/\partial x = 0 \text{ in } x = 0\big\}.$$

Since the region Γ_β is symmetric with respect to the tz-plane, the considerations of Chapter 3 in [2] imply that

$$\sigma_{ess}(-\Delta^D_{\Gamma_\beta}) = \sigma_{ess}(-\Delta^{DN}_{\Gamma^+_\beta}), \quad \sigma_{dis}(-\Delta^D_{\Gamma_\beta}) = \sigma(-\Delta^{DN}_{\Gamma^+_\beta}).$$

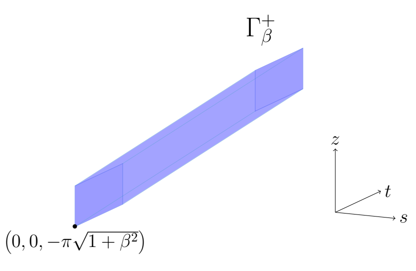
Then, from now on, the strategy is to study the discrete spectrum of the operator $-\Delta^{DN}_{\Gamma^+_\beta}$.

3 Change of coordinates

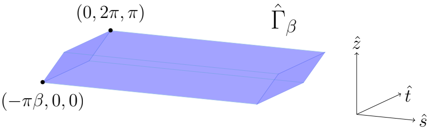
To study the operator $-\Delta^{DN}_{\Gamma^+_\beta}$, we are going to perform a change of coordinates in the following way. Consider the region $\hat{\Gamma}_\beta$ defined by

$$\hat{\Gamma}_\beta := \left\{(\hat{s}, \hat{t}, \hat{z}) \in (-\pi\beta, \infty) \times (0, 2\pi) \times (0, \pi) : \hat{z} < \frac{\hat{s}}{\beta} + \pi \text{ if } \hat{s} \in (-\pi\beta, 0)\right\}.$$

$\hat{\Gamma}_\beta$ is is isometrically affine to Γ^+_β; see Figure 3.1 (a).



(a) Waveguide Γ^+_β



(b) Waveguide $\hat{\Gamma}_\beta$

Figura 3.1: Isometric waveguides with $\beta = 0.7$.

Consider the quadratic form

$$Q^{DN}_{\hat{\Gamma}_\beta}(\psi) := \int_{\hat{\Gamma}_\beta} |\nabla\psi|^2 \mathrm{d}\hat{\mathbf{s}}, \tag{3.1}$$

$$\mathrm{dom}\, Q^{DN}_{\hat{\Gamma}_\beta} := \{\psi \in H^1(\hat{\Gamma}_\beta) : \psi = 0 \text{ in } \partial_D\hat{\Gamma}_\beta\};$$

$\hat{\mathbf{s}} = (\hat{s}, \hat{t}, \hat{z})$ denotes a point of $\hat{\Gamma}_\beta$, and $\partial_D\hat{\Gamma}_\beta := \{\mathbb{R}\times(0,2\pi)\times(0,\pi)\}\cap\partial\hat{\Gamma}_\beta$. Denote by $-\Delta^{DN}_{\hat{\Gamma}_\beta}$ the self-adjoint operator associated with $Q^{DN}_{\hat{\Gamma}_\beta}$.

Now, consider the dilation mapping $\mathcal{F}_\beta : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $\mathcal{F}_\beta(x, y_1, y_2) := (\beta x, y_1, y_2)$, and define $\hat{\Gamma} := \mathcal{F}_\beta^{-1}(\hat{\Gamma}_\beta)$, i.e.,

$$\hat{\Gamma} := \{(x, y_1, y_2) \in (-\pi, \infty) \times (0, 2\pi) \times (0, \pi) : y_2 < x + \pi \text{ if } x \in (-\pi, 0)\}.$$

Finally, consider the unitary operator

$$\begin{array}{rlcl} \hat{\mathcal{U}}_\beta : & L^2(\hat{\Gamma}_\beta) & \to & L^2(\hat{\Gamma}) \\ & \psi & \mapsto & \sqrt{\beta}(\psi \circ \mathcal{F}_\beta) \end{array}, \tag{3.2}$$

and the quadratic form

$$\hat{Q}_\beta(\psi) := Q^{DN}_{\hat{\Gamma}_\beta}(\hat{\mathcal{U}}_\beta^{-1}\psi) = \int_{\hat{\Gamma}} \left(\frac{1}{\beta^2}|\partial_x\psi|^2 + |\partial_{y_1}\psi|^2 + |\partial_{y_2}\psi|^2\right) \mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2$$

$$\mathrm{dom}\, \hat{Q}_\beta := \hat{\mathcal{U}}_\beta(\mathrm{dom}\, Q^{DN}_{\hat{\Gamma}_\beta}) = \{\psi \in H^1(\hat{\Gamma}) : \psi = 0 \text{ in } \partial_D\hat{\Gamma}\};$$

