Matemática
Contemporânea

Vol. 57, 210–235
http://doi.org/10.21711/231766362023/rmc577

Positive solutions to Yamabe-type equations by shooting methods

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Abstract. We will survey certain multiplicity results obtained for positive solutions of the Yamabe equation on closed manifolds. Many of these results were obtained using bifurcation theory. The strongest results appeared when the Yamabe equation can be reduced to an ordinary differential equation and global bifurcation theory can be applied. We will discuss how these results can be prove applying shooting methods, without using bifurcation theory.

Keywords: Yamabe equation; Shooting methods.

2020 Mathematics Subject Classification: 53C21, 58J55.

1 Introduction

Consider a closed Riemannian manifold (M, g) of dimension $n \geq 3$. It is a classical, important problem to understand the family of metrics $\hat{g}$, conformally equivalent to g, which have constant scalar curvature.

The first basic question is of course if there is at least one such conformal metric. Research on this problem started with the work of H. Yamabe in the classical article [30]. Yamabe considered the infimum of the (normalized) total scalar functional $\mathbf{S}$, restricted to the conformal class of g,

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which is the family of metrics conformally equivalent to g and is denoted by $[g]$. Namely, consider

$$Y(M,[g]) := \inf_{h\in[g]} \frac{\int_M s_h \; dvol_h}{(Vol(M,h))^{\frac{n-2}{n}}} = \inf_{h\in[g]} \mathbf{S}(h),$$

where $dvol_h$ denotes the volume element of h and s_h is the scalar curvature of h. $Y(M,[g])$ is now usually called the *Yamabe constant* of the conformal class $[g]$. By a direct computation one can see that the critical points of $\mathbf{S}$ restricted to $[g]$ are the metrics in $[g]$ which have constant scalar curvature. Yamabe then tried to show the existence of a metric of constant scalar curvature in $[g]$ by proving that the infimum $Y(M,[g])$ is realized. While the main ideas in the argument were correct, there was an important error in the proof. This was pointed out (and fixed under certain conditions) by N. Turdinger in [29]. Important progress was later obtained by T. Aubin [1]. Aubin showed in particular that $Y(M,[g]) \leq Y(\mathbb{S}^n,[g_0^n])$, where n is the dimension of M and g_0^n is the round metric on the sphere: this is now called Aubin's inequality. Aubin also showed that $Y(M,[g])$ is realized if the inequality is strict, and proved that this is actually the case in many situations. The problem was finally settled by R. Schoen [25], proving the strict inequality in the remaining cases and therefore giving the proof that in all conformal classes of metrics in any closed Riemannian manifold there exists at least one metric of constant scalar curvature.

The total scalar curvature functional $\mathbf{S}$ restricted to the conformal class of a metric g can be expressed in terms of g and the conformal factor. If we write the conformal metric $h\in[g]$ as $h = u^{4/(n-2)}g$, where $u \in \mathcal{C}^\infty(M)$, $u>0$, we obtain

$$\mathbf{S}(h) = Y_g(u) := \frac{\int_M \frac{4(n-1)}{n-2}|\nabla u|_g + s_g u^2 \; dvol_g}{\left(\int |u|^{p_n+1} \; dvol_g\right)^{2/(p_n+1)}},$$

where $p_n := \frac{n+2}{n-2}$ is the critical Sobolev exponent. Y_g is called the *Yamabe functional* and its Euler-Lagrange equation is the *Yamabe equation*:

$$-\frac{4(n-1)}{n-2}\Delta_g u + s_g u = \lambda u^{p_n}, \qquad u \in \mathcal{C}^\infty(M), \tag{1.1}$$

where Δ_g is the Laplace-Beltrami operator and $\lambda \in \mathbb{R}$. In fact, h has constant scalar curvature λ if and only if u is a positive solution to this equation.

It is easy to see that in case $Y(M,[g]) \leq 0$ then a constant scalar curvature metric that realizes the Yamabe constant is, up to scaling, the only metric of constant (non-positive) scalar curvature in $[g]$. But solutions in general are not unique in the case $Y(M,[g]) > 0$, and there has been many important results studying the space of positive solutions of the Yamabe equation in this case. The most important example in the theory is perhaps the conformal class of the metric of constant sectional on the sphere, which we will denote as before by $(\mathbb{S}^n, g_0^n)$. In this example there is a noncompact family of conformal diffeomorphisms, which give a noncompact family of positive solutions to the Yamabe equation. These are actually all the solutions to the Yamabe equation on $(\mathbb{S}^n, g_0^n)$. The corresponding conformal metrics are of course all isometric to g_0^n and in particular are minimizers for the Yamabe funcional. R. Schoen asked if $(\mathbb{S}^n, g_0^n)$ was actually the only closed Riemannian manifold with a non-compact set of positive solutions to the Yamabe equation [26, 27]. This question is very deep and rich. It turned out that there are other examples of Riemannian manifolds for which the set of positive solutions of the Yamabe equation is not compact, in dimension at least 25. See [6, 7, 8] and the references in these articles for more details on this problem.

There are also some particular situations when one can prove that solutions are not unique. For instance under the presence of symmetries, studying the equivariant problem as in the work by E. Hebey and M. Vaugon [15]. Also in the case of Riemannian products. If (M,g), (N,h) are closed Riemannian manifolds of constant scalar curvature and s_g is positive, then by a direct computation it is easy to check that $\lim_{\delta\to 0} \mathbf{S}(h+\delta g) = +\infty$. Therefore for δ small, the product metric cannot be a Yamabe minimizer since it does not verify Aubin's inequality. It then follows that there is at least one other solution, a Yamabe minimizer. An important particular example is given by the cylinders $(\mathbb{S}^1, dt^2) \times (\mathbb{S}^n, \delta g_0^n)$. One can study the Yamabe equation in these cases by looking at the universal Riemannian covering, which is $\mathbb{R} \times \mathbb{S}^n$. Since $\mathbb{R} \times \mathbb{S}^n$ is conformal to the punctured Euclidean $n+1$-space, $\mathbb{R}^{n+1} - \{0\}$, studying the Yamabe equation on $\mathbb{R} \times \mathbb{S}^n$ is equivalent to the study of solutions to the Yamabe equation on the Euclidean space with singularities at 0 and ∞. But it is known that all such solutions are radial by a result in [9]. This says that all solutions of the Yamabe equation on the products $(S^1, dt^2) \times (\mathbb{S}^n, \delta g_0^n)$ are actually functions which depend only on the circle, as was pointed out in [19, 28]. The problem is then reduced to an ordinary differential equation on the circle. R. Schoen in [28] gave a detailed study the solutions by considering the phase space of the corresponding autonomous system, in particular showing that the number of solutions goes to infinity as $\delta \to 0$.

For general Riemannian products one does not know that solutions depend only on one of the factors but many multiplicity results have been obtained by looking for solutions which do depend on only one of the factors. This is done mostly using bifurcation theory. We wil give a brief discussion on bifurcation theory and its applications to the Yamabe equation in the next section.

