


Symplectic structures on low dimensional 2-step nilmanifolds

Gabriela P. Ovando¹ and Mauro Subils ¹

¹Departamento de Matemática, ECEN - FCEIA, Universidad Nacional de Rosario. Pellegrini 250, 2000 Rosario, Santa Fe, Argentina.

Abstract. The aim of this work is the study of symplectic structures on 2-step nilmanifolds. We concentrate in the closedness condition, proving that the existence of a closed 2-form of type II is necessary to get a symplectic structure. In low dimensions, this condition is sufficient in most cases.

Keywords: Symplectic structures, 2-step nilmanifolds, closed 2-forms.

2020 Mathematics Subject Classification: 53D05, 57T15, 22E25.

1 Introduction

A 2-step nilmanifold is a smooth manifold $M = \Lambda \backslash N$, where N is a simply connected 2-step nilpotent Lie group and Λ is a co-compact discrete subgroup of N . After a result of Nomizu [20] any symplectic structure on the nilmanifold M is cohomologous to an invariant one on N . In this work we search for symplectic structures, by concentrating the attention on the closedness condition of the corresponding 2-form. Closed 2-forms on $M =$

Both authors were partially supported by ANPCyT and SCyT (UNR).

e-mail: gabriela@fceia.unr.edu.ar

e-mail: subils@fceia.unr.edu.ar

$\Gamma \backslash N$ are called *magnetic fields*, and they are deeply related to magnetic trajectories (see for instance [23] and references herein). In fact, magnetic trajectories are curves on a Riemannian manifold $(M, \langle \cdot, \cdot \rangle)$, solutions of the equation

$$\nabla_{\gamma'} \gamma' = F \gamma', \quad (1.1)$$

where ∇ is the Levi-Civita connection on M and F is a $(1,1)$ - tensor giving rise to a closed 2-form $\omega = \langle F \cdot, \cdot \rangle$, which could be degenerate. This is the focus of our research.

Symplectic 2-forms on homogeneous spaces were studied by different authors (see for instance [5, 4, 10, 12, 21]), and specifically on nilpotent Lie groups, see results for instance in [1, 6, 8, 14]. Also a construction by a double extension procedure was given in [5].

Since the problem of determining whether an arbitrary Lie algebra admits symplectic structures is difficult in general, it was attacked by using different strategies. In a case-by-case study, after the classification of nilpotent Lie algebras, one gets results in dimension six. See [14] for some results in dimension eight. Several subfamilies of nilpotent symplectic Lie algebras have been described. For example, symplectic filiform algebras in [2, 11] or free nilpotent Lie algebras in [7].

Pouseele and Tirao showed that for a nilpotent Lie algebra associated to a graph, the condition

$$2 \dim C(\mathfrak{n}) \leq \dim \mathfrak{n}$$

implies the existence of symplectic structures [24], where $C(\mathfrak{n})$ denotes the commutator of \mathfrak{n} . Moreover, the graph encodes geometrical information. Previously, Dotti and Tirao in [8] proved that if the 2-step nilpotent Lie group N admits a symplectic structure, then

$$2 \dim C(\mathfrak{n}) \leq \dim \mathfrak{n} + 1.$$

As said above, the focus here is concentrated on closed 2-forms. By studying their existence, it was proved in [22] that if \mathfrak{n} is non-singular (see

Definition 2.4 below) and

$$\dim \mathfrak{n} > 3 \dim C(\mathfrak{n})$$

then any closed 2-form ω on \mathfrak{n} satisfies

$$(*) \quad \omega(Z, U) = 0, \quad \text{for all } Z \in \mathfrak{z}, U \in \mathfrak{n}.$$

Notice that in the non-singular case, the commutator $C(\mathfrak{n})$ coincides with the center of the Lie algebra \mathfrak{z} . Condition $(*)$ is equivalent to saying that for any metric, the corresponding skew-symmetric map induced by ω preserves the decomposition $(**)$, see below.

Condition $(*)$ above is the key of our results. In terms of forms, it says that there is no 2-form ω on \mathfrak{n} such that $\omega(\mathfrak{z}, \mathfrak{z}) = 0$ but $\omega(\mathfrak{z}, \mathfrak{n}) \neq 0$. This is what we call a 2-form of type II. Now, asking for symplectic structures, we get:

Lemma. *If a 2-step nilpotent Lie group admits a left-invariant symplectic structure, then it admits a closed 2-form of type II.*

This observation is the starting point to study the family of closed 2-forms of type II, which is done with the help of a metric. Recall that on a 2-step nilpotent Lie group N equipped with a left-invariant metric, one makes use of a natural decomposition of the corresponding Lie algebra

$$(**) \quad \mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}, \quad \text{where } \mathfrak{v} = \mathfrak{z}^\perp.$$

Now, given a left-invariant 2-form ω on N , one can find a skew-symmetric map $F : \mathfrak{n} \rightarrow \mathfrak{n}$, such that $\omega = \langle F \cdot, \cdot \rangle$. We study the existence of symplectic structures, in terms of conditions on the map F . Any such linear map F decomposes as $F = F_1 + F_2$, where F_1 preserves the decomposition above, while F_2 makes $F(\mathfrak{z}) \subseteq \mathfrak{v}$ and $F(\mathfrak{v}) \subseteq \mathfrak{z}$. This corresponds to 2-forms of type I or II respectively.

The closedness condition of ω gives two closedness conditions for the 2-forms of type I and II. Explicitly, given the constant structures of the Lie

algebra C_{ij}^k where v_1, \dots, v_n is an orthonormal basis of \mathfrak{v} , the closedness condition for type II reduces to

$$\sum_{s=1}^m (C_{ij}^s b_{sk} + C_{jk}^s b_{si} + C_{ki}^s b_{sj}) = 0,$$

where (b_{sj}) gives a $m \times n$ matrix and $m = \dim \mathfrak{z}$. One searches for non-trivial solutions.

Looking for the symplectic structures on low-dimensional Lie algebras, we prove

Theorem. *Let N denote a nilpotent Lie group of dimension $2n$, $n \leq 3$. With exception of the trivial extension of the Heisenberg Lie algebra of dimension five, any such Lie group admits a symplectic structure if and only if it admits a closed 2-form of type II.*

Finally we show examples in higher dimensions, where the result above is no longer true. This is provided by free 2-step nilpotent Lie algebras in n generators with $n \geq 4$. In fact, these Lie algebras admit non-trivial closed 2-forms of type II but no symplectic structures.

2 Lie groups of step two with a left-invariant metric

In this section we recall basic facts on 2-step nilpotent Lie groups equipped with a left-invariant metric. Firstly recall that a Lie group is called 2-step nilpotent if its Lie algebra \mathfrak{n} is 2-step nilpotent, that is, \mathfrak{n} is not abelian and the Lie bracket satisfies $[[U, V], W] = 0$ for all $U, V, W \in \mathfrak{n}$. Throughout this paper Lie groups, so as their Lie algebras are considered over \mathbb{R} .

Example 2.1. The smallest dimensional (non-abelian) 2-step nilpotent Lie group is the Heisenberg Lie group of dimension three, H_3 . Its Lie algebra \mathfrak{h}_3 is spanned by vectors e_1, e_2, e_3 satisfying the non-trivial Lie bracket relation

$$[e_1, e_2] = e_3.$$

The Lie group H_3 can be modeled on \mathbb{R}^3 equipped with the product operation given by

$$(v_1, z_1)(v_2, z_2) = (v_1 + v_2, z_1 + z_2 + \frac{1}{2}v_1^t J v_2),$$

where $v_i = (x_i, y_i)$, $i = 1, 2$ and $J : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is the linear map $J(x, y) = (y, -x)$. Another presentation of the Heisenberg Lie group is given by 3×3 -triangular real matrices with 1's on the diagonal with the usual multiplication of matrices.

A Riemannian metric $\langle \cdot, \cdot \rangle$ on the Lie group N is called *left-invariant* if translations on the left by elements of the group are isometries. Thus, a left-invariant metric on N is determined by an inner product on the corresponding Lie algebra \mathfrak{n} , usually identified with the tangent space at the identity element $T_e N$. We denote also by $\langle \cdot, \cdot \rangle$ the metric on \mathfrak{n} .