$\partial_D\hat{\Gamma} := \{\mathbb{R}\times(0,2\pi)\times(0,\pi)\}\cap\partial\hat{\Gamma}$. Denote by $\hat{H}_\beta$ the self-adjoint operator associated with $\hat{Q}_\beta$. Since the operators $-\Delta^{DN}_{\Gamma^+_\beta}$ and $\hat{H}_\beta$ are unitarily equivalent, one has $\sigma_{ess}(\hat{H}_\beta) = \sigma_{ess}\big(-\Delta^{DN}_{\Gamma^+_\beta}\big)$, and $\sigma_{dis}(\hat{H}_\beta) = \sigma_{dis}\big(-\Delta^{DN}_{\Gamma^+_\beta}\big)$.

4 Existence of discrete spectrum

Before proving Proposition 1.2 we fix some notations that will be used later. Let Q be a closed and lower bounded sesquilinear form with domain dom Q dense in a Hilbert space H. Denote by A the self-adjoint operator associated with Q. The Rayleigh quotients of A can de defined as

$$\lambda_j(A) = \inf_{\substack{G \subset \mathrm{dom}\ Q \\ \dim G = j}} \sup_{\substack{\psi \in G \\ \psi \neq 0}} \frac{Q(\psi)}{\|\psi\|_H^2}. \tag{4.1}$$

Let $\mu = \inf \sigma_{ess}(A)$. The sequence $\{\lambda_j(A)\}_{j\in\mathbb{N}}$ is non-decreasing and satisfies (i) If $\lambda_j(A) < \mu$, then it is a discrete eigenvalue of A; (ii) If $\lambda_j(A) \geq \mu$, then $\lambda_j(A) = \mu$; (iii) The j-th eigenvalue of A less than μ (it it exists) coincides with $\lambda_j(A)$.

Now, we present the prove of Proposition 1.2 which is inspired by [5].

Proof of Proposition 1.2. Consider the quadratic form

$$q_\beta(\psi) := \hat{Q}_\beta(\psi) - \frac{5}{4}\|\psi\|^2_{L^2(\hat{\Gamma})}, \quad \mathrm{dom}\, q_\beta := \mathrm{dom}\, \hat{Q}_\beta. \tag{4.2}$$

According to (4.1) and Proposition 1.1, it is enough to find a function $\psi \in \mathrm{dom}\, \hat{Q}_\beta \setminus \{0\}$ so that $q_\beta(\psi) < 0$.

The first step is to build a sequence $\{\psi_n\} \subset \mathrm{dom}\, q_\beta$ so that $q_\beta(\psi_n) \to 0$, as $n \to \infty$. For that, let $w \in C^\infty(\mathbb{R})$ be a real function so that $w = 1$, for $x \leq 1$, and $w = 0$, for $x \geq 2$. Define, for each $n \in \mathbb{N} \setminus \{0\}$,

$$w_n(x) = w\left(\frac{x}{n}\right), \quad \text{and} \quad \psi_n(x, y_1, y_2) = w_n(x)\chi(y_1, y_2),$$

where χ denotes the normalized eigenfunction correspondingly to the first eigenvalue (5/4) of the Dirichlet Laplacian in S; namely, $\chi(y_1, y_2) = (\sqrt{2}/\pi)\sin(y_1/2)\sin(y_2)$. In particular,

$$\int_0^\infty |w_n'|^2 \mathrm{d}x = \frac{1}{n}\int_0^\infty |w'|^2 \mathrm{d}x \to 0, \quad \text{as} \quad n \to \infty, \tag{4.3}$$

and

$$\int_S \left(|\partial_{y_1}\chi|^2 + |\partial_{y_2}\chi|^2\right) \mathrm{d}y_1 \mathrm{d}y_2 = \frac{5}{4}. \tag{4.4}$$