2 Multiplicity results for the Yamabe equation by bifurcation theory

Given Banach spaces X, Y, consider a map $F : X \times \mathbb{R} \to Y$ and the equation

$$F(x, \lambda) = 0.$$

We assume that we are in the situation when there is a canonical one parameter family of solutions, which depends on λ. To simplify the discussion, we assume now that this canonical family of solutions is given by $x = 0$, $F(0, \lambda) = 0$ for all λ. To study the problem one needs to assume certain regularity for the map F. Let us assume in this discussion that the map F is of class C^2, to avoid any technical issue. Fix a point $(0, \lambda_0)$

in the canonical family of solutions, which we will call the family of *trivial* solutions. Bifurcation theory deals with the problem of understanding the space of solutions close to $(0, \lambda_0)$. To do this consider the differential of F, $D_{(0,\lambda_0)}F$. Since the derivative in the direction of λ vanishes, $D_\lambda F(0, \lambda_0) = 0$, the information comes from $D_x F(0, \lambda_0)$. If $D_x F(0, \lambda_0)$ were an isomorphism then we can apply the Implicit Function Theorem for Banach spaces to prove that in a neighborhood of $(0, \lambda_0)$ all solutions are given by the canonical solutions in the neighborhood. In this situation the point $(0, \lambda_0)$ is called a *locally rigid* element of the family. In the opposite situation, when $(0, \lambda_0)$ is an accumulation point of non-trivial solutions, it is said that $(0, \lambda_0)$ is a *bifurcation point*. Then for $(0, \lambda_0)$ to be a bifurcation point it is necessary that $D_x F(0, \lambda_0)$ is not an isomorphism. But in general it is not a sufficient condition and it is often a very delicate issue to decide if a point where $D_x F f(0, \lambda_0)$ is not an isomorphism, is a bifurcation point or not.

Under certain general hypothesis, to see if such a point is indeed a bifurcation point, and to understand the space of solutions around such a point is determined by what is called the *bifurcation equation*: asume that $D_x F(0, \lambda_0)$ is a Fredholm operator, $X = V \oplus W$, where V is the kernel of $D_x F(0, \lambda_0)$ and it is finite dimensional and W is closed, and $Y = R \oplus Z$ where Z is finite dimensional and R is the range of $D_x F(0, \lambda_0)$ and it is closed. Let $P : Y \to Z$, $Q : Y \to R$ be the projections. The equation $F(x, \lambda) = 0$ is of course equivalent to the couple of equations

$$P(F(x, \lambda)) = 0, \qquad Q(F(x, \lambda)) = 0.$$

Applying the Implicit Function Theorem to $Q \circ F$ we obtain that in a neighborhood U of $(0, \lambda_0) \in V \oplus \mathbb{R}$ there exists a function $I : U \to W$, $I(0, \lambda_0) = 0$, such that locally $Q \circ F(v, w, \lambda) = 0$ if and only if $w = I(v, \lambda)$. Then locally the equation $F(v, w, \lambda) = 0$ is equivalent to the bifurcation equation:

$$B(v, \lambda) = P(F(v, I(v, \lambda), \lambda) = 0.$$

Then $(0, \lambda_0)$ is a bifurcation point if and only if the bifurcation equation has a sequence of nontrivial solutions converging to $(0, \lambda_0)$. Note that the bifurcation equation is a finite dimensional equation in a finite number of variables; this is usually called the Lyapunov-Schmidt reduction.

To apply bifurcation theory to the Yamabe equation one has to fix a family of constant scalar curvature metrics on a manifold, and consider the Yamabe equation on each metric in the family. The general setup was considered by L. L. de Lima, P. Piccione and M. Zedda in [13].

Consider a family g_λ of Riemannian metrics on the closed manifold M which have constant positive scalar curvature. The necessary condition to have bifurcation at g_{λ_0} is given in [13, Proposition 3.1]: If g_{λ_0} is a bifurcation point then $\frac{s_{g_{\lambda_0}}}{n-1}$ is an eigenvalue of $-\Delta_{g_{\lambda_0}}$.

In the previous situation, if g_{λ_0} satisfies that $\frac{s_{g_{\lambda_0}}}{n-1}$ is an eigenvalue of $-\Delta_{g_{\lambda_0}}$ we say that g_{λ_0} is a *degeneracy point* for the family. A sufficient condition to have bifurcation at a degeneracy point g_{λ_0} is obtained in [13, Theorem 3.3]: Let n_λ be the dimension of the space of eigenfunctions corresponding to eigenvalues strictly less than $\frac{s_{g_\lambda}}{n-1}$. Assume that $\varepsilon > 0$ is such that on $[\lambda_0 - \varepsilon, \lambda_0 + \varepsilon]$, g_{λ_0} is the only degeneracy point. If $n_{\lambda_0 - \varepsilon} \neq n_{\lambda_0 + \varepsilon}$ then g_{λ_0} is a bifurcation point.

It might happen that all points are degeneracy points and even if one has an isolated degeneracy point it might happen that there is no jump in n_λ as λ passes through the degeneracy point, so one could not decide if it is a bifurcation point using the previous result. But one sees that in general one can normalize the scalar curvatures to take the same constant value in the family and one could know which are the bifurcations points for the family if one knows the eigenvalues of the Laplacian for the metrics in the family. But, of course, to compute the Laplace spectrum is a very difficult problem in general.

One situation that can be worked out more explicitly is the case of the Riemannian products already mentioned in the introduction. If (M_1, g_1), (M_2, g_2) are Riemannian manifolds of constant scalar curvature there is a canonical family of metrics of constant scalar curvature: $\lambda > 0 \mapsto \mathbf{g}_\lambda = g_1 + \lambda g_2$. Note that $s_{\mathbf{g}_\lambda} = s_{g_1} + \lambda^{-1} s_{g_2}$ and the spectrum of $\mathbf{g}_\lambda$ is given by the combinations $\alpha_i + \lambda^{-1}\beta_j$, where α_i is an eigenvalue of $(M_1 g_1)$ and β_j is an eigenvalue of (M_2, g_2). Then one can understand the bifurcation problem for this family of metrics if one can compute the scalar curvature and Laplace spectrum of both factors, g_1 and g_2. Except for some degenerate situations, in the case when both s_{g_1} and s_{g_2} are positive, one can prove that the set of bifurcation points is discrete and consists of sequences of values of λ going to 0 and to ∞, see [13, Theorem 4.5].

A more general situation is to consider the total space of a Riemannian submersion. One obtains a canonical family of metrics on the total space by multiplying the metric along the fibers by a constant $\lambda > 0$. One needs to impose certain restrictions on the fibration to make sure that the scalar curvature is constant for the metrics in the family. But even in the most favorable cases the study of the spectrum of the Laplacian in the total space is much more complicated than in the case of Riemannian products. See the article by R. Bettiol and P. Piccione [4], for a discussion of this situation.

