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Some open problems in Noether-Lefschetz theory for toric varieties

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Abstract. On a quasi-smooth hypersurface X in an n-dimensional projective simplicial toric variety $\mathbb{P}^n_\Sigma$ associated to a fan Σ, the morphism $i^* : H^p(\mathbb{P}^n_\Sigma, \mathbb{Q}) \to H^p(X, \mathbb{Q})$ induced by the inclusion, is injective for $p = \dim X$ and an isomorphism for $p \leq \dim X - 1$. When $n = 2k+1$ one can define the Noether-Lefschetz locus NL_β as the locus of quasi-smooth hypersurfaces of degree β such that i^* restricted to the middle algebraic cohomology is not an isomorphism. In [6] Bruzzo and Grassi proved a Noether-Lefschetz type theorem: if the projective simplicial toric variety is Oda, i.e., the multiplication morphism

$$H^0(\mathcal{O}_{\mathbb{P}^{2k+1}_\Sigma}(\alpha)) \otimes H^0(\mathcal{O}_{\mathbb{P}^{2k+1}_\Sigma}(\gamma)) \to H^0(\mathcal{O}_{\mathbb{P}^{2k+1}_\Sigma}(\alpha + \gamma)))$$

is surjective whenever α and γ are an ample and nef class, respectively, then, on a very general hypersurface X,

$$i^* : H^{k,k}(\mathbb{P}^{2k+1}_\Sigma, \mathbb{Q}) \to H^{k,k}(X, \mathbb{Q})$$

is an isomorphism. Hence, NL_β is a countable union of closed subschemes in the projective complete linear system $\mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^{2k+1}_\Sigma}(\beta))$. The aims of this review article are to give a brief survey and to present some open problems of the Noether-Lefschetz locus and its components.

Keywords: Noether-Lefschetz components, toric varieties, Hodge theory.

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1 Introduction

What is nowadays the Noether-Lefschetz theorem was stated in 1882 by Max Noether, and was proved in 1920 by Salomon Lefschetz using algebraic topological methods. In Lefschetz's words:

"It was my lot to plant the harpoon of algebraic topology into the body of the whale of algebraic geometry".

The classical Noether-Lefschetz theory is about the Picard number of surfaces in the 3-dimensional projective space. Let $\mathcal{U}_d \subset \mathbb{P}H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d))$ be the locus of smooth surfaces of degree d in $\mathbb{P}^3$, with $d \geq 4$; then the very general surface in $\mathcal{U}_d$ has Picard number 1. For a historical perspective of the Noether-Lefschetz problem and exhaustive references the reader may consult [5].

In Section 3, we present an extension of the previous result, a Noether-Lefschetz type theorem for toric varieties, proved by Bruzzo and Grassi in [6]:

Let $X = \{f_0 = 0\}$ be a quasi-smooth hypersurface of an odd-dimensional projective simplicial toric variety $\mathbb{P}^{2k+1}_\Sigma$.

Theorem 3.8 If the multiplication morphism on the Jacobian ring of X

$$\gamma_k : R(f_0)_\beta \otimes R(f_0)_{k\beta-\beta_0} \to R(f_0)_{(k+1)\beta-\beta_0}$$

is surjective, where β_0 is the anticanonical class of $\mathbb{P}^{2k+1}$ then, for f in the complement of a countable union of closed subschemes of positive codimension, one has

$$H^{k.k}(X_f, \mathbb{Q})/i^*(H^{k,k}(\mathbb{P}^{2k+1}_\Sigma, \mathbb{Q})) = 0.$$

In Section 4, a survey of [11], we establish lower and upper bounds for the codimension of the irreducible components of the Noether-Lefschetz locus. In subsection 4.1, we obtain the lower bound, which, following the terminology in [5], we call the "explicit Noether-Lefschetz theorem for toric varieties", namely:

Theorem 4.1 Let $\mathbb{P}_{\Sigma}^{2k+1}$ be a Gorenstein projective simplicial toric variety, η a 0-regular primitive ample Cartier class, and β a Cartier class such that $k\beta - \beta_0 = n\eta$ $(n > 0)$, where β_0 is the anticanonical class of $\mathbb{P}_{\Sigma}^{2k+1}$. Assume that the multiplication morphism $S_\beta \otimes S_{n\eta} \to S_{\beta+n\eta}$ is surjective, and that $H^q(\mathbb{P}_{\Sigma}^{2k+1}, \mathcal{O}_{\mathbb{P}_{\Sigma}^{2k+1}}(\beta - q\eta)) = 0$ for $q = 1, \ldots, 2k$; then

$$n + 1 \leq \operatorname{codim} Z$$

for every irreducible component Z of the Noether-Lefschetz locus NL_β.