Due to (4.4), and since

$$\int_{-\pi}^0 \int_0^{2\pi} \int_0^{x+\pi} \left(|\nabla_y \chi|^2 - 5/4\,|\chi|^2\right) \mathrm{d}y_2 \mathrm{d}y_1 \mathrm{d}x = 0,$$

one has

$$
\begin{aligned}
q_\beta(\psi_n) &= \hat{Q}_\beta(\psi_n) - \frac{5}{4}\|\psi_n\|^2_{L^2(\hat\Gamma)} \\
&= \int_{\hat\Gamma} \left[\frac{1}{\beta^2}\left|w_n'\chi\right|^2 + |w_n|^2\left(|\nabla_y\chi|^2 - \frac{5}{4}|\chi|^2\right)\right]\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2 \\
&= \int_{-\pi}^0\int_0^{2\pi}\int_0^{x+\pi}\left(|\nabla_y\chi|^2 - \frac{5}{4}|\chi|^2\right)\mathrm{d}y_2\mathrm{d}y_1\mathrm{d}x + \frac{1}{\beta^2}\int_0^\infty \left|w_n'\right|^2\mathrm{d}x \\
&= \frac{1}{\beta^2}\int_0^\infty |w_n'|^2\mathrm{d}x.
\end{aligned}
$$

By (4.3), it follows that $q_\beta(\psi_n) \to 0$, as $n \to \infty$.

Now, fix $\varepsilon \in \mathbb{R}$. For each $n \in \mathbb{N}\setminus\{0\}$, define

$$
\psi_{n,\varepsilon}(x, y_1, y_2) := \psi_n(x, y_1, y_2) + \varepsilon\phi(x, y_1, y_2),
$$

for some $\phi \in \operatorname{dom} q_\beta$. In this case,

$$
q_\beta(\psi_{n,\varepsilon}) = q_\beta(\psi_n) + 2\varepsilon\,\mathrm{Re}\left(q_\beta(\psi_n, \phi)\right) + \varepsilon^2 q_\beta(\phi).
$$

The second step of the proof is to find ϕ so that

$$
\lim_{n\to\infty} q_\beta(\psi_n, \phi) \neq 0. \tag{4.5}
$$

In fact, if (4.5) holds true, it is enough to choose a convenient ε so that $q_\beta(\psi_{n,\varepsilon}) < 0$, for n large enough.

The construction of ϕ is as follows. Consider $\eta \in C_0^\infty(\mathbb{R})$, with $\operatorname{supp}\eta \subset (-\pi, 0)$, and $\eta(0) = 0$. Take $h(y)$, with $y \in S$, satisfying $h(y_1, 0) = 0$ (the definition of h will be detailed later). Define $\phi(x, y_1, y_2) := \eta(x)h(y_1, y_2)$. Then,

$$
\begin{aligned}
&q_\beta(\psi_n, \phi) = \\
&\quad= \int_{\hat\Gamma}\left[\frac{1}{\beta^2}(w_n'\chi)(\eta' h) + w_n\eta\left(\partial_{y_1}\chi\partial_{y_1}h + \partial_{y_2}\chi\partial_{y_2}h - \frac{5}{4}\chi h\right)\right]\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2 \\
&\quad= \int_{-\pi}^0\int_0^{2\pi}\int_0^{x+\pi}\eta\left(\partial_{y_1}\chi\partial_{y_1}h + \partial_{y_2}\chi\partial_{y_2}h - \frac{5}{4}\chi h\right)\mathrm{d}y_2\mathrm{d}y_1\mathrm{d}x \\
&\quad= \int_{-\pi}^0 \eta\Bigg(-\frac{\sqrt{2}}{\pi}\cos(x)\int_0^{2\pi}\sin\left(\frac{y_1}{2}\right)h(y_1, x+\pi)\mathrm{d}y_1 \\
&\quad- \frac{\sqrt{2}}{2\pi}\int_0^{x+\pi}\sin(y_2)\left(h(2\pi, y_2) + h(0, y_2)\right)\mathrm{d}y_2\Bigg)\mathrm{d}x;
\end{aligned}
$$

in the last equality was performed a parts integration and used the explicit form of χ.

Finally, take $h(y_1, y_2) := \eta(y_2 - \pi)\sin(y_1/2)\cos(y_2 - \pi)$; note that $h(y_1, 0) = 0$. Then,

$$\begin{aligned} q_\beta(\psi_n, \phi) &= -\frac{\sqrt{2}}{\pi}\int_{-\pi}^{0} \eta^2(x)\cos^2(x)\left(\int_0^{2\pi}\sin^2\left(\frac{y_1}{2}\right)\mathrm{d}y_1\right)\mathrm{d}x \\ &= -\sqrt{2}\int_{-\pi}^{0}\eta^2(x)\cos^2(x)\mathrm{d}y_x \neq 0. \end{aligned}$$

Consequently, (4.5) is true, and the result is proven. □

5 Number of discrete eigenvalues

In this section we present the poofs of Propositions 1.4 and 1.5; they are inspired by [14] and [6], respectively.