There is a way to simplify the previous discussion and which still captures many of the multiplicity results obtained. In the case of a Riemannian product $(M_1, g_1) \times (M_2, g_2)$ of closed Riemannian manifolds of constant positive scalar curvature we can consider (this was already mentioned in the introduction when discussing the case of the cylinders) the Yamabe equation of the product $\mathbf{g}_\lambda = g_1 + \lambda g_2$ restricted to functions on M_1. In the case of the canonical variation of a Riemannian submersion, we can do the same trick assuming that the submersion is harmonic [22], so that the Laplacians of the total space and of the base commute, as long as the scalar curvature of the total space stays constant. One therefore studies solutions of an equation of the form

$$-\Delta_{g_1} u + \lambda u = \lambda u^q. \tag{2.1}$$

In the equation λ is a positive constant, related to the scalar curvature of the total space, and $q = p_{n_1+n_2}$, where n_i is the dimension of M_i. Note that $p_{n_1+n_2} < p_{n_1}$.

An equation like (2.1) is called a *Yamabe type equation.* The equation is called critical if $q = p_n$, subcritical if $q < p_n$ and supercritical if $q > p_n$. We will discuss bifurcation for these equations in the next section. From the previous comments the results will imply multiplicity results for the Yamabe equation on certain Riemannian products or total spaces of Riemannian submersions.

3 Yamabe-type equations: reduction to an ODE and global bifurcation

We consider now the Yamabe-type equation (2.1) with $\lambda > 0$, $q > 1$, on a fixed closed Riemannian manifold (M^n, g). To apply bifurcation theory in this situation we consider the map $F : C^{2,\alpha}(M) \times (0, \infty) \to C^{0,\alpha}(M)$, given by $F(u, \lambda) = -\Delta u + \lambda u - \lambda u^q$: and study bifurcation, for the equation $f(u, \lambda) = 0$, from the trivial family of solutions $(1, \lambda)$.

Note that $D_u F(1, \lambda)[v] = -\Delta v + \lambda(1 - q)v$. Following the discussion in the previous section it is easy to see that the bifurcation points of the family are exactly the points $(1, \lambda_i)$, where $\lambda_i(q - 1)$ is an eigenvalue of $-\Delta$. Therefore we can understand which are the bifurcation points for the family as long as one knows the eigenvalues of the Laplacian. We will discuss now how we can obtain global bifurcation results in certain cases by reducing the equation to an ordinary differential equation.

The simplest case where one can reduce Equation (2.1) to an ordinary differential is on Riemannian manidols which admit an isometric cohomogeneity one action. If we restrict the equation to functions which are invariant by the action then we reduce the equation to an ordinary differential equation on the orbit space, with singularities in the singular orbits. More generally, we can consider functions which are constant along the level sets of an *isoparametric function.* If (M^n, g) is a Riemannian manifold, then a smooth function $f : M \to [c, d]$ is called an *isoparametric function* if there exist a continuous function a and a smooth function b

such that

$$|\nabla f|_g^2 = b \circ f, \qquad \text{and} \qquad \Delta_g f = a \circ f.$$

An isoparametric function is called *proper* if its level sets are connected. We will assume that the isoparametric functions we consider are proper, although it will be necessary only in a few situations. The level sets of an isoparametric function are called *isoparametric hypersurfaces.* An immediate consequence of the definition of an isoparametric function is the following reduction of a partial differential equation on (M, g) into an ordinary differential equation on the closed interval $[c, d] \subset \mathbb{R}$.

Proposition 3.1. *Let (M, g) be a closed Riemannian manifold and $f : M \to [c, d]$ be an isoparametric function with $c < d$ and $|\nabla f|_g^2 = b \circ f$ and $\Delta_g f = a \circ f$. Then, for any function $\varphi : \mathbb{R} \to \mathbb{R}$, $\phi : [c, d] \to \mathbb{R}$ is a solution to the problem*

$$-b\phi'' - a\phi' = \varphi(\phi) \ \textit{in} \ [c, d], \tag{3.1}$$

if and only if $u = \phi \circ f$ is a solution to the problem

$$-\Delta_g u = \varphi(u) \quad \textit{on} \ \ M.$$

The proof of this Proposition follows directly from the identity $\Delta_g(v \circ f) = (v'' \circ f)|\nabla f|_g^2 + (v' \circ f)\Delta_g f$.

These ideas can be applied to general Riemannian manifolds, as it was done for instance in [3], but we will restrict now to the case of the round sphere $(\mathbb{S}^n, g_0^n)$. We will make this restriction to simplify the discussion, and also because the round sphere admits a very rich family of isoparametric functions.

E. Cartan [10] proved that, in the case of a space form (M, g), a hypersurface $S \subset M$ is isoparametric (according to the previous definition) if and only if it has constant principal curvatures. In particular, the regular orbits of a cohomogeneity one action are examples of isoparametric hypersurfaces: they are called homogeneous.

We will give now a brief discussion of the theory of isoparametric hypersurfaces in the sphere $(\mathbb{S}^n, g_0^n)$. See [11] for more details on this and on isoparametric hypersurfaces in general. If we denote by ℓ the number of distinct principal curvatures of a fixed isoparametric hypesurface in the sphere $(\mathbb{S}^n, g_0^n)$, we have that that $\ell = 1, 2, 3, 4$ or 6. If ℓ is odd, all the multiplicities of the principal curvatures are the same, while if ℓ is even, there are, at most, two different multiplicities m_1 and m_2 (see the articles by H. F. Münzner, [20, 21]).

It is easy to see that if f is a proper isoparametric function and φ is a monotone function defined on the range of f then $\varphi \circ f$ is also a proper isoparametric function, with the same level sets. In the case of the sphere there is a way to pick certain particular isoparametric function within this family. After composition as before, all proper isoparametric functions on the sphere are the restrictions of Cartan-Münzner polynomials on $\mathbb{R}^{n+1}$ ([20, 21]). A Cartan-Münzner polynomial is a polynomial F in $\mathbb{R}^{n+1}$ of degree d which satisfies the Cartan-Münzner equations:

$$\langle \nabla F, \nabla F \rangle = d^2 \|x\|^{2d-2}$$

$$\Delta F = \frac{1}{2} c d^2 \|x\|^{d-2},$$

where c is a constant, which we will describe below. Then $f = F_{|_{S^n}}$ is an isoparametric function on the sphere: it verifies

$$\langle \nabla f, \nabla f \rangle = d^2 (1 - f^2)$$

$$\Delta f = -d(n + d - 1) f + \frac{1}{2} c d^2.$$

It is easy to see that $f : S^n \to [-1, 1]$ and its only critical values are -1 and 1. For $t \in (-1, 1)$ $f^{-1}(t)$ is called an *isoparametric hypersurface* of degree d, and $d = \ell$, the number of distinct principal curvatures. In case d is odd we have that $c = 0$. If $d = 2, 4$ or 6 then half of principal curvatures have multiplicity m_1 and the other half have multiplicity m_2, and the constant c is $m_2 - m_1$.

We will denote an isoparametric function on the sphere by $f^{d,c}$ (but note that there are different Cartan-Münzner polynomials with the same values of d and c).