In subsection 4.2, using the Hodge theory for hypersurfaces in complete simplicial toric varieties, and the orbifold structure of the quasi-smooth hypersurfaces (see [3]), extending the ideas in [7] we establish an upper bound, specifically:

Theorem 4.3 $\operatorname{codim} Z \leq h^{k-1,k+1}(X_f)$ for every irreducible component Z of the Noether-Lefschetz locus $NL_{\lambda,U}^{k,\beta}$.

In Section 5, we show a Noether-Lefschetz type theorem for quasi-smooth intersection subvarieties, Theorem 2.5 in [11].

Finally, in Section 6 we present some open problems related to all the previous sections which we divide into the following subsections:

- 6.1 Oda and Hodge conditions.

- 6.2 Constructing Noether-Lefschetz components of a given codimension.

- 6.3 Components of the Noether-Lefschetz locus with maximal codimension and density.

- 6.4 A prediction of the Hodge conjecture.

- 6.5 An extended Noether-Lefschetz locus.

2 Preliminaries and Notation

2.1 The Cox ring and toric varieties

Definition 2.1. Let Y be a complete normal variety with finitely generated Class group. The Cox ring of Y is the graded ring

$$\mathrm{Cox}(Y) := \bigoplus_{[D]\in \mathrm{Cl}(Y)} H^0(\mathcal{O}_Y(D))$$

Definition 2.2. A toric variety is an irreducible variety Y containing a torus $T \simeq (\mathbb{C}^*)^n$ as a Zariski open subset such that the action, $T \times T \to T$, of T on itself extends to an algebraic action of T on Y.

Theorem 2.3 (Corollary 4.4 in [4])**.** *Let Y be a complete normal variety where any two points of Y are contained in a common open affine neighborhood and it has finitely generated class group. Then* $\mathrm{Cox}(Y)$ *is a polynomial ring if and only if Y is a toric variety.*

Remark 2.4. The Cox ring encodes a lot of information of a given variety, see [1] for an exhaustive study of this ring. So, for example, Mori dream spaces can be characterized via its Cox ring, see [20] for more details.

2.2 Construction of toric varieties via fans

Let M be a free abelian group of rank n, let $N = Hom(M, \mathbb{Z})$, and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$.

Definition 2.5.

- A convex subset $\sigma \subset N_{\mathbb{R}}$ is a rational k-dimensional simplicial cone if there exist k linearly independent primitive elements $e_1, \dots, e_k \in N$ such that $\sigma = \{\mu_1 e_1 + \cdots + \mu_k e_k\}$.
- The generators e_i are integral if for every i and any nonnegative rational number μ the product μe_i is in N only if μ is an integer.
- Given two rational simplicial cones σ, σ', one says that σ' is a face of σ ($\sigma' < \sigma$) if the set of integral generators of σ' is a subset of the set of integral generators of σ.

- A finite set $\Sigma = \{\sigma_1, \dots, \sigma_t\}$ of rational simplicial cones is called a rational simplicial complete n-dimensional fan if:

  1. all faces of cones in Σ are in Σ;
  2. if $\sigma, \sigma' \in \Sigma$ then $\sigma \cap \sigma' < \sigma$ and $\sigma \cap \sigma' < \sigma'$;
  3. $N_{\mathbb{R}} = \sigma_1 \cup \cdots \cup \sigma_t$.