Whenever the Lie algebra \mathfrak{n} is 2-step nilpotent, its commutator $C(\mathfrak{n})$ is contained in the center \mathfrak{z} . So the metric $\langle \cdot, \cdot \rangle$ determines an orthogonal decomposition as vector spaces on the Lie algebra:

$$\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}, \quad \text{where} \quad \mathfrak{v} = \mathfrak{z}^\perp. \quad (2.1)$$

The decomposition in Equation (2.1) induces the skew-symmetric maps $j(Z) : \mathfrak{v} \rightarrow \mathfrak{v}$, for every $Z \in \mathfrak{z}$, implicitly defined by

$$\langle Z, [V, W] \rangle = \langle j(Z)V, W \rangle \quad \text{for all } Z \in \mathfrak{z}, V, W \in \mathfrak{v}. \quad (2.2)$$

Note that $j : \mathfrak{z} \rightarrow \mathfrak{so}(\mathfrak{v})$ is a linear map, which has a corresponding kernel. As said the commutator $C(\mathfrak{n})$ is contained in the center and one has the splitting

$$\mathfrak{z} = C(\mathfrak{n}) \oplus \ker(j)$$

as orthogonal direct sum of vector spaces. In fact,

- since $\langle Z, [U, V] \rangle = 0$ for all $U, V \in \mathfrak{v}$ and $Z \in \ker(j)$, then $\ker(j) \perp C(\mathfrak{n})$.
- $\dim \mathfrak{z} = \dim \ker(j) + \dim C(\mathfrak{n})$.

- The restriction $j : C(\mathfrak{n}) \mapsto \mathfrak{so}(\mathfrak{v})$ is injective.

See the proof of the next result in Proposition 2.7 in [9].

Proposition 2.2. [9] *Let $(N, \langle \cdot, \cdot \rangle)$ denote a 2-step nilpotent Lie group with a left-invariant metric. Then*

- *the subspaces $\ker j$ and $C(\mathfrak{n})$ are commuting ideals in \mathfrak{n} .*
- *Let $E = \exp(\ker(j))$. Then E is the Euclidean de Rham factor of N and N is isometric to the Riemannian product of the totally geodesic submanifolds E and \bar{N} where $\bar{N} = \exp(\mathfrak{v} \oplus C(\mathfrak{n}))$.*

Example 2.3. Let \mathfrak{h}_3 denote the Heisenberg Lie algebra of dimension three with basis e_1, e_2, e_3 as in Example 2.1. The canonical metric makes this basis orthonormal. It is not hard to see that the center is the subspace $\mathfrak{z} = \text{span}\{e_3\}$, while its orthogonal complement is the subspace $\mathfrak{v} = \text{span}\{e_1, e_2\}$ and moreover the map $j : \mathfrak{z} \rightarrow \mathfrak{so}(\mathfrak{v})$ is generated by

$$j(e_3) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

in the basis e_1, e_2 of \mathfrak{v} .

Let $\Gamma \subset N$ denote a discrete subgroup of the nilpotent Lie group N , such that the quotient $M = \Gamma \backslash N$ is compact. This is called a *compact nilmanifold*. Important geometrical properties of M are obtained from the Lie algebra of N , \mathfrak{n} . In fact, by a theorem of Nomizu [20] every de Rham cohomology group $H^i(M, \mathbb{R})$ is isomorphic to the group $H^i(\mathfrak{n})$.

As a consequence, results obtained at the Lie algebra level and related to forms on \mathfrak{n} , are applicable for the corresponding Lie group and their quotients.

Definition 2.4. A 2-step nilpotent real Lie algebra \mathfrak{n} with center \mathfrak{z} is called *non-singular* if $\text{ad}(X) : \mathfrak{n} \rightarrow \mathfrak{z}$ is onto for any $X \notin \mathfrak{z}$ [9]. The corresponding 2-step nilpotent Lie group will be called non-singular.

See the next examples of non-singular Lie algebras.

Example 2.5. *Heisenberg Lie algebras.* Let $n \geq 1$ be any integer and let $X_1, Y_1, X_2, Y_2, \dots, X_n, Y_n$ be any basis of a real vector space \mathfrak{v} isomorphic to \mathbb{R}^{2n} . Let Z be an element generating a one dimensional space \mathfrak{z} . Define a Lie bracket by $[X_i, Y_i] = -[Y_i, X_i] = Z$ and the other Lie brackets by zero. The Lie algebra $\mathfrak{h}_{2n+1} = \mathfrak{v} \oplus \mathfrak{z}$ is the $(2n+1)$ -dimensional Heisenberg Lie algebra.

By making use of the maps $j(Z) : \mathfrak{v} \rightarrow \mathfrak{v}$ defined in Equation (2.2), one may described the non-singularity notion. See [9].

Proposition 2.6. *Let \mathfrak{n} denote a 2-step nilpotent Lie algebra. The following properties are equivalent:*

1. \mathfrak{n} is non singular,
2. For every inner product $\langle \cdot, \cdot \rangle$ and every non-zero element $Z \in \mathfrak{z}$ the map $j(Z)$ is non-singular.
3. For some inner product $\langle \cdot, \cdot \rangle$ and every non-zero element $Z \in \mathfrak{z}$ the map $j(Z)$ is non-singular.

Non-singular Lie algebras are known as *fat* algebras because they are the symbol algebra [Definition 3.1.7 in [3]] or nilpotentization [19] of fat distributions.

A 2-step nilpotent Lie algebra $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ equipped with a inner product $\langle \cdot, \cdot \rangle$ is *singular* if $j(Z)$ is singular for every nonzero $Z \in \mathfrak{z}$.

The Lie algebra \mathfrak{n} is *almost non-singular* if $j(Z)$ is non-singular for every Z in an open dense subset of \mathfrak{z} .

Every 2-step nilpotent Lie algebra is non-singular, almost non-singular or singular. See [9].

Given a metric on a 2-step nilpotent Lie algebra \mathfrak{n} , if two non-zero elements $Z, Z' \in \mathfrak{z}$ can be found such that $j(Z)$ is non-singular and $j(Z')$ is singular, then the Lie algebra \mathfrak{n} is almost non-singular.

Among other examples, see [16, 17], a family of non-singular Lie algebras is provided by H -type Lie algebras, which are defined as follows.

Let $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$ denote a 2-step nilpotent Lie algebra equipped with a metric. If the map $j(Z) : \mathfrak{v} \rightarrow \mathfrak{v}$ is orthogonal for every $Z \in \mathfrak{z}$ with $\langle Z, Z \rangle = 1$, then the Lie algebra \mathfrak{n} is a *Lie algebra of type H* [15] (also known as H -type Lie algebras). Equivalently, the 2-step nilpotent Lie algebra \mathfrak{n} is of type H if and only if

$$j(Z)^2 = -\langle Z, Z \rangle Id, \quad \text{for every } Z \in \mathfrak{z},$$

which is equivalent to $j(Z)j(\tilde{Z}) + j(\tilde{Z})j(Z) = -2\langle Z, \tilde{Z} \rangle Id$, for $Z, \tilde{Z} \in \mathfrak{z}$. By making use of this identity one proves that

$$[X, j(Z)X] = \langle X, X \rangle Z$$

for every $X \in \mathfrak{v}$ and $Z \in \mathfrak{z}$.

In particular, the real, complex and quaternionic Heisenberg algebras are examples of Lie algebras of type H .

3 Left-invariant 2-forms and magnetic fields

In this section we study closed left-invariant 2-forms, called *magnetic fields*, on any 2-step nilpotent Lie group. We shall see that the closedness condition imposes several restrictions.

Let ω denote a left-invariant 2-form on a Lie group $(N, \langle \cdot, \cdot \rangle)$ equipped with a left-invariant metric. It is *closed* if

$$\omega([U, V], W) + \omega([V, W], U) + \omega([W, U], V) = 0 \quad \text{for all } U, V, W \in \mathfrak{n}.$$

Note that a left-invariant 2-form on N is determined by its values at the identity. Thus, we say this is a 2-form on \mathfrak{n} . The closed 2-form ω is *symplectic* whenever it is non-degenerate, that is, if $\omega(U, V) = 0$ for all $V \in \mathfrak{n}$, then $U = 0$.

Let $\langle \cdot, \cdot \rangle$ denote the metric on \mathfrak{n} . Let ω be a left-invariant 2-form on N , then there exists a unique skew-symmetric endomorphism $F : \mathfrak{n} \rightarrow \mathfrak{n}$

satisfying

$$\omega(X, Y) = \langle F(X), Y \rangle, \quad \text{for all } X, Y \in \mathfrak{n}.$$

Conversely, let $F : \mathfrak{n} \rightarrow \mathfrak{n}$ denote a skew-symmetric endomorphism on the Lie algebra. Define the associated 2-form ω as $\omega(X, Y) = \langle F(X), Y \rangle$. When ω is closed, such skew-symmetric map F is known as a *Lorentz force*.

Now, the condition of asking the 2-form ω to be closed is equivalent to asking the skew-symmetric map F to satisfy the equation:

$$\langle F(U), [V, W] \rangle + \langle F(V), [W, U] \rangle + \langle F(W), [U, V] \rangle = 0, \quad \text{for all } U, V, W \in \mathfrak{n}. \tag{3.1}$$

Furthermore, ω is non-degenerate, if and only if F is non-singular.