Note that if $\varphi \in C^2[-1,1]$ then $\Delta(\varphi \circ f^{d,c}) = d^2(1-(f^{d,c})^2)\ (\varphi'' \circ f^{d,c})\ + (-d(n+d-1)f^{d,c} + (1/2)cd^2)\ (\varphi' \circ f^{d,c})$. And therefore, as in Proposition 3.1, $\varphi \circ f^{d,c}$ satisfies Equation (2.1) if and only if φ satisfies

$$d^2(1-t^2)\varphi''(t)+(-d(n+d-1)t+(1/2)cd^2)\varphi'(t)-\lambda(\varphi(t)-\varphi(t)^q) = 0. \quad (3.2)$$

Given an isoparametric function f, let $S_f = \{\varphi \circ f : \varphi : [-1,1] \to \mathbb{R}\}$. Let $C_f^{k,\alpha} = C^{k,\alpha} \cap S_f$. We will look for solutions of Equation (2.1) in S_f applying bifurcation theory. We will sometines denote a function $\varphi \circ f \in S_f$ simply by φ: we hope that this will not cause any confusion.

Then consider $F : C_f^{2,\alpha} \times (0,\infty) \to C_f^{0,\alpha}$, $F(\varphi \circ f, \lambda) =$

$$\left[\ d^2(1-t^2)\varphi''(t) + (-d(n+d-1)t + (1/2)cd^2)\varphi'(t) - \lambda(\varphi(t) - \varphi(t)^q)\ \right] \circ f.$$

Note that $D_\varphi F(1,\lambda)[v] = d^2(1-t^2)v''(t)+(-d(n+d-1)t+(1/2)cd^2)v'(t) - \lambda(1-q)v$. To find the possible bifurcation points we have to solve the equation

$$d^2(1-t^2)v''(t) + (-d(n+d-1)t + (1/2)cd^2)v'(t) - \lambda(1-q)v(t) = 0, \quad (3.3)$$

where $t \in [-1,1]$. A solution must verify the initial condition

$$(-d(n+d-1)(-1) + (1/2)cd^2)v'(-1) - \lambda(1-q)v(-1) = 0.$$

The space of solutions of the initial value problem verifying this condition has dimension 1. One solves the initial value problem with any pair of initial values $(v(-1), v'(-1))$ satisfying this condition, and checks if the solution is defined on all of $[-1,1]$, verifying

$$(-d(n+d-1)(1) + (1/2)cd^2)v'(1) - \lambda(1-q)v(1) = 0.$$

This linear problem is actually well known. A nontrivial solution of the equation exists if and only if $\lambda(q-1) = dk(n+dk-1)$ for some nonnegative integer k. The corresponding equation for $\lambda = \lambda_k = \frac{dk(n+dk-1)}{q-1}$ is the classical Jacobi equation and a solution is a polynomial of degree k called a Jacobi polynomial, which we will denote by P_k (see [2, Sections 1,2] for the details on Jacobi polynomials used in this article). It is easy to see from the discussion in the previous two sections that all these λ_k's give bifurcation points for the Equation (3.2). But more explicit information can be given in this situation. It is easy to check that $D_\varphi F(1,\lambda) : C^{2,\alpha} \to C^{0,\alpha}$ is self adjoint with respect to a weighted L^2 inner product on $[-1,1]$, see [2, Lemma 3.1]. In particular both the kernel and cokernel are 1-dimensional and the bifurcation equation from Section 2 is given by a real valued function B of two variables. The point $(1,\lambda_k)$ is a critical point of the function B. As $F(1,\lambda) = 0$ we have that $\partial^2 B/\partial\lambda^2 = 0$. And a straightforward computation shows that $\partial^2 B/\partial\lambda\partial v \neq 0$ (see the proof in [2, Theorem 3.2]. It follows that $(1,\lambda_k)$ is a nondegenerate critical point of the bifurcation function B, of index one. Then we can apply the Morse Lemma to show that up to diffeomorphism the space of solution around $(1,\lambda_k)$ looks like the space of solutions of $x^2 - y^2 = 0$ near $(0,0) \in \mathbb{R}^2$. Therefore the space of solutions in a neighborhood consists of two curves. One is the curve $(1,\lambda)$ of trivial solutions and there is another curve, of non-trivial solutions. This is essentially the content of the classical result of Crandall and Rabinowitz on the bifurcation from simple eigenvalues, see [12]. But in this case one can also say something about the global structure of the space of solutions. The first important point is that the number of critical points of a nontrivial solution of Equation (3.2) is locally constant (in the space of solutions). It is a classical result that the Jacobi polynomial of degree k, P_k, has k simple roots in the interval $(-1,1)$. This of course implies that P_k has exactly $k-1$ critical points in (-1,1). Since the path of solutions near the bifurcation point has the form $s \mapsto 1 + sP_k + o(s^2)$ (see [12]) it follows that the nontrivial solutions near the k-th bifurcation point have exactly $k-1$ critical points in $(-1,1)$. Let C_k be the connected component (in the space of nontrivial solutions) of a path of nontrivial solutions appearing at $(1,\lambda_k)$ (note that there are actually two paths of nontrivial solutions appearing at $(1, \lambda_k)$, which together with $(1, \lambda_k)$ form single path). From the previous comments all the solutions in C_k have exactly $k-1$ critical points in $(-1,1)$, and therefore in particular $C_k \cap C_j = \emptyset$ if $j \neq k$. This implies by the global bifurcation theorem of Rabinowitz, [24], that the connected component C_k is not compact. It was proved in [5] that there exists $\lambda_0 > 0$ such that if $\lambda \in (0, \lambda_0]$ the only positive solution of Equation (2.1) is the constant solution. It is also known that for fixed $0 < \lambda_1^+ < \lambda_2^*$ the space of solutions of Equation (2.1) with $\lambda \in [\lambda_1^*, \lambda_2^*]$ is compact (see for instance [17, Lemma 2.4]).

If there is $\lambda^* > \lambda_k$ such that for any solution $(u, \lambda) \in C_k$ we have that $\lambda \neq \lambda^*$ then C_k would be contained in $[\lambda_0, \lambda^*]$, and therefore would be compact by the result mentioned above. Since this would give a contradiction we have that for any $\lambda > \lambda_k$ there is a solution $(u, \lambda) \in C_k$.

We point out that for a solution φ of Equation (3.2) the space of critical points of the corresponding solution $\varphi \circ f$ of Equation (2.1) is given by the critical points of f and the preimages by f of the critical points of φ. So in particular for the solutions of Equation (3.2) in C_k the set of critical points of the corresponding solution of Equation (2.1) has $k+1$ connected components.

We have sketched the main ideas in the proof of [16, Theorem 1.1]:

Theorem A: Let $f^{d,c}$ be an isoparametric function on $\mathbb{S}^n$. For each positive integer k there exists at least k positive nonconstant solutions of Equation (2.1) for $\lambda \in (\lambda_k, \lambda_{k+1}]$. For each integer i, $1 \leq i \leq k$, $\lambda \in (\lambda_k, \lambda_{k+1}]$, there is a positive solution u_i of Equation (2.1) for which the set of critical points has exactly $i+1$ connected components.