Proof of Proposition 1.4. Here, the strategy is to study the operator $\hat{H}_\beta$. At first, suppose $\lambda_1(\hat{H}_\beta)$ and $\lambda_2(\hat{H}_\beta)$ discrete eigenvalues of $\hat{H}_\beta$. To analyze these values, we define the following auxiliary problem. Let W be the region limited by the faces

$$\begin{aligned} T_1 &:= \{(x, y_1, y_2) \in (-\pi, 0) \times \{2\pi\} \times (0, \pi) : y_2 < x + \pi\}, \\ T_2 &:= \{(x, y_1, y_2) \in (-\pi, 0) \times \{0\} \times (0, \pi) : y_2 < x + \pi\}, \\ S_1 &:= \{(x, y_1, y_2) \in (-\pi, 0) \times (0, 2\pi) \times \{0\}\}, \\ S_2 &:= \{(x, y_1, y_2) \in \{0\} \times (0, 2\pi) \times (0, \pi)\}; \end{aligned}$$

see Figure 5.1. Consider the operator

$$A(\beta) := -\frac{1}{\beta^2}\partial_x^2 - \partial_{y_1}^2 - \partial_{y_2}^2, \tag{5.1}$$

where $\partial_x := \partial/\partial x$, $\partial_{y_1} := \partial/\partial y_1$, $\partial_{y_2} := \partial/\partial y_2$, and the problem

$$\begin{cases} A(\beta)\psi = \lambda\psi, & \text{in } W, \\ \psi = 0, & \text{in } T_1 \cup T_2 \cup S_1, \\ \partial_x\psi = 0, & \text{in } S_2, \\ \partial_\nu\psi = 0, & \text{in } \partial W \setminus (\overline{T_1} \cup \overline{T_2} \cup \overline{S_1} \cup \overline{S_2}). \end{cases} \tag{5.2}$$

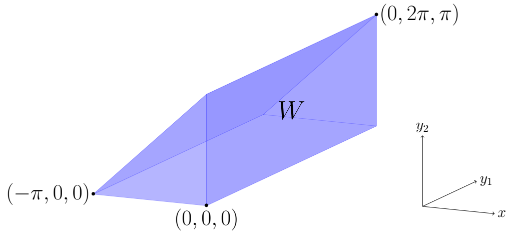


Figura 5.1: Region W.

$\partial\nu$ denotes the directional derivative along the exterior normal.

Denote by μ_1^β and μ_2^β the first two eigenvalues of the problem (5.2), and by ψ_1^β and ψ_2^β the correspondingly eigenfunctions; note that $A(\beta)$ has purely discrete spectrum. In particular, $A(1) = -\Delta$, and, in this case,

$$\mu_1^1 = \frac{3}{4}, \quad \psi_1^1(x,y_1,y_2) = \frac{2}{\pi\sqrt{\pi}}\cos\left(\frac{x}{2}\right)\sin\left(\frac{y_1}{2}\right)\sin\left(\frac{y_2}{2}\right),$$
$$\mu_2^1 = \frac{3}{2}, \quad \psi_2^1(x,y_1,y_2) = \frac{2}{\pi\sqrt{\pi}}\cos\left(\frac{x}{2}\right)\sin\left(y_1\right)\sin\left(\frac{y_2}{2}\right).$$

Suppose

$$\mu_2^\beta \geq \frac{5}{4}. \tag{5.3}$$

Consider the subspace

$$E_\perp^\beta = \left\{\varphi \in \operatorname{dom}\hat{Q}_\beta : \int_W \psi_1^\beta \varphi \, \mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2 = 0\right\}.$$

If $\varphi \in \operatorname{dom}\hat{Q}_\beta$, denote by ψ its restriction in W. By min-max Principle, one has

$$\begin{aligned}\frac{5}{4}\int_W |\psi|^2 \mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2 &\leq \mu_2^\beta \int_W |\psi|^2 \mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2 \\ &\leq \int_W \left(\frac{1}{\beta^2}\left|\partial_x\psi\right|^2 + \left|\partial_{y_1}\psi\right|^2 + \left|\partial_{y_2}\psi\right|^2\right)\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2.\end{aligned} \tag{5.4}$$