Assume now that the 2-step nilpotent Lie group N has Lie algebra \mathfrak{n} , which decomposes into the orthogonal splitting $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ as in Equation (2.1). Write $F_{\mathfrak{v}} = \pi_{\mathfrak{v}} \circ F$ and $F_{\mathfrak{z}} = \pi_{\mathfrak{z}} \circ F$, where $\pi_{\mathfrak{v}} : \mathfrak{n} \rightarrow \mathfrak{v}$ and $\pi_{\mathfrak{z}} : \mathfrak{n} \rightarrow \mathfrak{z}$ denote the corresponding orthogonal projections onto the subspaces \mathfrak{v} and \mathfrak{z} , respectively. So, the 2-form ω associated to F is closed if and only if the following conditions hold

$$(C1) \quad F_{\mathfrak{z}}(Z) \in C(\mathfrak{n})^{\perp}, \quad \text{for all } Z \in \mathfrak{z},$$

and for all $U, V, W \in \mathfrak{v}$:

$$(C2) \quad \langle F_{\mathfrak{z}}(U), [V, W] \rangle + \langle F_{\mathfrak{z}}(V), [W, U] \rangle + \langle F_{\mathfrak{z}}(W), [U, V] \rangle = 0,$$

which can be obtained by analyzing Condition (3.1) in terms of the projections onto the subspaces \mathfrak{v} and \mathfrak{z} .

Remark 3.1. For any exact form, the corresponding Lorentz map satisfies $F_{\mathfrak{z}} \equiv 0$. Furthermore, any skew-symmetric map on the Lie algebra $F : \mathfrak{n} \rightarrow \mathfrak{n}$ such that the projection $F_{\mathfrak{z}} \equiv 0$ trivially satisfies Equations (C1) and (C2).

The skew-symmetric map $F : \mathfrak{n} \rightarrow \mathfrak{n}$ decomposes as

$$F = F_1 + F_2,$$

where F_1 and F_2 are skew-symmetric maps such that, with respect to the orthogonal splitting $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ in Equation (2.1), one has:

- F_1 preserves the decomposition: $F_1(V + Z) = F_{\mathfrak{v}}(V) + F_{\mathfrak{z}}(Z)$,
- F_2 interchanges the subspaces \mathfrak{v} and \mathfrak{z} : $F_2(V + Z) = F_{\mathfrak{z}}(V) + F_{\mathfrak{v}}(Z)$ for all $V \in \mathfrak{v}, Z \in \mathfrak{z}$.

In fact, for any skew-symmetric map $F \in \mathfrak{so}(\mathfrak{n})$, one has the decomposition $F = F_{\mathfrak{z}} + F_{\mathfrak{v}}$ and so,

$$F(V + Z) = (F_{\mathfrak{v}}(V) + F_{\mathfrak{z}}(Z)) + (F_{\mathfrak{v}}(Z) + F_{\mathfrak{z}}(V)), \quad \text{for all } V \in \mathfrak{v}, Z \in \mathfrak{z}.$$

The skew-symmetry property from F implies that both F_1 and F_2 are skew-symmetric.

Notice that

- F_1 trivially satisfies Condition (C2) and
- F_2 trivially satisfies Condition (C1).

Thus the skew-symmetric map F gives rise to a closed 2-form if and only if for both F_1 and F_2 it holds:

- F_1 satisfies Condition (C1), that is $F_1(\mathfrak{z}) \subseteq \ker(j)$ and
- F_2 satisfies Condition (C2).

Definition 3.2. Let F denote a skew-symmetric map on a 2-step nilpotent Lie algebra $(\mathfrak{n}, \langle \cdot, \cdot \rangle)$. We say that

1. F is of type I, if F preserves the decomposition $\mathfrak{v} \oplus \mathfrak{z}$ (so, $F = F_1$)
2. F is of type II, if $F(\mathfrak{v}) \subseteq \mathfrak{z}$ and $F(\mathfrak{z}) \subseteq \mathfrak{v}$ (so, $F = F_2$ above).

Next, we shall study exact 2-forms. Start with a left-invariant 1-form η . Consider the linear isomorphism between the Lie algebra \mathfrak{n} and its dual space \mathfrak{n}^* given by the metric, that is, sending $U \rightarrow \ell_U$, where $\ell_U(V) = \langle U, V \rangle$.

By considering the decomposition of the Lie algebra \mathfrak{n} given in Equation (2.1) one can write any left-invariant 1-form η as $\eta = \ell_{Z+V}$. One verifies easily that the differential satisfies

$$d\ell_{Z+V}(U, V) = \langle Z, [U, W] \rangle = \langle j(Z)U, W \rangle.$$

The kernel of the differential operator $d : \mathfrak{n}^* \rightarrow \Lambda^2 \mathfrak{n}^*$ contains the subspace $\{\ell_V, \text{ with } V \in \mathfrak{v}\}$. For the description of the rank of d notice that $d\eta \neq 0$ if and only if $j(Z) \neq 0$. This implies Z belongs to $C(\mathfrak{n})$. Thus, for the 2-form $d\ell_Z$, the corresponding Lorentz force F is given by $F = j(Z)$ for some non-trivial $Z \in C(\mathfrak{n})$.

We have therefore proved the next result.

Proposition 3.3. *Let $(N, \langle \cdot, \cdot \rangle)$ denote a 2-step-nilpotent Lie group equipped with a left-invariant metric and Lie algebra \mathfrak{n} with orthogonal splitting $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ as in (2.1). Let $F : \mathfrak{n} \rightarrow \mathfrak{n}$ denote a linear map. Write the map F as*

$$F = F_1 + F_2,$$

where $F_1(\mathfrak{v}) \subseteq \mathfrak{v}$ and $F_1(\mathfrak{z}) \subseteq \mathfrak{z}$, while $F_2(\mathfrak{z}) \subseteq \mathfrak{v}$ and $F_2(\mathfrak{v}) \subseteq \mathfrak{z}$. Then

(i) *The skew-symmetric map F gives rise to a closed 2-form if and only if both maps F_1 and F_2 are skew-symmetric and*

- F_1 satisfies Condition (C1) and
- F_2 satisfies Condition (C2).

(ii) *The 2-form associated with F is exact if and only if there is $Z \in C(\mathfrak{n})$ such that $F = j(Z)$.*

Remark 3.4. Note that if the 2-step nilpotent Lie group N has no Euclidean factor, then $\mathfrak{z} = C(\mathfrak{n})$. Thus, any skew-symmetric map F of type I satisfies $F_{\mathfrak{z}} \equiv 0$.

On the other hand, if N has Euclidean factor then $\ker(j) \neq \{0\}$ and any non-trivial skew-symmetric map $F : \mathfrak{n} \rightarrow \mathfrak{n}$ that satisfies $F(\mathfrak{v}) \subseteq \ker(j)$ and $F(\mathfrak{z}) \subseteq \mathfrak{v}$ gives rise to a closed left-invariant 2-form of type II on N .

Example 3.5. Let $V_0 + Z_0$ be any element on a 2-step nilpotent Lie algebra \mathfrak{n} . Then a natural choice for F is the skew-symmetric part of $\text{ad}(V_0 + Z_0)$: $\text{ad}(V_0 + Z_0) - \text{ad}(V_0 + Z_0)^* = \text{ad}(V_0) - \text{ad}(V_0)^*$, where $\text{ad}(X)^*$ denotes the adjoint of $\text{ad}(X)$ with respect to the metric. Notice that $\text{ad}(V_0)(\mathfrak{v}) \subseteq \mathfrak{z}$

and $\text{ad}(V_0)^*(\mathfrak{z}) \subset \mathfrak{v}$. Thus $\text{ad}(V_0) - \text{ad}(V_0)^*$ gives rise to a closed 2-form if and only if Condition (C2) holds, equivalently:

$$\langle [V_0, V], [U, W] \rangle + \langle [V_0, U], [W, V] \rangle + \langle [V_0, W], [V, U] \rangle = 0 \text{ for all } U, V, W \in \mathfrak{v}.$$

Which kind of elements $V_0 \in \mathfrak{v}$ may satisfy this equation? Take the Heisenberg Lie algebra of dimension $2n + 1$, $n \geq 2$, in Example 2.5. Assume $V_0 = \sum_i x_i X_i + \sum_i y_i Y_i$. By taking $U = Y_i, V = X_j$ and $W = Y_j$ with $i \neq j$, one gets $x_i = 0$ for every i . Analogously, by taking $U = X_i$, one finally obtains $V_0 = 0$.

Notice that if a Lie group N admits a symplectic structure, then the dimension of N is an even integer.