In [23] a partial result in the direction of Theorem A was proved using only techniques from ordinary differential equations. In [14] a double shooting method was used to find multiplicity results for nodal solutions of the Yamabe equation on the sphere. In the next two sections we will describe this method and show that it can be used to give a new proof of Theorem A, without using bifurcation theory.

4 Double shooting

Given an isoparametric function $f = f^{d,c}$ on the sphere $\mathbb{S}^n$ to prove Theorem A we have to find solutions to the ordinary differential Equation (3.2). We will first give a different normalization of the equation. In Equation (3.2) we let $t = \cos(s)$, $s \in [0, \pi]$ and call $w(s) = \varphi(\cos(s))$. Note that the critical points of w are $0, \pi$ and the values of $s \in (0, \pi)$ such that $\cos(s)$ is a critical point of φ. By a direct computation we see that Equation (3.2) is equivalent to

$$w''(s) + \left(\frac{n-1}{d}\cos(s) - \frac{c}{2}\right)\frac{w'(s)}{\sin(s)} + \frac{\lambda}{d^2}(w^q - w) = 0. \tag{4.1}$$

As before, we let m_1, m_2, be the multiplicities of the principal curvatures of the regular level sets of f. We recall that we have that $d(m_1 + m_2) = 2(n-1)$ and $c = m_2 - m_1$. Then we write the previous equation as:

$$w'' + \frac{h(s)}{\sin s} w' + \frac{\lambda}{d^2}[w^q - w] = 0 \text{ on } [0, \pi], \tag{4.2}$$

where $h(s) = \frac{m_1+m_2}{2}\cos s - \frac{m_2-m_1}{2}$.

Observe this equation becomes singular at $s = 0$ and $s = \pi$, and that the natural boundary conditions in order to obtain a smooth solution on $\mathbb{S}^n$ are $w'(0) = w'(\pi) = 0$. Also notice that the function h satisfies $h(0) = m_1$, $h(\pi) = -m_2$, it is strictly decreasing, it has a unique zero $a_0 \in (0, \pi)$ and $h(s) > 0$ in $[0, a_0)$, while $h(s) < 0$ in $(a_0, \pi]$. Moreover, the function $\widetilde{h}(s) := -h(\pi - s) = \frac{m_1+m_2}{2}\cos s + \frac{m_2-m_1}{2}$ has the same properties with m_1 and m_2 interchanged and a unique zero at $\pi - a_0$.

For any initial value $x \in [0, \infty)$ denote by w_x the solution of Equation (4.2) with initial conditions $w_x'(0) = 0$, $w_x(0) = x$.

Define the energy function

$$E(s, x) := \frac{(w_x'(s))^2}{2} + \frac{\lambda}{d^2}\left(\frac{w_x^{q+1}}{q+1} - \frac{w_x^2}{2}\right).$$

The function E is nonincreasing on the first variable in the interval

$[0, a_0]$ and nondecreasing in $[a_0, \pi]$, since

$$E'(s,x) = -\frac{h(s)}{\sin s}(w_x'(s))^2.$$

In particular we have that $E(s,x) \leq E(0,x)$ for all $s \in [0, a_0]$. This implies that w_x and w_x' are uniformly bounded on $[0, a_0]$ which implies that w_x is defined on $[0, a_0]$. A similar argument can be made to show that if initial conditions are given on π then the corresponding solution is defined on $[a_0, \pi]$.

To prove Theorem A we will consider the solutions w_x, $\widetilde{w}_y$ of Equation (4.2) with initial conditions $w_x'(0) = \widetilde{w}_y'(\pi) = 0$, $w_x(0) = x$, $\widetilde{w}_y(\pi) = y$ and consider the maps $I(x) = (w_x(a_0), w_x'(a_0))$ and $J(y) = (\widetilde{w}_y(a_0), \widetilde{w}_y'(a_0))$. $I, J : \mathbb{R} \to \mathbb{R}^2$ and if $I(x) = J(y)$ then $w_x = \widetilde{w}_y$ is a solution of Equation (4.2) with $w_x'(0) = w_x'(\pi) = 0$. To understand the intersections of the curves I, J one needs to obtain information of the functions w_x, $\widetilde{w}_y$.

Therefore we will consider the initial value problem

$$\begin{cases} w_i''(s) + \frac{h(s)}{\sin s} w_i'(s) + \frac{\lambda}{d^2}(w_i^q - w_i) = 0 \quad \text{in } [0, a_0], \\ w_i(0) = x, w_i'(0) = 0, \end{cases} \tag{4.3}$$

and the "final" value problem

$$\begin{cases} w_f''(s) + \frac{h(s)}{\sin s} w_f'(s) + \frac{\lambda}{d^2}(w_f^q - w_f) = 0 \quad \text{in } [a_0, \pi], \\ w_f(\pi) = y, w_f'(\pi) = 0, \end{cases} \tag{4.4}$$

As we mentioned before, Problem (4.4) can actually be written as an initial condition problem having the form of Problem (4.3), by considering the function $\widetilde{h}(s) = -h(\pi - s) = \frac{m_1+m_2}{2}\cos s + \frac{m_2-m_1}{2}$. Then w_f solves Problem (4.4) if and only if $\omega(s) = w_f(\pi - s)$ solves the initial value problem

$$\begin{cases} \omega''(s) + \frac{\widetilde{h}(s)}{\sin s}\omega'(s) + \frac{\lambda}{d^2}(\omega^q - \omega) = 0 \quad \text{in } [0, \pi - a_0], \\ \omega(0) = y, \omega'(0) = 0, \end{cases} \tag{4.5}$$

So we will discuss Problem (4.3), which will also give the corresponding information for Problem (4.4).

We first point out the following:

Lemma 4.1. *If* $0 < x \leq 1$, *then* $w_x(s) > 0$ *for all* $s \in [0, a_0]$

Proof. Let $x \in (0, 1]$ and suppose, to get a contradiction, that w_x has a zero s_0 in $(0, a_0]$. Notice that $w_x'(s_0) \neq 0$, otherwise $w_x \equiv 0$ by uniqueness of the solution. Observe $E(0) = \frac{\lambda}{d^2}(\frac{x^{q+1}}{q+1} - \frac{x^2}{2}) < 0$, since $0 < x \leq 1$ and $q > 1$. As the energy $E(s, x)$ is non increasing on the first variable, we have that $0 > E(0, x) \geq E(s_0, x) = \frac{(w_x'(s_0))^2}{2} > 0$, which is a contradiction. □

Remark 4.2. *Note in general that if* $x > 0$ *and* $w_x(s_1) = 0$ *then* $E(s_1, x) > 0$. *Also if* $w_x'(s_2) = 0$ *and* $w_x(s_2) \in (0, 1)$, *then* $E(s_2, x) < 0$. *So, for instance, it cannot happen that* $s_2 < s_1 \leq a_0$.

Next we prove:

Lemma 4.3. *There exists* $D_0 > 1$ *such that for all* $x \in (0, D_0)$ *the solution* w_x *is strictly positive in* $[0, a_0]$. $u_{D_0}(a_0) = 0$, u_{D_0} *is strictly positive and decreasing in* $[0, a_0)$.