Furthermore, for each $\varphi \in \operatorname{dom}\hat{Q}_\beta$,

$$\begin{aligned}&\frac{5}{4}\int_{\hat{\Gamma}\backslash W} |\varphi|^2 \mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2 \\ &\leq \int_{\hat{\Gamma}\backslash W} \left(\frac{1}{\beta^2}\left|\partial_x\varphi\right|^2 + \left|\partial_{y_1}\varphi\right|^2 + \left|\partial_{y_2}\varphi\right|^2\right)\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2.\end{aligned} \tag{5.5}$$

By (5.4) and (5.5), we obtain

$$\frac{5}{4}\int_{\hat{\Gamma}} |\varphi|^2 \mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2 \leq \int_{\hat{\Gamma}} \left(\frac{1}{\beta^2}|\partial_x \varphi|^2 + |\partial_{y_1}\varphi|^2 + |\partial_{y_2}\varphi|^2\right)\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2,$$

for all $\varphi \in E_\perp^\beta$. Since $E_\perp^\beta$ has codimension 1, one has

$$\lambda_2(\hat{H}_\beta) \geq \inf_{\substack{\psi \in \mathrm{dom}\,\hat{Q}_\beta \cap E_\perp^\beta \\ \psi \neq 0}} \frac{\hat{Q}_\beta(\psi)}{\|\psi\|^2_{L^2(\hat{\Gamma})}} \geq \frac{5}{4},$$

which contradicts the hypothesis that $\lambda_2(\hat{H}_\beta)$ is a discrete eigenvalue.

Now, we are going to find values β so that (5.3) holds true. By min-max Principle, we have

$$\mu_2^\beta = \sup_{F_1}\left\{\inf_{\substack{\psi \in \mathcal{C}\cap F_1^\perp \\ \psi \neq 0}} \frac{\int_W \left(\frac{1}{\beta^2}|\partial_x\psi|^2 + |\partial_{y_1}\psi|^2 + |\partial_{y_2}\psi|^2\right)\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2}{\|\psi\|^2_{L^2(W)}}\right\},$$

where F_1 denotes any subspace of dimension at most 1. Then,

$$\begin{aligned}
\frac{3}{2} = \mu_2^1 &= \sup_{F_1}\left\{\inf_{\substack{\psi \in \mathcal{C}\cap F_1^\perp \\ \psi \neq 0}} \frac{\int_W \left(|\partial_x\psi|^2 + |\partial_{y_1}\psi|^2 + |\partial_{y_2}\psi|^2\right)\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2}{\|\psi\|^2_{L^2(W)}}\right\} \\
&\leq \max\{\beta^2, 1\}\sup_{F_1}\left\{\inf_{\substack{\psi \in \mathcal{C}\cap F_1^\perp \\ \psi \neq 0}} \frac{\int_W \left(\frac{1}{\beta^2}|\partial_x\psi|^2 + |\partial_{y_1}\psi|^2 + |\partial_{y_2}\psi|^2\right)\mathrm{d}x\mathrm{d}y_1\mathrm{d}y_2}{\|\psi\|^2_{L^2(W)}}\right\} \\
&= \max\{\beta^2, 1\}\mu_2^\beta.
\end{aligned}$$

Consequently,

$$\mu_2^\beta \geq \frac{3}{2}\min\{1/\beta^2, 1\}.$$

This inequality shows that if $\beta \in (0, \sqrt{6/5}]$, then (5.3) holds true. □

Proof of Proposition 1.5. The strategy in this proof is to work with the original region Γ_β defined in the Introduction by (1.1). Consider the parallelepiped $P_\beta \subset \Gamma_\beta$ restricted by $t = 0$, $t = 2\pi$, $z = -\alpha\pi$, $z = 0$, and

$s = \pm\tau\pi$, where α and τ satisfy

$$\alpha \in \left(0, \sqrt{1+\beta^2}\right) \quad \text{and} \quad \tau\pi = \left(-\alpha\frac{1}{\sqrt{1+\beta^2}} + 1\right)\pi\frac{\sqrt{1+\beta^2}}{\beta};$$

see Figure 5.2. By the monotonicity of the eigenvalues, one has

$$\lambda_j(-\Delta^D_{P_\beta}) \geq \lambda_j(-\Delta^D_{\Gamma_\beta}), \quad \forall\, j \geq 1. \tag{5.6}$$