Lemma 3.6. *Let N denote a 2-step nilpotent Lie group admitting a symplectic structure. Then N admits a left-invariant Lorentz force of type II.*

Proof. Choose a left-invariant metric on N and fix this metric at the Lie algebra level. Take the decomposition $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ and in Equation (2.1). Take the linear map $F \in \mathfrak{so}(\mathfrak{n})$ such that $\omega(u, v) = \langle F(u), v \rangle$, and write $F = F_1 + F_2$, where F_1 preserves the decomposition, but $F_2(\mathfrak{z}) \subseteq \mathfrak{v}$ and $F_2(\mathfrak{v}) \subseteq \mathfrak{z}$. Assume F_2 is trivial, so $F = F_1$ is of type I. The closedness condition says that $F(\mathfrak{z}) \subseteq \ker(j) \subsetneq \mathfrak{z}$. But since $F(\mathfrak{v}) \subseteq \mathfrak{v}$, it follows that F cannot be non-singular, which is a contradiction. \square

Remark 3.7. The proof of the previous lemma shows that a Lorentz force of type I never induces a symplectic form.

Moreover, let \mathfrak{n} be a nilpotent Lie algebra and let η denote a 1-form. Thus, $d\eta(Z, U) = 0$ for every $Z \in \mathfrak{z}, U \in \mathfrak{n}$. This says that $d\eta$ cannot be symplectic. This was already known, see for instance [13].

Example 3.8. Let \mathfrak{h}_3 denote the Heisenberg Lie algebra of dimension three. Let $e^i, i = 1, 2, 3$ denote the dual basis of the orthonormal basis e_1, e_2, e_3 . With the convention $e^{ij} = e^i \wedge e^j$, clearly the 2-forms

e^{12}, e^{13}, e^{23} give a basis of the space of 2-forms on \mathfrak{n} . Since $\langle j(e_3)e_1, e_2 \rangle = \langle e_3, [e_1, e_2] \rangle = 1$, then e^{12} is exact.

It is not hard to see that any 2-form $\omega = \alpha e^{12} + \beta e^{13} + \gamma e^{23}$ is closed.

Notice that any skew-symmetric map of type II, namely F_2 , gives rise to a 2-form ω satisfying the condition

$$\omega(Z, \tilde{Z}) = 0, \quad \text{for all } Z, \tilde{Z} \in \mathfrak{z}.$$

This means, that the center is “isotropic” for ω . In this situation only one of the two following conditions is true:

- (i) either $\omega(\mathfrak{z}, \mathfrak{n}) = 0$ or
- (ii) there is $U \in \mathfrak{n} - \mathfrak{z}$ such that $\omega(Z, U) \neq 0$ for some $Z \in \mathfrak{z}$.

Indeed the non-trivial cases correspond to the second condition (ii). Due to the correspondence between 2-forms and skew-symmetric maps, one can obtain a description of 2-forms in terms of skew-symmetric maps of type I or II, as follows.

As above, let $\langle \cdot, \cdot \rangle$ denote a metric on the 2-step nilpotent Lie algebra \mathfrak{n} and take the orthogonal decomposition $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ as in Equation (2.1). Let $F : \mathfrak{n} \rightarrow \mathfrak{n}$ be the skew-symmetric map on \mathfrak{n} such that $\omega(V + Z, \tilde{V} + \tilde{Z}) = \langle F(V + Z), \tilde{V} + \tilde{Z} \rangle$.

The condition of the center to be isotropic says that $\langle F(Z), \tilde{Z} \rangle = 0$ for all $Z, \tilde{Z} \in \mathfrak{z}$, that is $F_{\mathfrak{z}}(Z) = 0$, for any $Z \in \mathfrak{z}$, which in terms of the families we introduced previously, gives:

$$F(Z + V) = F_1(V) + F_2(Z) + F_2(V) \quad \text{for all } V + Z \in \mathfrak{n}.$$

But in this situation, the corresponding 2-form ω is closed if and only if F_2 satisfies Condition (C2). In fact, any skew-symmetric linear map F_1 with $F_1(\mathfrak{z}) \equiv 0$ induces a closed 2-form as already noticed in Remark 3.1.

On the other hand, $\omega(Z, U) \neq 0$ if and only if $\langle F(Z), U \rangle \neq 0$ if and only if $\langle F_2(Z), U \rangle \neq 0$ for some $Z \in \mathfrak{z}$ and some $U \in \mathfrak{n} - \mathfrak{z}$. And this must occur for any metric.

Proposition 3.9. *Let \mathfrak{n} denote a 2-step nilpotent Lie algebra.*

- (i) *There is a non-trivial closed 2-form ω for which either $\omega(\mathfrak{z}, \mathfrak{z}) \neq 0$ or $\omega(\mathfrak{z}, \mathfrak{n}) = 0$ if and only if for any metric $\langle \cdot, \cdot \rangle$ on the Lie algebra \mathfrak{n} there is a non-trivial skew-symmetric map of type I satisfying Condition (C1).*
- (ii) *There is a non-trivial closed 2-form ω for which $\omega(\mathfrak{z}, \mathfrak{z}) = 0$ but $\omega(\mathfrak{z}, \mathfrak{n}) \neq 0$ if and only if for any metric $\langle \cdot, \cdot \rangle$ on the Lie algebra \mathfrak{n} there is a non-trivial skew-symmetric map of type II satisfying Condition (C2).*

Moreover, in terms of Definition 3.2, the proposition above enables us to distinguish 2-forms into two families. We shall say that a 2-form ω on the Lie algebra \mathfrak{n} is

- (i) *of type I:* if it satisfies either $\omega(\mathfrak{z}, \mathfrak{z}) \neq 0$ or $\omega(\mathfrak{z}, \mathfrak{n}) = 0$
- (ii) *of type II:* if it satisfies $\omega(\mathfrak{z}, \mathfrak{z}) = 0$ but $\omega(\mathfrak{z}, \mathfrak{n}) \neq 0$.

Example 3.10. Let $\mathfrak{h}(\mathbb{C})$ denote the Heisenberg Lie algebra over \mathbb{C} . Consider the underlying real Lie algebra of dimension six, that we denote in the same way. It has a center of dimension two spanned by Z_1, Z_2 and complementary subspace of dimension four spanned by the vectors X_1, Y_1, X_2, Y_2 . They satisfy the non-trivial Lie bracket relations

$$[X_1, Y_1] = -[X_2, Y_2] = Z_1 \quad [X_1, Y_2] = [X_2, Y_1] = Z_2.$$

The left multiplication by i induces a real linear map $J : \mathfrak{h}(\mathbb{C}) \rightarrow \mathfrak{h}(\mathbb{C})$ satisfying $J^2 = -Id$ and $J \circ \text{ad}(U) = \text{ad}(U) \circ J$ for all $U \in \mathfrak{h}(\mathbb{C})$. Explicitly, in the basis, one has

$$J(Z_1) = Z_2 \quad J(X_1) = X_2 \quad J(Y_1) = Y_2.$$

Take the metric on $\mathfrak{h}(\mathbb{C})$ making the set Z_i, X_i, Y_i for $i=1,2$, an orthonormal basis. Thus, $\mathfrak{h}(\mathbb{C})$ is a Lie algebra of type H . Clearly the complex structure

J is skew-symmetric with respect to this metric $\langle \cdot, \cdot \rangle$. Since this Lie algebra is non-singular, the kernel of j is trivial, $\ker j = \{0\}$.

Assume that F is a skew-symmetric map on the Lie algebra giving rise to a closed 2-form. Write $F = F_1 + F_2$ as above. Since $\mathfrak{h}(\mathbb{C})$ is non-singular, Condition (C1) imposes the linear map F_1 is trivial on \mathfrak{z} . Thus, F_1 corresponds to a skew-symmetric map $\mathfrak{v} \rightarrow \mathfrak{v}$.

On the other hand, for $V, W \in \mathfrak{v}$, the condition (C2) for F_2 gives

$$\begin{aligned} 0 &= \langle F_2(V), [JV, W] \rangle + \langle F_2(JV), [W, V] \rangle + \langle F_2 W, [V, JV] \rangle \\ &= \langle F_2(V), J[V, W] \rangle + \langle F_2(JV), [W, V] \rangle + \langle F_2 W, J[V, V] \rangle \\ &= \langle F_2(V), J[V, W] \rangle - \langle F_2(JV), [V, W] \rangle \\ &= \langle -JF_2(V) - F_2(JV), [V, W] \rangle. \end{aligned}$$

Since W is an arbitrary element and $\text{ad}(V)$ is onto the center \mathfrak{z} we get $F_2(JV) = -JF_2(V)$, for every $V \in \mathfrak{v}$. And since F_2 is skew-symmetric it holds on \mathfrak{n} :

$$F_2 \circ J = -J \circ F_2. \tag{3.2}$$

Conversely, any skew-symmetric map $F_2 : \mathfrak{h}(\mathbb{C}) \rightarrow \mathfrak{h}(\mathbb{C})$ that verifies Equation (3.2) will give rise to a closed 2-form of type II.