A rational simplicial complete n-dimensional fan Σ defines an n-dimensional toric variety $\mathbb{P}^n_\Sigma$ having only orbifold singularities which we assume to be projective. Moreover, $T := N \otimes_{\mathbb{Z}} \mathbb{C}^* \simeq (\mathbb{C}^*)^n$ is the torus action on $\mathbb{P}^n_\Sigma$. We denote by $\Sigma(i)$ the i-dimensional cones of Σ and each $\rho \in \Sigma$ corresponds to an irreducible T-invariant Weil divisor D_ρ on $\mathbb{P}^n_\Sigma$. Let $\mathrm{Cl}(\Sigma)$ be the group of Weil divisors on $\mathbb{P}^n_\Sigma$ modulo rational equivalences.

The Cox ring of $\mathbb{P}^n_\Sigma$ is the polynomial ring $S = \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)]$, S has the $\mathrm{Cl}(\Sigma)$-grading, a Weil divisor $D = \sum_{\rho \in \Sigma(1)} u_\rho D_\rho$ determines the monomial $x^u := \prod_{\rho \in \Sigma(1)} x_\rho^{u_\rho} \in S$ and conversely $\deg(x^u) = [D] \in \mathrm{Cl}(\Sigma)$.

For a cone $\sigma \in \Sigma$, $\hat{\sigma}$ is the set of 1-dimensional cones in Σ that are not contained in σ and $x^{\hat{\sigma}} := \prod_{\rho \in \hat{\sigma}} x_\rho$ is the associated monomial in S.

Definition 2.6. The irrelevant ideal of $\mathbb{P}^n_\Sigma$ is the monomial ideal $B_\Sigma :=< x^{\hat{\sigma}} \mid \sigma \in \Sigma >$ and the zero locus $Z(\Sigma) := \mathbb{V}(B_\Sigma)$ in the affine space $\mathbb{A}^{\#\Sigma(1)} := \mathrm{Spec}(S)$ is the irrelevant locus.

Proposition 2.7 (Theorem 5.1.11 [17])**.** *The toric variety $\mathbb{P}^n_\Sigma$ is a categorical quotient $\mathbb{A}^{\#\Sigma(1)} \setminus Z(\Sigma)$ by the group $Hom(\mathrm{Cl}(\Sigma), \mathbb{C}^*)$ and the group action is induced by the $\mathrm{Cl}(\Sigma)$-grading of S.*

Let us denote by $U(\Sigma)$ the open set $\mathbb{A}^{\#\Sigma(1)} \setminus Z(\Sigma)$ and by $D(\Sigma)$ the group $Hom(\mathrm{Cl}(\Sigma), \mathbb{C}^*)$.

Remark 2.8. The rank of the class group, the number of rays and the dimension of the toric variety are related by the following equation $rk(\mathrm{Cl}(\Sigma)) = \#\Sigma(1) - n$.

2.3 Quasi-smooth subvarieties.

Definition 2.9. A subvariety $X \subset \mathbb{P}^n_\Sigma$ is quasi-smooth if $\mathbb{V}(I_X) \subset \mathbb{A}^{\#\Sigma(1)}$ is smooth outside $Z(\Sigma)$.

Example 2.10. Quasi-smooth hypersurfaces or more generally quasi-smooth intersections are quasi-smooth subvarieties (see [3] or [23] for more details).

Remark 2.11. Quasi-smooth subvarieties are suborbifolds of $\mathbb{P}^n_\Sigma$ in the sense of Satake in [27]. Intuitively speaking they are subvarieties whose only singularities come from the ambient space.

Proposition 2.12 (Proposition 4.15 in [3]). *If $f \in H^0(\mathcal{O}_{\mathbb{P}^n_\Sigma}(\beta))$ is a general section for β an ample class, then its zero locus $X \subset \mathbb{P}^n_\Sigma$ is a quasi-smooth hypersurface.*