Proof. Let $P = \{x \in (1, \infty) : w_x$ is strictly positive on $[0, a_0]\}$. Since w_1 is constant equal to 1 it follows that there exists $\varepsilon > 0$ such that $(1, 1 + \varepsilon) \subset P$. In particular $P \neq \emptyset$. It is also clear that P is open. It can be seen that for x large enough w_x will have zeroes in $(0, a_0)$ (see for instance [14, Theorem 3.1]), so $D_0 = \sup P < \infty$. Since P is open we have that $D_0 \notin P$. But since D_0 is in the closure of P we have that $w_{D_0} \geq 0$ on $[0, a_0]$. If $s \in (0, a_0)$ and $w_{D_0}(s) = 0$ then s would be a local minimum, and therefore $w_{D_0}'(s) = 0$. By uniqueness of solutions this would imply that w_{D_0} is constant equal to 0, which is a contradiction. Therefore we must have that $w_{D_0}(a_0) = 0$. Note that if w_{D_0} had a local minimum at $s_0 \in (0, a_0)$ then we would have that $w_{D_0}(s_0) \in (0, 1)$ and we would get a contradiction from Remark (4.2). It follows that w_{D_0} is strictly decreasing in $[0, a_0]$. □

Similarly we have

Lemma 4.4. *There exists* $\widetilde{D_0} > 1$ *such that for all* $y \in (0, \widetilde{D_0})$ *the solution* $\widetilde{w}_y$ *is strictly positive in* $[a_0, \pi]$. $\widetilde{w}_{\widetilde{D_0}}(a_0) = 0$, $\widetilde{w}_{\widetilde{D_0}}$ *is strictly positive and increasing in* $[0, a_0]$.

We will restrict the curves I, J to $[0, D_0]$ and $[0, \widetilde{D_0}]$, respectively. If for $x \in (0, D_0)$, $y \in (0, \widetilde{D_0})$ we have that $I(x) = J(y)$ then the corresponding solution of Equation (4.2) is positive. We distinguish four cases:

$\mathbf{C_1}$: $x, y \in (0, 1)$. In this case 0, and π are local minima.

$\mathbf{C_2}$: $x, y > 1$. In this case 0, and π are local maxima.

$\mathbf{C_3}$: $x < 1$, $y > 1$. In this case 0 is a local minimum and π is a local maximum.

$\mathbf{C_4}$: $x > 1$, $y < 1$. In this case 0 is a local maximum and π is a local minimum.

If $a_0 = \pi/2$ and h is antisymmetric around a_0, then one can see that the solutions in cases $\mathbf{C_3}$ and $\mathbf{C_4}$ are equivalent; they are the reflection around $\pi/2$ of each other. Note also that the solutions in cases $\mathbf{C_1}$ and $\mathbf{C_2}$ have an odd number of critical points, and the solutions in cases $\mathbf{C_3}$ and $\mathbf{C_4}$ have an even number of critical points. Solutions of case $\mathbf{C_1}$ come form the intersections of I and J restricted to $(0, 1)$. Case $\mathbf{C_2}$ comes from the restriction of I and J to $(1, A_0)$ and $(1, \widetilde{A_0})$ respectively. Case $\mathbf{C_3}$ comes form restricting I to $(0, 1)$ and J to $(1, \widetilde{A_0})$ and Case $\mathbf{C_4}$ comes from the restriction of I to $(1, A_0)$ and J to $(0, 1)$.

Theorem A follows from proving: Fix a positive integer k. For any positive integer $i \leq k$ and $\lambda > \lambda_k$ there exists:

(1): If i is odd, a positive solution of Equation (4.2) having exactly $i - 1$ critical points in $(0, \pi)$ coming from $\mathbf{C_3}$ and a positive solution of Equation (4.2) having exactly $i - 1$ critical points in $(0, \pi)$ coming from $\mathbf{C_4}$.

(2): If i is even, a positive solution of Equation (4.2) having exactly $i - 1$ critical points in $(0, \pi)$ coming from $\mathbf{C_1}$ and a positive solution of Equation (4.2) having exactly $i - 1$ critical points in $(0, \pi)$ coming from $\mathbf{C_2}$.

Note that actually we get twice the number of solutions stated in Theorem A. As mentioned before, in case (1) the two solutions obtained might be equivalent, but the solutions in (2) are always different.

In the next section we will give all the details to prove the case $\mathbf{C_1}$. Namely;

Theorem B: Fix a positive integer k. For any positive even integer $i \leq k$ and $\lambda > \lambda_k$ there exist $x, y \in (0,1)$ such that $I(x) = J(y)$ and the corresponding solution of Equation (4.2) has exactly $i+1$ critical points.

All the other cases are proved in a very similar way. We restrict to the case $\mathbf{C_1}$ to make the proof as short and readable as possible.

5 Proof of Theorem B

In this section we consider Equation (4.2) and prove Theorem B. We keep the notation from the previous section.

Let a_0 be the unique zero of h in $(0,\pi)$. For $x \in [0,1]$ let w_x be the solution of Equation (4.2) with initial conditions $w_x(0) = d$, $w_x'(0) = 0$. Note that w_0 and w_1 are constant functions. Note also that if $x \neq 0, 1$ and s is a critical point of w_x then $w_x(s) \neq 0, 1$; moreover s is a local minimum if and only if $w_x(s) \in (0,1)$ and a local maximum if and only if $w_x(s) \in (1,\infty)$. Recall that we defined the path $I : [0,1] \to \mathbb{R}^2$ by $I(x) = (w_x(a_0), w_x'(a_0))$.

Note that $I(1) = (1,0)$ and $I(x) \neq (1,0)$ if $x \neq 1$. It is then easy to see that we have a well defined continuous function $\theta : [0,1) \to \mathbb{R}$ such that $\theta(0) = -\pi$, and and $\theta(x)$ gives an angle between $I(x)$ and the half-line $[1,\infty)$. Also notice that $(w_x)'(a_0) = 0$ if and only if $\theta(x) = k\pi$ for some integer k. If k is odd then $w_x(a_0) < 1$ and a_0 is a local minimum for w_x. If k is even then $w_x(a_0) > 1$ and a_0 is a local maximum of w_x.

For $x \neq 0, 1$ define $n(x)$ as the number critical of w_x in $(0, a_0)$. We will see that $\theta(x)$ determines $n(x)$. To prove this we start with the following observation:

Lemma 5.1. *Suppose $\theta(x_*) = k\pi$ for some $x_* > 0$ and some integer k, and that $n(x_*) = m \geq 0$. Then, given $0 < \varepsilon < \pi$ there exists $\delta > 0$ such that if $|x - x_*| < \delta$, then $\theta(x) \in (k\pi - \varepsilon, k\pi + \varepsilon)$ and*

1. $\theta(x) < k\pi$ *iff* $n(x) = m + 1$,

2. $\theta(x) \geq k\pi$ *iff* $n(x) = m$.

Proof. First choose $\delta_1 > 0$ such that if $|x - x_*| < \delta_1$, then $\theta(x) \in (k\pi - \varepsilon, k\pi + \varepsilon)$.