The next result proved in [22] determines a condition on the dimension of the Lie algebra and its center for the non-existence of magnetic fields of type II, that is, the corresponding skew-symmetric map is of type II.

Lemma 3.11. [22] *Let \mathfrak{n} denote a non-singular 2-step nilpotent Lie algebra such that $\dim \mathfrak{n} > 3 \dim \mathfrak{z}$. Then any closed 2-form on \mathfrak{n} satisfies*

$$\omega(Z, U) = 0, \quad \text{for all } Z \in \mathfrak{z}, U \in \mathfrak{n}.$$

Proof. Firstly, note that since the Lie algebra is non-singular, its commutator coincides with the center, $C(\mathfrak{n}) = \mathfrak{z}$. Let ω denote a closed 2-form on \mathfrak{n} , the non-singularity property implies that $\omega(\mathfrak{z}, \mathfrak{z}) = 0$. In fact, let $Z, \tilde{Z} \in \mathfrak{z}$ with $Z = [U, V]$ for $U, V \in \mathfrak{n}$. The closedness condition says that $\omega(\tilde{Z}, [U, V]) = 0$.

By contradiction, assume that there exists a non-trivial closed 2-form ω on the non-singular 2-step nilpotent Lie algebra \mathfrak{n} such that there are $Z \in \mathfrak{z}$ and $U \in \mathfrak{n} - \mathfrak{z}$ satisfying $\omega(Z, U) \neq 0$, that is ω is of type II.

Let $\langle \cdot, \cdot \rangle$ be a metric on \mathfrak{n} inducing an orthogonal decomposition $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ as in Equation (2.1). Notice that by hypothesis, $\dim \mathfrak{v} > 2 \dim \mathfrak{z}$.

Let F denote the skew-symmetric map on the Lie algebra \mathfrak{n} such that $\omega(X, Y) = \langle F(X), Y \rangle$. Then

$$\omega(Z, U) \neq 0 \text{ if and only if } \langle F(Z), U \rangle \neq 0 \text{ if and only if } \langle F_2(Z), U \rangle \neq 0,$$

where F_1 and F_2 are taken as in Proposition 3.3. Since $\langle Z, F_2(U) \rangle \neq 0$ says that the image of $F_2|_{\mathfrak{v}}$ is non-trivial, we may assume (changing Z if necessary) that $F_2(U) = Z$, so that $\langle Z, F_2(U) \rangle = \langle Z, Z \rangle \neq 0$.

Denote the kernel of F_2 by $\mathcal{W} = \ker(F_2|_{\mathfrak{v}})$. Thus one has:

$$\dim \mathcal{W} = \dim \mathfrak{v} - \dim \text{Image}(F_2|_{\mathfrak{v}}) \geq \dim \mathfrak{v} - \dim \mathfrak{z} > \dim \mathfrak{v}/2$$

Since $j(Z)$ is non-singular, it holds $\dim j(Z)\mathcal{W} > \dim \mathfrak{v}/2$, which implies that the intersection is nontrivial, $\mathcal{W} \cap j(Z)\mathcal{W} \neq \{0\}$. Let elements $W, \widetilde{W} \in \mathcal{W}$ such that $j(Z)W = \widetilde{W} \neq 0$. Now, the closedness condition for the 2-form ω is equivalent to Condition (C2) for F_2 . And for W, \widetilde{W}, U we get

$$\begin{aligned} 0 &= \langle F_2(W), [\widetilde{W}, U] \rangle + \langle F_2(\widetilde{W}), [U, W] \rangle + \langle F_2(U), [W, \widetilde{W}] \rangle \\ &= \langle Z, [W, \widetilde{W}] \rangle = \langle \widetilde{W}, \widetilde{W} \rangle \neq 0, \end{aligned}$$

which is a contradiction. Thus, there are no closed 2-forms of type II under the hypothesis. \square

Example 3.12. A singular example. Let \mathfrak{n} denote the Lie algebra $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$, where \mathfrak{v} is spanned by the vectors V_1, V_2, V_3, V_4, V_5 and \mathfrak{z} is spanned by Z_1, Z_2 , and they obey the non-trivial Lie bracket relations:

$$[V_1, V_2] = Z_1, \quad [V_3, V_4] = Z_2 = [V_4, V_5].$$

Take the metric on \mathfrak{n} that makes the previous basis orthonormal.

Let $F : \mathfrak{n} \rightarrow \mathfrak{n}$ denote the skew-symmetric map given by

$$F(V_3) = Z_2 = -F(V_5), \quad F(V_4) = Z_1, \quad F(V_i) = 0, \quad i = 1, 2.$$

Usual computations show that the 2-form given as $\omega(X, Y) = \langle F(X), Y \rangle$ is closed.

This is an example of a singular Lie algebra (that is, every $j(Z)$ is singular) admitting a closed 2-form of type II. In fact, every map $j(z_1 Z_1 + z_2 Z_2) : \mathfrak{v} \rightarrow \mathfrak{v}$ has a matrix of the form

$$\begin{pmatrix} 0 & -z_1 & 0 & 0 & 0 \\ z_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -z_2 & 0 \\ 0 & 0 & z_2 & 0 & -z_2 \\ 0 & 0 & 0 & z_2 & 0 \end{pmatrix}$$

in the basis of \mathfrak{v} given above. And $\dim \mathfrak{n} > 3 \dim \mathfrak{z}$.

In [22] the authors determine the algebras of type H admitting Lorentz forces of type II.

Theorem 3.13. *Let $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ be a Lie algebra of type H. Then \mathfrak{n} admits a Lorentz force of type II if and only if \mathfrak{n} is the 3-dimensional Heisenberg algebra, the 6-dimensional complex Heisenberg algebra or the 7-dimensional quaternionic Heisenberg algebra.*

The results of this section give obstructions for the existence of left-invariant symplectic structures on 2-step nilpotent Lie groups. This follows as an application.

Corollary 3.14. *Let N denote a 2-step nilpotent Lie group of dimension $2n$. Whenever the Lie group N is non-singular and $\dim \mathfrak{n} > 3 \dim \mathfrak{z}$, there is no left-invariant symplectic structure on N .*

In particular, the only Lie group of type H admitting a left-invariant symplectic structure corresponds to the complex Heisenberg with Lie algebra $\mathfrak{h}(\mathbb{C})$.

Proof. By Lemma 3.11 any non-singular Lie algebra \mathfrak{n} satisfying $\dim \mathfrak{n} > \dim \mathfrak{z}$ cannot admit a closed-symplectic structure of type II. Thus any closed 2-form corresponds to a 2-form of type I. Such 2-form corresponds to a skew-symmetric map on \mathfrak{n} satisfying Condition (C1), that is $F_3(Z) \in \ker j$ for all $Z \in \mathfrak{z}$. Thus by the non-singularity property $F_3(Z) = 0$ for all $Z \in \mathfrak{z}$. So, any closed 2-form of type I is degenerate.

Among Lie algebras of type H admitting closed 2-form of type II, only the complex Heisenberg Lie algebra $\mathfrak{h}(\mathbb{C})$ is even-dimensional. According to computations in Example (3.10), any Lorentz force on $\mathfrak{h}(\mathbb{C})$ has a matrix presentation of the form in the basis $X_1, X_2, Y_1, Y_2, Z_1, Z_2$:

$$\begin{pmatrix} 0 & a & b & c & g & h \\ -a & 0 & d & e & h & -g \\ -b & -d & 0 & f & i & j \\ -c & -e & -f & 0 & j & -i \\ -g & -h & -i & -j & 0 & 0 \\ -h & g & -j & i & 0 & 0 \end{pmatrix}$$

and it is not difficult to see that there are non-singular examples. Thus, there exist symplectic structures in this case. □

Remark 3.15. Dotti and Tirao proved in [8]:

Let $M = T^t \times \Gamma \backslash H$ be an even dimensional H -type nilmanifold, where H is an H -type group whose Lie algebra is not isomorphic to $\mathfrak{h}_3, \mathfrak{h}(\mathbb{C})$ or $\mathfrak{h}(\mathbb{H})$. Then there is no symplectic structure on M .

By Remark 3.4, if $t \geq 1$ we get examples of Lie groups that do not admit symplectic structures but admitting a non-trivial closed left-invariant 2-form of type II. In 4.2 we show more interesting examples on indecomposable Lie algebras.

4 Closed 2-forms on low dimensional Lie algebras

In this section we study closed 2-forms of type II on 2-step nilpotent Lie algebras of dimension n with $n \leq 6$. The main goal is to compare

the existence of closed 2-forms of type II with the existence of symplectic structures.