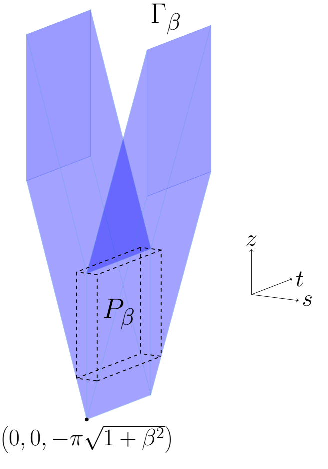


Figura 5.2: Parallelepiped $P_\beta \subset \Gamma_\beta$, with $\beta = 4$ and $\alpha = 3$.

The eigenvalues of the Dirichlet Laplacian in P_β can be explicitly calculated, and they are given by

$$\lambda_{l,m,n}(-\Delta^D_{P_\beta}) = \frac{1}{4}\left(\frac{l^2}{\tau^2} + m^2 + \frac{4n^2}{\alpha^2}\right), \quad l, m, n \in \mathbb{N}.$$

Since $\sigma_{ess}(-\Delta^D_{\Gamma_\beta}) = [5/4, \infty)$, the strategy is to find index $l, m, n \in \mathbb{N}$ so that $\lambda_{l,m,n}(-\Delta^D_{P_\beta}) < 5, 4$, i.e.,

$$\frac{l^2}{\tau^2} + m^2 + \frac{4n^2}{\alpha^2} < 5. \tag{5.7}$$

By choosing $\alpha > 1$, one has

$$\tau < \frac{\sqrt{1+\beta^2} - 1}{\beta} < 1, \quad \beta \in (0, \infty)\,.$$

Consequently, $l = 1$ (recall $\tau < 1$) and the values $m, n \in \mathbb{N}$ that satisfy (5.7) are necessarily of the form

$$\frac{1}{\tau^2} + m^2 + \frac{4n^2}{\alpha^2} = \frac{\beta^2}{(\sqrt{1+\beta^2} - \alpha)^2} + m^2 + \frac{4n^2}{\alpha^2}.$$

Now, fix $m \in \mathbb{N}$. The minimum of the value $\lambda_n(-\Delta_{P_\beta}^D)$ is obtained by analyzing the equation

$$\frac{d}{d\alpha}\left(\lambda_n(-\Delta_{P_\beta}^D)\right) = 0 \quad \Longleftrightarrow \quad \frac{2\beta^2}{(\sqrt{1+\beta^2} - \alpha)^3} - \frac{8n^2}{\alpha^3} = 0.$$

Some straightforward calculations show that α must be

$$\alpha = \frac{4^{1/3} n^{2/3} \beta^{1/3}}{\frac{\beta}{\sqrt{1+\beta^2}}}.$$

For β large enough, one has $\beta/\sqrt{1+\beta^2} \approx 1$. Then,

$$\alpha = 4^{1/3} n^{2/3} \beta^{1/3},$$

and

$$\lambda_n(-\Delta_{P_\beta}^D) = \frac{1}{4}\left[\frac{1}{(1 - 4^{1/3} n^{2/3} \beta^{-2/3})^2} + m^2 + 4^{1/3} n^{2/3} \beta^{-2/3}\right], \quad n \in \mathbb{N}.$$

Let $Z := 4^{1/3} n^{2/3} \beta^{-2/3}$. The last equation turns to be

$$\frac{1}{(1-Z)^2} + m^2 + Z = 5.$$

If $m = 1$, we obtain the equation

$$\frac{1}{(1-Z)^2} + Z = 4,$$

whose roots are $Z_1 \approx 0.46791$, $Z_2 \approx 1.6527$, and $Z_3 \approx 3.87938$. For the value Z_1, one has $\lambda_n(-\Delta_{P_\beta}^D) < 5/4$, and, consequently $\lambda_n(-\Delta_{\Gamma_\beta}^D) < 5/4$. In this case,

$$n \leq \left(0.46791 \cdot 4^{-1/3} \beta^{2/3}\right)^{3/2} \simeq 0.160034\beta.$$

Then, the maximum number $N \in \mathbb{N}$ so that $\lambda_N(-\Delta_{P_\beta}^D) < 5/4$ is bigger than 0.160034β. Finally, we can see that the number of eigenvalues $-\Delta_{\Gamma_\beta}^D$ strictly less than $5/4$ increases as $\beta \to \infty$. $\square$

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