If k is even, then $w_{x_*}(a_0) > 1$. For $\epsilon_1 > 0$ small enough, $w_{x_*}(t) > 1$ for all $s \in (a_0 - \epsilon_1, a_0 + \epsilon_1)$. Then there exists a positive $\delta_2 < \delta_1$ such that if $|x - x_*| < \delta_2$, then $w_x(s) > 1$ for all $s \in (a_0 - \epsilon_1, a_0 + \epsilon_1)$. We can also assume that $w_x{}'$ has exactly one zero in $(a_0 - \epsilon_1, a_0 + \epsilon_1)$, which is a local maximum; $w_x{}' > 0$ before the critical point and $w_x{}' < 0$ after the critical point. There exists also $\delta < \delta_2$ such that if $x \in (x_* - \delta, x_* + \delta)$, then w_x has exactly m critical points in $[0, a_0 - \epsilon_1]$. For $x \in (x_* - \delta, x_* + \delta)$ we have that if $\theta(x) \geq k\pi$ then $w_x{}'(a_0) \geq 0$. This implies that the critical point of w_x in $(a_0 - \epsilon_1, a_0 + \epsilon_1)$ is $\geq a_0$. Therefore $n(x) = m$. If instead $\theta(x) < k\pi$ then $w_x{}'(a_0) < 0$. This implies that the critical point of w_x in $(a_0 - \epsilon_1, a_0 + \epsilon_1)$ is $< a_0$. Therefore $n(x) = m + 1$.

The argument in the case k is odd is similar. □

As usual for $x \in \mathbb{R}$ let $[x]$ be the maximum integer $\leq x$. Then for $x > 0$ define

$$\overline{n}(x) = -\left[\frac{\theta(x)}{\pi}\right] - 2.$$

Proposition 5.2. *For* $x \in (0, 1)$ *we have that* $\theta(x) < -\pi$ *and* $n(x) = \overline{n}(x)$.

Proof. Let $A = \{x \in (0, 1) : \theta(x) < -\pi,\ n(x) = \overline{n}(x)\}$. There is $x_0 \in (0, 1)$ such that for all $x \in (0, x_0]$ the solution w_x is strictly increasing in $[0, a_0]$, and therefore $\theta(x) \in (-\pi, -2\pi)$. It follows that $n(x) = \overline{n}(x) = 0$ and $(0, x_0] \subset A$. If there is $x_1 \in (0, 1)$ such that $\theta(x_1) = -\pi$ and $\theta(x) < -\pi$ for all $x \in (0, x_1)$ if follows from Lemma 5.1 that for x close to x_1, w_x is strictly increasing and less than 1 in $[0, a_0]$. Therefore w_{x_1} would be nondecreasing in $[0, a_0]$ and have a critical point with value at most 1 in a_0, which is crearly not possible. Therefore $\theta(x) < -\pi$ for all $x \in (0, 1)$. For any negative integer k both $n(x)$ and $\overline{n}(x)$ are constant on an interval where $\theta(x) \in ((k+1)\pi, k\pi)$. And Lemma 5.1 says that both $n(x)$ and $\overline{n}(x)$

change in the same way as $\theta(x)$ passes through $k\pi$. Therefore $A = (0,1)$ and the proposition is proved. □

An alternative way to define $\theta(x)$ is the following: for $x \in (0,1)$ consider the continuous function $\varphi : [0, a_0] \to \mathbb{R}$ such that $\varphi(0) = -\pi$ and $\varphi(t)$ gives an angle between $(w_x(t), w_x{}'(t))$ and the half line $[1, \infty)$. Let $\phi(x) = \varphi(a_0)$. Then both θ and ϕ are continuous functions which at each x coincide up to an integer multiple of 2π. Moreover, it is easy to see that $\phi(x) = \theta(x)$ for x close to 0. It follows that $\theta = \phi$. The last proposition also follows from this description.

We want to obtain information of w_x for x close to 1. More precisely, we need to understand $\lim_{x\to 1} \theta(x)$. For this we consider $v_\lambda(t) = \frac{\partial w}{\partial x}(s,1)$. Note that v_λ satisfies the linearized equation

$$v''(s) + \frac{h(s)}{\sin s} v'(s) + \lambda(q-1)v(s) = 0 \tag{5.1}$$

with initial conditions $v_\lambda(0) = 1$, $v'_\lambda(0) = 0$. To match the previous normalization we consider $-v_\lambda$ and define a continuous function $\varphi_\lambda : [0, a_0] \to \mathbb{R}$ such that $\varphi_\lambda(0) = -\pi$ and for each $s \in [0, \pi)$ $\varphi_\lambda(s)$ is an angle between $(-v_\lambda(s), -v'_\lambda(s))$ and the positive real axis $[0, \infty)$. Let $\theta_\lambda = \varphi_\lambda(a_0)$.

Remark 5.3. *Since* $w_{1-\delta}(t) = 1 + \delta(-v_\lambda)(t) + O(\delta^2)$ *in* $[0, a_0]$ *it is easy to see that* $\lim_{d\to 1} \theta(d) = \theta_\lambda$

Note that Equation (5.1) is the renormalization of Equation (3.3). Let $\lambda_k = \frac{dk(n+dk-1)}{q-1}$ be the eigenvalues of the problem. v_{λ_k} be the corresponding eigenfunction with $v_{\lambda_k}(0) = 1$. Note that v_{λ_k} is a multiple of $P_k(\cos(s)$, where P_k is the Jacobi polynomial, as in Section 3. Then for each $k \geq 1$ v_{λ_k} has exactly k zeroes in $(0, \pi)$. This implies that v_{λ_k} has exactly $k-1$ critical points in $(0, \pi)$. It might happen that a_0 is one of the critical points. The other ones fall in either $(0, a_0)$ or in (a_0, π).

We will need the following application of Sturm comparison theory:

Lemma 5.4. *If* $\lambda > \lambda_k$ *then* $\theta_\lambda < \theta_{\lambda_k}$.

Proof. Consider the functions $\varphi_\lambda(t)$ and $\varphi_{\lambda_k}(t)$. We have to prove that $\varphi_\lambda(a_0) < \varphi_{\lambda_k}(a_0)$. We have that $\varphi_\lambda(0) = \varphi_{\lambda_k}(0) = -\pi$. Both functions are strictly decreasing. For a positive integer j, $\varphi_\lambda(t_j) = -\pi/2 - j\pi$ if and only if $t_j = t_j(\lambda)$ is the j-th zero of v_λ. The most classical Sturm Comparison Theorem says that $t_j(\lambda_k) > t_j(\lambda)$. If for some $t \in (0, a_0]$ we have that the number of zeroes of v_λ in $(0, t]$ is strictly greater than the number of zeroes of v_{λ_k} in $(0, t)$ then $\varphi_\lambda(t) < \varphi_{\lambda_k}(t)$. Note that this happens in particular if $v_{\lambda_k}(t) = 0$. Therfore if $v_{\lambda_k}(a_0) = 0$ then the lemma is proved. If $v_{\lambda_k}(a_0) \neq 0$ we can also assume that $v_\lambda(a_0) \neq 0$ and v_{λ_k} and v_{λ_k} have the same number of zeroes in $(0, a_0)$. In this situation we can apply the Second Comparison Theorem of Sturm theory ([18, 10.41]) which says that

$$\frac{{v_{\lambda_k}}'(a_0)}{v_{\lambda_k}(a_0)} > \frac{{v_\lambda}'(a_0)}{v_\lambda(a_0)}.$$