Notice that the non-existence of closed 2-forms of type II on the Heisenberg Lie algebra of dimension 5 is a consequence of Lemma 3.11, while the existence for non-singular Lie algebras such as the Heisenberg Lie algebra of dimension 3, and the complex Heisenberg Lie algebra in dimension six, is stated in Theorem 3.13.

Firstly, recall the list of 2-step nilpotent Lie algebras up to dimension 6, where we write the non-trivial Lie brackets:

Dimension 3

1. $\mathfrak{h}_1(\mathbb{R})$ Heisenberg Lie algebra of dimension three: $[e_1, e_2] = e_3$.

Dimension 4

1. $\mathfrak{h}_1(\mathbb{R}) \oplus \mathbb{R} = \text{span}\{e_1, e_2, e_3, e_4\}$ with $[e_1, e_2] = e_3$.

Dimension 5

1. $\mathfrak{h}_1(\mathbb{R}) \oplus \mathbb{R}^2 = \text{span}\{e_1, e_2, e_3, e_4, e_5\}$ with $[e_1, e_2] = e_3$.
2. $\mathfrak{h}_2(\mathbb{R})$ Heisenberg Lie algebra of dimension 5: $\text{span}\{e_1, e_2, e_3, e_4, e_5\}$ with $[e_1, e_2] = [e_3, e_4] = e_5$.
3. $\mathfrak{g}_5 = \text{span}\{e_1, e_2, e_3, e_4, e_5\}$ with $[e_1, e_3] = e_4$, $[e_2, e_3] = e_5$, called Star.

Dimension 6: the Lie algebra is spanned by vectors $e_1, e_2, e_3, e_4, e_5, e_6$

1. $\mathfrak{h}_1(\mathbb{R}) \oplus \mathbb{R}^3$ with $[e_1, e_2] = e_3$.
2. $\mathfrak{h}_2(\mathbb{R}) \oplus \mathbb{R}$ with $[e_1, e_2] = [e_3, e_4] = e_5$.
3. $\mathfrak{g}_5 \oplus \mathbb{R}$ with $[e_1, e_3] = e_4$, $[e_2, e_3] = e_5$.
4. $\mathfrak{h}_1(\mathbb{R}) \oplus \mathfrak{h}_1(\mathbb{R})$ with $[e_1, e_2] = e_5$, $[e_3, e_4] = e_6$.

- 5. \mathfrak{f}_6 : where $[e_1, e_2] = e_4$ $[e_1, e_3] = e_5$ $[e_2, e_3] = e_6$. This is the free 2-step nilpotent Lie algebra in three generators.
- 6. \mathfrak{k}_6 : where $[e_1, e_4] = e_5$ $[e_2, e_3] = -e_5$ $[e_3, e_4] = e_6$.
- 7. $\mathfrak{h}_1(\mathbb{C})$ with $[e_1, e_2] = e_5 = -[e_3, e_4]$, $[e_1, e_4] = e_6 = [e_2, e_3]$. It is the complex Heisenberg Lie algebra.

Proposition 4.1. *Every 2-step nilpotent Lie group of dimension n with $n \leq 6$ admits a non-trivial closed 2-form of type II, with exception of the real Heisenberg Lie algebra of dimension 5.*

Proof. For the general approach to the proof, let \mathfrak{n} denote a 2-step nilpotent Lie algebra equipped with any metric. Then the Lie algebra admits a decomposition $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ as in Equation (2.1).

Let v_1, \dots, v_m be a orthonormal basis of \mathfrak{v} and z_1, \dots, z_n a orthonormal basis of \mathfrak{z} . As usual, write C_{ij}^k for the constant structures of the Lie algebra, which in our cases is

$$[v_i, v_j] = \sum_{s=1}^n C_{ij}^s z_s.$$

Now, let ω denote a closed 2-form of type II satisfying $\omega = \langle F \cdot, \cdot \rangle$. Thus $F(v_k) = \sum_{t=1}^n b_{tk} z_t$ for any $k = 1, \dots, m$. Notice that $b_{sk} = -b_{ks}$ from the skew-symmetric property of F . Now, the closedness condition says

$$\langle [v_i, v_j], F(v_k) \rangle + \langle [v_j, v_k], F(v_i) \rangle + \langle [v_k, v_i], F(v_j) \rangle, \quad \text{for all } i, j, k,$$

which explicitly gives

$$\sum_{s=1}^n (C_{ij}^s b_{sk} + C_{jk}^s b_{si} + C_{ki}^s b_{sj}) = 0.$$

Indeed trivial solutions $b_{si} = 0$ for all i, s are always possible.

The next table shows the solutions of the equation above in every case.

We denote by e^i , the dual forms of the elements e_j in the corresponding Lie algebras. And as usual $e^{ij} := e^i \wedge e^j$.

□

Lie algebra	closed 2-forms
\mathfrak{h}_1	$a_{13}e^{13} + a_{23}e^{23}, \quad a_{i3} \in \mathbb{R}.$
$\mathfrak{h}_1 \oplus \mathbb{R}$	$a_{13}e^{13} + a_{23}e^{23} + a_{14}e^{14} + a_{24}e^{24}, \quad a_{ij} \in \mathbb{R}.$
$\mathfrak{h}_1 \oplus \mathbb{R}^2$	$\sum_{j=3}^5 a_{1j}e^{1j} + \sum_{j=3}^5 a_{2j}e^{2j}, \quad a_{ij} \in \mathbb{R}.$
\mathfrak{h}_2	no
\mathfrak{g}_5	$\sum_{i=1}^3 \sum_{j=4}^5 a_{ij}e^{ij}, \quad a_{ij} \in \mathbb{R}.$
$\mathfrak{h}_1 \oplus \mathbb{R}^3$	$\sum_{j=3}^6 a_{1j}e^{1j} + \sum_{j=3}^6 a_{2j}e^{2j}, \quad a_{ij} \in \mathbb{R}.$
$\mathfrak{h}_2 \oplus \mathbb{R}$	$\sum_{i=1}^4 a_{i6}e^{i6}, \quad a_{i6} \in \mathbb{R}.$
$\mathfrak{g}_5 \oplus \mathbb{R}$	$\sum_{i=1}^3 \sum_{j=4}^6 a_{ij}e^{ij}, \quad a_{ij} \in \mathbb{R}.$
$\mathfrak{h}_1 \oplus \mathfrak{h}_1$	$\sum_{i=1}^2 a_{i5}e^{i5} + \sum_{j=3}^4 a_{i6}e^{i6}, \quad a_{ij} \in \mathbb{R}.$
\mathfrak{f}_6	$\sum_{i=1}^3 \sum_{j=4}^6 a_{ij}e^{ij}, a_{25} = a_{16} + a_{34}, \quad a_{ij} \in \mathbb{R}.$
\mathfrak{k}_6	$a(e^{16} + e^{35}) + b(e^{26} + e^{45}) + \sum_{i=3}^4 a_{i6}e^{i6}, \quad a_{ij} \in \mathbb{R}.$
$\mathfrak{h}(\mathbb{C})$	$\sum_{i=1,3} a_{i5}(e^{i5} - e^{i+16}) + \sum_{i=1,3} a_{i6}(e^{i6} + e^{i+15}), \quad a_{ij} \in \mathbb{R}.$

Table 4.1: Closed 2-forms of type II

Let us now determine the 2-closed forms on each of the nilpotent Lie algebras of dimension $n \leq 6$ listed above. They are obtained from canonical computations by asking the 2-form to be closed (see (3.1)).

Recall that $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ with $\mathfrak{v} = \mathfrak{z}^\perp$. And any skew-symmetric map F of type I preserves the decomposition, while if F is of type II, then $F(\mathfrak{z}) \subseteq \mathfrak{v}$ and $F(\mathfrak{v}) \subseteq \mathfrak{z}$.

Dimension 3: In $\mathfrak{h}_1(\mathbb{R})$ every 2-form is closed and it is represented by any skew-symmetric map $F : \mathfrak{h}_1(\mathbb{R}) \rightarrow \mathfrak{h}_1(\mathbb{R})$. Thus in the canonical basis, it has a matricial presentation

$$\begin{pmatrix} A & B \\ -B^t & 0 \end{pmatrix} \quad A \in \mathfrak{so}(2, \mathbb{R}), \quad B^t = \begin{pmatrix} b & c \end{pmatrix}.$$

Dimension 4: In $\mathfrak{h}_1(\mathbb{R}) \oplus \mathbb{R}$ every closed 2-form is represented by a matrix of the form:

$$\begin{pmatrix} A & B \\ -B^t & 0 \end{pmatrix} \quad A \in \mathfrak{so}(2, \mathbb{R}), \quad B^t = \begin{pmatrix} b & c \\ d & e \end{pmatrix}.$$

Notice that there are symplectic structure, whenever the matrix above is non-singular.