2.4 Cayley trick

The Cayley trick is a way to associate to a quasi-smooth intersection subvariety a quasi-smooth hypersurface. Let $L_1, \dots, L_s$ be line bundles on $\mathbb{P}^n_\Sigma$ and let $\pi : \mathbb{P}(E) \to \mathbb{P}^n_\Sigma$ be the projective space bundle associated to the vector bundle $E = L_1 \oplus \cdots \oplus L_s$. It is known that $\mathbb{P}(E)$ is a $(n+s-1)$-dimensional simplicial toric variety whose fan depends on the degrees of the line bundles and the fan Σ. Furthermore if the Cox ring, without considering the grading, of $\mathbb{P}^n_\Sigma$ is $\mathbb{C}[x_1, \dots, x_r]$ then the Cox ring of $\mathbb{P}(E)$ is

$$\mathbb{C}[x_1, \dots, x_r, y_1, \dots, y_s]$$

Moreover for X a quasi-smooth intersection subvariety cut off by $f_1, \dots, f_s$ such that $\deg(f_i) = [L_i]$, we relate the hypersurface Y cut off by $F = y_1 f_1 + \cdots + y_s f_s$ which turns out to be quasi-smooth, for more details see Section 2 in [23] and for an application of this trick see [24].

For simplicial toric varieties we have a Hilbert's Nullstellensatz theorem, i.e., there is a $1-1$ correspondence between closed subvarieties and radical homogeneous ideals. Furthermore, all the closed subvarieties arise in this way.

We denote $\mathbb{P}(E)$ as $\mathbb{P}^{n+s-1}_{\Sigma,X}$ to keep track of its relation with X and $\mathbb{P}^n_\Sigma$.

Remark 2.13. There is a morphism $\iota : X \to Y \subset \mathbb{P}^{n+s-1}_{\Sigma,X}$. Moreover every point $z = (x, y) \in Y$ with $y \neq 0$ has a preimage. Hence any subvariety $W = \mathbb{V}(I_W) \subset X \subset \mathbb{P}^n_\Sigma$ has a natural interpretation in Y, i.e., $\mathbb{V}(I_W) =: W' \subset Y \subset \mathbb{P}^{n+s-1}_{\Sigma,X}$ such that $\pi(W') = W$.

2.5 Oda varieties

Definition 2.14. A toric variety $\mathbb{P}^n_\Sigma$ is an Oda variety if the multiplication map $S^\alpha \otimes S^\gamma \to S^{\alpha+\gamma}$ is surjective whenever α is an ample class and γ is a nef one.

This definition was introduced in [26] by Oda in a more general setting and it can be stated in terms of the Minkowski sum of polytopes, i.e., the sum $P_\alpha + P_\gamma$ of the polytopes associated with the line bundles $\mathcal{O}_{\mathbb{P}^n_\Sigma}(\alpha)$ and $\mathcal{O}_{\mathbb{P}^n_\Sigma}(\gamma)$ is equal to the Minkowski sum $P_{\alpha+\gamma}$, the polytope associated with the line bundle $\mathcal{O}_{\mathbb{P}^n_\Sigma}(\alpha + \gamma)$.

Proposition 2.15 (Corollary 4.2 in [21])**.**

1. *A smooth toric variety with Picard number 2 is an Oda variety*
2. *A total space of a toric projective bundle over an Oda variety is also an Oda variety*

3 A Noether-Lefschetz type theorem

This section is an overview of the work of Bruzzo and Grassi in [6].

3.1 Primitive cohomology of a hypersurface

Let X be a quasi-smooth hypersurface in $\mathbb{P}^n_\Sigma$, then the morphism $i^* : H^{n-1}(\mathbb{P}^n_\Sigma, \mathbb{C}) \to H^{n-1}(X, \mathbb{C})$ induced by the inclusion is injective by propositionosition 10.8 in [3].

Definition 3.1. The primitive cohomology $H^{n-1}_{\text{prim}}(X)$ is the quotient

$$H^{n-1}(X,\mathbb{C})/i^*(H^{n-1}(\mathbb{P}^n_\Sigma,\mathbb{C})).$$

Remark 3.2. Let $i_*: H^{n-1}(X,\mathbb{Q}) \to H^{n+1}(\mathbb{P}^n_\Sigma,\mathbb{Q})$ be the Gysin map. The $\ker i_*$ is known in the literature as the variable cohomology or vanishing cohomology, see for example [12] and [31], respectively. In degree $n-1$ the variable or vanishing cohomology and the primitive cohomology coincide.