Note that we are assuming that $v_{\lambda_k}(a_0)$ and $v_\lambda(a_0)$ have the same sign. If both are positive, as the vector $\left(1, \frac{{v_{\lambda_k}}'(a_0)}{v_{\lambda_k}(a_0)}\right)$ is above the vector $\left(1, \frac{{v_\lambda}'(a_0)}{v_\lambda(a_0)}\right)$, we have that $\theta_\lambda < \theta_{\lambda_k}$. If both are negative, as the vector $\left(1, \frac{{v_{\lambda_k}}'(a_0)}{v_{\lambda_k}(a_0)}\right)$ is above the vector $\left(1, \frac{{v_\lambda}'(a_0)}{v_\lambda(a_0)}\right)$, the direction of $(v_{\lambda_k}(a_0), {v_{\lambda_k}}'(a_0))$ is under the direction of the vector $(v_\lambda(a_0), {v_\lambda}'(a_0))$. This again implies that $\theta_\lambda < \theta_{\lambda_k}$. □

Now we proceed to consider the second curve, corresponding to the solutions of Equation (4.2) with condition $w'(\pi) = 0$. Let $\widetilde{h}(s) = - - h(\pi - s) = \frac{m_1+m_2}{2}\cos s + \frac{m_2-m_1}{2}$ and consider the initial conditions Problem (4.5). As it was mentioned in Section 4, ω is a solution to Problem (4.5) if and only if $\widetilde{w}(s) := \omega(\pi - s)$ solves the "final" conditions Problem (4.4). Then we can apply the same ideas to this problem.

For $y \in \mathbb{R}$, we denote by $\widetilde{w}_y$ the solution to the Problem 4.4 and define the map $J(y) := (\widetilde{w}_y(a_0), \widetilde{w}'_y(a_0))$.

We have that $J(1) = (1, 0)$, $J(0) = (0, 0)$, $J(y) \neq (1, 0)$ if $y \neq 1$. So, there is a well define argument function ϑ such that $\vartheta(0) = -\pi$ and $\vartheta(y)$ gives and angle between $J(y)$ and the half line $[1, \infty)$.

Then it follows from the same ideas used to study the curve I that $\vartheta(y) > -\pi$ for every $y \in (0,1)$ and that if $N(y)$ denotes the number of critical points of $\widetilde{w}_y$ in (a_0, π), then

$$N(y) = \left[\frac{\vartheta(y)}{\pi}\right] + 1.$$

Now consider the solution $\widetilde{v}$ of Equation (5.1) with boundary conditions $\widetilde{v}_\lambda(\pi) = -1$, $\widetilde{v}'_\lambda(\pi) = 0$. Define a continuous function $\widetilde{\varphi}_\lambda : [a_0, \pi] \to \mathbb{R}$ such that $\widetilde{\varphi}_\lambda(\pi) = -\pi$ and for each $s \in [a_0, \pi]$, $\widetilde{\varphi}_\lambda(s)$ is an angle between $(\widetilde{v}_\lambda(s), \widetilde{v}'_\lambda(s))$ and the positive real axis $[0, \infty)$. Let $\vartheta_\lambda = \widetilde{\varphi}_\lambda(a_0)$.

Then as in Lemma 5.4, we have:

Lemma 5.5. *If $\lambda > \lambda_k$ then $\vartheta_\lambda > \vartheta_{\lambda_k}$.*

Next, we define curves in the argument-radius plane $R, S : [0,1) \to \mathbb{R} \times \mathbb{R}_{>0}$ given by

$$R(x) := (\theta(x), |I(x) - (1,0)|) \qquad \text{and} \qquad S(y) := (\vartheta(y), |J(y) - (1,0)|)$$

We have:

Lemma 5.6. *The curves R and S are simple and they intersect only at the point $(-\pi, 1)$.*

Proof. The fact that R and S are simple follows immediately from the uniqueness of the solutions to the Problems (4.3) and (4.4). For an $x, y \in (0,1)$ we have seen that $\theta(x) < -\pi$ and $\vartheta(y) > -\pi$. Therefore, $R \cap S = \{R(0) = S(0)\} = \{(-\pi, 1)\}$. □

We have now everything we need to prove Theorem B:

Proof of Theorem B. Fix an integer $k \geq 1$ and $\lambda > \lambda_k$. Consider the curves R and S corresponding to λ. R and S together for a simple curve α_λ which lies in the upper half plane and goes from $(\theta_\lambda, 0)$ to $(\vartheta_\lambda, 0)$. Let i be an even integer, $i \leq k$. Note that v_{λ_k} has k zeroes in $(0, \pi)$

and critical points in 0 and π. Therefore it has exactly $k+1$ critical in $[0,\pi]$ and therefore $\varphi_{\lambda_k}(\pi) = -(k+1)\pi$. But $\varphi_{\lambda_k}(\pi) = \theta_{\lambda_k} - \vartheta_{\lambda_k} - \pi$. Since $\lambda > \lambda_k$ we have from Lemma 5.4 and Lemma 5.5 that $\vartheta_\lambda - \theta_\lambda > \vartheta_{\lambda_k} - \theta_{\lambda_k} = k\pi$. And since $i \le k$ the curve $\alpha_\lambda - i(\pi,0)$ which goes from $(\theta_\lambda - i\pi, 0)$ to $(\vartheta_\lambda - i\pi, 0)$ must intersect φ_λ (since $\vartheta_\lambda - i\pi > \theta_\lambda$). Note that R cannot intersect $R - i(\pi,0)$: If $R(x_1) = R(x_2) - (i\pi, 0)$ then $I(x_1) = I(x_2)$ which would imply that $x_1 = x_2$, reaching a contradiction. Similarly, S cannot intersect $S - i(\pi,0)$. And since $\theta(x) < -\pi$ and $\vartheta(y) > -\pi$ for all $x, y \in (0,1)$ it follows that $R - i(\pi,0)$ cannot intersect S. It follows that R must intersect $S - (i\pi,0)$. Let $x, y \in (0,1)$ be points such that $R(x) = S(y) - (i\pi,0)$. Then $I(x) = J(y)$. Therefore $w_x = \widetilde{w}_y$ is a well-defined solution of Equation (4.2), on $[0,\pi]$. Note that since $R(x) = S(y) - (i\pi,0)$, we have that $\theta(x) = \vartheta(y) - i\pi$. Note also that the function φ_x is defined in the whole intervale $[0,\pi]$, and we have that $\varphi_x(\pi) = \varphi_x(a_0) - (\widetilde{\varphi}_y(a_0) + \pi) = \theta(x) - \vartheta(y) - \pi = -(i+1)\pi$. This implies that w_x has exactly $i+1$ critical points in $[0,\pi]$. $\square$

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