Dimension 5:

• In $\mathfrak{h}_1(\mathbb{R}) \oplus \mathbb{R}^2$ every closed 2-form is represented by a matrix of the form:

$$\begin{pmatrix} A & B \\ -B^t & C \end{pmatrix} \quad A \in \mathfrak{so}(2, \mathbb{R}), \quad B^t \in M_{2 \times 3}(\mathbb{R}) \quad C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & h \\ 0 & -h & 0 \end{pmatrix}.$$

• In $\mathfrak{h}_2(\mathbb{R})$ every closed 2-form is represented by a matrix of the form:

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \quad A \in \mathfrak{so}(4, \mathbb{R}).$$

• In \mathfrak{g}_5 every closed 2-form is represented by a matrix of the form:

$$\begin{pmatrix} A & B \\ -B^t & 0 \end{pmatrix} \quad A \in \mathfrak{so}(3, \mathbb{R}), \quad B^t \in M_{2 \times 3}(\mathbb{R}).$$

Dimension 6

• In $\mathfrak{h}_1(\mathbb{R}) \oplus \mathbb{R}^3$ every closed 2-form is represented by a matrix of the form:

$$\begin{pmatrix} A & B \\ -B^t & C \end{pmatrix} \quad A \in \mathfrak{so}(2, \mathbb{R}), \quad B \in M_{2 \times 4}(\mathbb{R}), \quad C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -e & -f \\ 0 & e & 0 & -g \\ 0 & f & g & 0 \end{pmatrix}.$$

The 2-form $\omega = e^1 \wedge e^3 + e^2 \wedge e^4 + e^5 \wedge e^6$ gives a symplectic structure.

• In $\mathfrak{h}_2(\mathbb{R}) \oplus \mathbb{R}$ every closed 2-form is represented by a matrix of the form:

$$\begin{pmatrix} A & B \\ -B^t & 0 \end{pmatrix} \quad A \in \mathfrak{so}(4, \mathbb{R}), \quad B^t = \begin{pmatrix} 0 & 0 & 0 & 0 \\ h & g & j & i \end{pmatrix}.$$

There is no symplectic structure.

- In $\mathfrak{g}_5 \oplus \mathbb{R}$ every closed 2-form is represented by a matrix of the form:

$$\begin{pmatrix} A & B \\ -B^t & 0 \end{pmatrix} \quad A \in \mathfrak{so}(3, \mathbb{R}), \quad B \in M_{3 \times 3}(\mathbb{R}).$$

The 2-form $\omega = e^1 \wedge e^4 + e^2 \wedge e^5 + e^3 \wedge e^6$ gives a symplectic structure.

- In $\mathfrak{h}_1(\mathbb{R}) \oplus \mathfrak{h}_1(\mathbb{R})$ every closed 2-form is represented by a 6×6 matrix of the form:

$$\begin{pmatrix} A & B \\ -B^t & 0 \end{pmatrix} \quad A \in \mathfrak{so}(4, \mathbb{R}), \quad B^t = \begin{pmatrix} a & b & 0 & 0 \\ 0 & 0 & c & d \end{pmatrix}.$$

The 2-form $\omega = e^1 \wedge e^5 + e^2 \wedge e^4 + e^3 \wedge e^6$ gives a symplectic structure.

- In \mathfrak{f}_6 every closed 2-form is represented by a matrix of the form:

$$\begin{pmatrix} A & B \\ -B^t & 0 \end{pmatrix} \quad A \in \mathfrak{so}(3, \mathbb{R}), \quad B = \begin{pmatrix} d & e & f \\ g & f+i & h \\ i & j & k \end{pmatrix}.$$

The 2-form $\omega = e^1 \wedge e^6 + 2e^2 \wedge e^5 + e^3 \wedge e^4$ gives a symplectic structure.

- In \mathfrak{k}_6 every closed 2-form is represented by a 6×6 skew-symmetric matrix of the form:

$$\begin{pmatrix} A & B \\ -B^t & 0 \end{pmatrix} \quad A \in \mathfrak{so}(4, \mathbb{R}), \quad B^t = \begin{pmatrix} 0 & 0 & g & h \\ g & h & i & j \end{pmatrix}.$$

The 2-form $\omega = e^1 \wedge e^6 + e^2 \wedge e^4 + e^3 \wedge e^5$ gives a symplectic structure.

- In $\mathfrak{h}_1(\mathbb{C})$ every closed 2-form is represented by a matrix of the form:

$$\begin{pmatrix} A & B \\ -B^t & 0 \end{pmatrix} \quad A \in \mathfrak{so}(4, \mathbb{R}), \quad B^t = \begin{pmatrix} g & h & i & j \\ h & -g & j & -i \end{pmatrix}.$$

The 2-form $\omega = e^1 \wedge e^6 + e^2 \wedge e^5 + e^3 \wedge e^4$ gives a symplectic structure.

The results of the previous paragraphs establish the proof of the following result.

Theorem 4.2. *Every 2-step nilpotent Lie group of dimension $2n$ with $n \leq 3$ admits a left-invariant symplectic structure if and only if it admits a closed left-invariant 2-form of type II, with exception of the trivial extension of the Heisenberg Lie group of dimension five.*

4.1 The action of the group of orthogonal automorphisms

As proved in [23] there is an action of the group of orthogonal automorphisms that facilitates to find the solutions: in fact, assume γ is a solution of Equation (1.1). Take an orthogonal automorphism φ , then

$$\nabla_{d\varphi\gamma'}d\varphi\gamma' = d\varphi\nabla_{\gamma'}\gamma' = d\varphi F \gamma' = d\varphi F d\varphi^{-1}d\varphi\gamma',$$

see more details in [23]. Thus, the magnetic trajectories for F are related via $d\varphi$ with the magnetic trajectories for $d\varphi F d\varphi^{-1}$.

Orthogonal automorphisms of the 2-step nilpotent Lie group N are in correspondence with orthogonal automorphisms of the corresponding Lie algebra \mathfrak{n} . That is, $\psi : \mathfrak{n} \rightarrow \mathfrak{n}$, such that $\langle \psi x, \psi y \rangle = \langle x, y \rangle$ for any $x, y \in \mathfrak{n}$.

By using the orthogonal decomposition $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ we have that

- $\psi(\mathfrak{z}) \subseteq \mathfrak{z}$ and $\psi(\mathfrak{v}) \subseteq \mathfrak{v}$,
- $\psi = (A, B)$ with $A \in O(\mathfrak{v})$ and $B \in O(\mathfrak{z})$ satisfy $Aj(Z)A^{-1} = j(BZ)$, for all $Z \in \mathfrak{z}$.

In particular, the last equality above says that exact magnetic fields are preserved by this action.

It is not hard to see that this action preserves the type of the magnetic field. That is, if F is of type I (resp. II), then $\psi \circ F \circ \psi^{-1}$ is of type I (resp. II), for any $\psi : \mathfrak{n} \rightarrow \mathfrak{n}$ orthogonal automorphism.

Remark 4.3. In [23] the authors considered a more general action to compute solutions. In fact given a magnetic trajectory $\gamma : I \rightarrow M$ solution of the magnetic equation (1.1) for a Lorentz force F , then the curve $\phi \circ \gamma(\nu s)$ is solution of the magnetic equation for the Lorentz force $\nu d\phi F d\phi^{-1}$, for $\nu \in \mathbb{R} - \{0\}$ and ϕ a isometry of M .

Example 4.4. In the Heisenberg Lie algebra of dimension three the group of orthogonal automorphisms $Auto(\mathfrak{h}_1(\mathbb{R}))$ consists of maps $\psi : \mathfrak{n} \rightarrow \mathfrak{n}$, where $\psi(v + z) = Av + \det(A)z$, for $v \in \mathfrak{v}, z \in \mathfrak{z}, A \in O(2)$.

Under this action any closed 2-form

- of type I is equivalent to $j(e_3)$;
- of type II it is equivalent to

$$\begin{pmatrix} 0 & B \\ -B^t & 0 \end{pmatrix} \quad B^t = \begin{pmatrix} r & 0 \end{pmatrix} \quad r \in \mathbb{R}^+.$$

The action of $\text{Auto}(\mathfrak{h}_1(\mathbb{R}))$ preserves the norm $\|v\|$ for any $v \in \mathfrak{v}$. Thus $r = \|v\|$.

The natural action is that of the automorphism group on the second cohomology group $H^2(\mathfrak{n}, \mathbb{R})$ which enables to classify the extensions of \mathfrak{n} . See for instance [18].

4.2 Lie algebras admitting closed 2-forms of type II but no symplectic structure.

Now we present examples of indecomposable Lie algebras admitting a closed 2-form of type II but no symplectic structure.

There is a family of 2-step nilpotent Lie algebras which are constructed from graphs. Let G be a directed graph with at least one edge. Denote the vertices of G by $S = \{X_1, \dots, X_m\}$ and its edges by $E = \{Z_1, \dots, Z_q\}$. The Lie algebra $L(G)$ is the vector space direct sum $\mathfrak{v} \oplus \mathfrak{z}$, where we let E be a basis over \mathbb{R} for \mathfrak{z} and S be a basis over \mathbb{R} for \mathfrak{v} . Define the bracket relations among elements of S according to adjacency rules:

- if Z_k is a directed edge from vertex X_i to vertex X_l then define the skew-symmetric bracket $[X_i, X_l] = Z_k$.
- If there is no edge between two vertices, then define the bracket of those two elements in S to be zero.

Extend the bracket relation to all of \mathfrak{v} by using bilinearity of the bracket. The existence problem of symplectic structures in this family of 2-step nilpotent Lie algebras was solved by Pouseele and Tirao in [24]:

Theorem 4.5. [24]. *Let G be a graph. Then the Lie algebra $L(G)$ associated with G is symplectic if and only if $|V| + |E|$ is even, and, in each connected component of G , the number of edges does not exceed the number of vertices.*

Consider the complete graph K_4 : it has four vertices V_1, V_2, V_3, V_4 and six edges $Z_1, Z_2, Z_3, Z_4, Z_5, Z_6$, giving rise to a Lie algebra of dimension ten with non-trivial Lie brackets

$$[V_1, V_2] = Z_1, \quad [V_1, V_3] = Z_2, \quad [V_1, V_4] = Z_3,$$

$$[V_2, V_3] = Z_4, \quad [V_2, V_4] = Z_5, \quad [V_3, V_4] = Z_6.$$

Since $|E| = 6 > 4 = |V|$ the corresponding Lie algebra does not admit a symplectic structure.

However it admits a 2-form of type II: one has to solve four linear equations for the coordinates a_{ij} such that $F(V_i) = \sum_{j=1}^6 a_{ji} Z_j$:

- $\langle [V_1, V_2], F(V_3) \rangle + \langle [V_2, V_3], F(V_1) \rangle + \langle [V_3, V_1], F(V_2) \rangle = 0$ is equivalent to

$$a_{13} + a_{41} - a_{22} = 0,$$

- $\langle [V_1, V_2], F(V_4) \rangle + \langle [V_2, V_4], F(V_1) \rangle + \langle [V_4, V_1], F(V_2) \rangle = 0$ is equivalent to

$$a_{14} + a_{51} - a_{32} = 0,$$

- $\langle [V_1, V_3], F(V_4) \rangle + \langle [V_3, V_4], F(V_1) \rangle + \langle [V_4, V_1], F(V_3) \rangle = 0$ is equivalent to

$$a_{24} + a_{61} - a_{33} = 0.$$

- $\langle [V_2, V_3], F(V_4) \rangle + \langle [V_3, V_4], F(V_2) \rangle + \langle [V_4, V_2], F(V_3) \rangle = 0$ is equivalent to

$$a_{44} + a_{62} - a_{53} = 0.$$

This is a linear homogeneous system with 24 variables and 4 equations. That is, there always exists non-trivial solutions. Notice that the indices at the beginning of each equations show that the corresponding column has a 1 only in this position. Thus, the system has column rank equals four. Moreover the result of Pouseele and Tirao says:

Among complete graphs K_n , the only one giving rise to a symplectic Lie algebra is K_3 .

The ideas of the previous example for the complete graph K_4 can be extended to other complete graphs K_n to get closed 2-forms of type II. One has to solve $\frac{n(n-1)(n-2)}{3!}$ equations for $n\frac{n(n-1)}{2}$ coefficients a_{ij}^k , for $n \geq 3$, where $i < j$. In fact the corresponding matrix has column rank equals $\frac{n(n-1)(n-2)}{3!}$, which is verified at the column of a_{ij}^k for all possible i, j, k with $i < j < k$, and $1 \leq i$.

The associated Lie algebra for K_n is the free 2-step nilpotent Lie algebra in n generators.

Remark 4.6. The Lie algebra induced from the graph K_3 is the Lie algebra \mathfrak{f}_6 . It admits closed 2-forms of type II. Moreover there exists symplectic forms induced from 2-forms of type II. In fact, take for instance the 2-form induced from a matrix as in page 15 with $d = j = h = 1$ and the other coefficients equal zero, is symplectic.

Acknowledgements

The authors kindly thank the reviewers for the careful reading of the work and for helpful suggestions to improve the first version.

References

- [1] C. BENSON, C. S. GORDON, Kähler and symplectic structures on nilmanifolds, *Topology* **27** (1988), No. 4, 513–518 .

- [2] D. BURDE, Characteristically nilpotent Lie algebras and symplectic structures, *Forum Mathematicum* **18** (2006), No. 5, 769–787.
- [3] A. CĂP, J. SLOVAK, *Parabolic geometries I, Background and general theory*, Math. Surveys and Monographs, Vol 154, AMS, Providence, R.I., 2019.
- [4] B. Y. CHU, Symplectic homogeneous spaces, *Trans. Am. Math. Soc.* **197** (1974), 145–159.
- [5] J. M. DARDIÉ, A. MEDINA, Double extension symplectique d'un groupe de Lie symplectique (in French) [Symplectic double extension of a symplectic Lie group], *Adv. Math.* **117** (1996), 208–227.
- [6] V. DEL BARCO, Symplectic structures on nilmanifolds: an obstruction for their existence, *J. Lie Theory* **24** (2014), No. 3, 889–908.
- [7] V. DEL BARCO, Symplectic structures on free nilpotent Lie algebras, *Proc. Japan Acad. Ser. A Math. Sci.*, **95** (2019), No. 8, 88–90.
- [8] I. DOTTI, P. TIRAO, Symplectic structures on Heisenberg-type nilmanifolds, *Manuscripta Math.* **102** (2000), 383–401.
- [9] P. EBERLEIN, Geometry of 2-step nilpotent Lie groups with a left-invariant metric, *Ann. Sci. E. N. S. 4 série* **27** (1994), No. 5, 611–660.
- [10] M. FERNÁNDEZ, A. GRAY, Compact symplectic solvmanifolds not admitting complex structures, *Geom. Dedicata* **34** (1990), 295–299.
- [11] J. GÓMEZ, A. JIMÉNEZ-MERCHÁN, Y. KHAKIMDJANOV, Symplectic structures on filiform Lie algebras, *Journal of Pure and Applied Algebra*, **156** (2001), No. 1, 15–31 .
- [12] M. GOZE, A. BOUYAKOUB, Sur les algèbres de Lie munies d'une forme symplectique (in French) [About Lie algebras with a symplectic form], *Rend. Sem. Fac. Sci. Univ. Cagliari* **57** (1987), 85–97.

- [13] M. GOZE, E. REMM, Contact and Frobeniusian forms on Lie groups, *Differ. Geom. Appl.* **35** (2014), 74–94.
- [14] M. GOZE, E. REMM, Symplectic structures on 2-step nilpotent Lie algebras, arXiv:1510.08212 (2015).
- [15] A. KAPLAN, Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms, *Trans. Am. Math. Soc.* **258** (1980), 147–153.
- [16] J. LAURET, D. OSCARI, On nonsingular two-step nilpotent Lie algebras, *Math. Res. Lett.* **21** (2014), No. 3, 553–583 .
- [17] F. LEVSTEIN, A. TIRABOSCHI, Regular metabelian Lie algebras, in: *Geometry and Representation Theory of Real and p -adic groups. Progress in Mathematics*, vol. **158**, Birkhäuser Boston, 1998, pp. 197–207.
- [18] D. V. MILLIONSHCHIKOV, R. JIMENEZ, Geometry of central extensions of nilpotent Lie algebras, *Proc. Steklov Inst. Math.* **305** (2019), 209–231.
- [19] R. MONTGOMERY, *A tour of subriemannian geometries, their geodesics and applications*, AMS Mathematical Surveys and Monographs 91, AMS, Providence, R.I., 2002.
- [20] K. NOMIZU, On the cohomology of compact homogeneous spaces of nilpotent Lie groups, *Ann. of Math.* **59** (1954), 531–538.
- [21] G. OVANDO, Four Dimensional Symplectic Lie Algebras, *Beiträge zur Algebra und Geometrie* **47** (2006), No. 2, 419–434.
- [22] G. OVANDO, M. SUBILS, Magnetic fields on non-singular 2-step nilpotent Lie groups, ArXiv:2210.12180 (2022).
- [23] G. OVANDO, M. SUBILS, Magnetic Trajectories on 2-Step Nilmanifolds, *J. Geom. Anal.* **33** (2023), 186.

- [24] H. POUSEELE, P. TIRAO, Compact symplectic nilmanifolds associated with graphs, *J. Pure Appl. Algebra* **213** (2009), No. 9, 1788–1794.