The primitive cohomology has a pure Hodge structure

$$H^{n-1}_{\text{prim}}(X) = \bigoplus_{p=0}^{n-1} H^{p,n-1-p}_{\text{prim}}(X).$$

inherited from the pure Hodge structures of $H^{n-1}(X,\mathbb{C})$ and $H^{n-1}(\mathbb{P}^n_\Sigma,\mathbb{C})$.

Proposition 3.3 (Proposition 2.10 in [6])**.** *There is a natural isomorphism*

$$H^{p,n-1-p}_{\text{prim}}(X) \simeq \frac{H^0(\Omega^n_{\mathbb{P}^n_\Sigma}(n+1-p)X)}{H^0(\Omega^n_{\mathbb{P}^n_\Sigma}(n-p)X) + dH^0(\Omega^n_{\mathbb{P}^n_\Sigma}(n-p)X)}$$

The resulting projection map multiplied by the factor $(-1)^{p-1}/(n+1-p)!$ we denote by

$$r_p : H^0(\Omega^n_{\mathbb{P}^n_\Sigma}(n+1-p)X) \to H^{p,n-1-p}_{\text{prim}}(X)$$

and we call it the p^{th}-residue map.

3.2 The moduli space of ample hypersurfaces

This is a summary of the principal results of Section 13 in [3] which are key points of the proof of the Noether-Lefschetz theorem of this section.

Let $Aut(\mathbb{P}^n_\Sigma)$ be the automorphism group of $\mathbb{P}^n_\Sigma$. Given $\beta \in \mathrm{Cl}(\Sigma)$, we denote by $Aut_\beta(\mathbb{P}^n_\Sigma)$ the subgroup of automorphism preserving β.

When we describe $\mathbb{P}^n_\Sigma$ as the quotient $U(\Sigma)/D(\Sigma)$, it is clear that $Aut(\mathbb{P}^n_\Sigma)$ does not act on $U(\Sigma)$ but Cox in [15] proved that there exists a short exact sequence

$$1 \to D(\Sigma) \to \widetilde{Aut}(\mathbb{P}^n_\Sigma) \to Aut(\mathbb{P}^n_\Sigma) \to 1$$

where $\widetilde{Aut}(\mathbb{P}^n_\Sigma)$ is the group of automorphisms of $\mathbb{A}^{\#\Sigma(1)}$ which preserves $U(\Sigma)$ and normalizes $D(\Sigma)$. Any element $\psi \in \widetilde{Aut}(\mathbb{P}^n_\Sigma)$ induces an automorphim $\psi : S \to S$ satisfying $\psi(S_\gamma) = S_{\psi(\gamma)}$.

Definition 3.4. Given $\beta \in \mathrm{Cl}(\Sigma)$, let $\widetilde{Aut}_\beta(\mathbb{P}^n_\Sigma)$ be the subgroup of $\widetilde{Aut}(\mathbb{P}^n_\Sigma)$ preserving β.

Let $\widetilde{Aut}^0(\mathbb{P}^n_\Sigma)$ be the connected component of the identity of $\widetilde{Aut}(\mathbb{P}^n_\Sigma)$. It is canonically isomorphic to the group $Aut_g(S)$ of $\mathrm{Cl}(\Sigma)$-graded automorphisms of S.

If $\beta \in \mathrm{Cl}(\Sigma)$ is an ample class then,

$$\mathcal{U}_\beta/\widetilde{Aut}_\beta(\mathbb{P}^n_\Sigma) := \{f \in S^\beta \mid f \text{ is quasi-smooth}\}/\widetilde{Aut}_\beta(\mathbb{P}^n_\Sigma)$$

should be a coarse moduli space. The problem is that $\widetilde{Aut}_\beta(\mathbb{P}^n_\Sigma)$ does not need to be a reductive group, i.e., the quotient may not exist. However, there is a non-empty open set U such that the quotient

$$\mathcal{M}^0_\beta := U/\widetilde{Aut}_\beta(\mathbb{P}^n_\Sigma)$$

exists (See Section 2 in [16] for more details).

Proposition 3.5 (Proposition 13.7 in [3]). *If β is ample and $f \in S^\beta$ is generic then $R(f)_\beta$ is naturally isomorphic to $T_X\mathcal{M}_\beta$, the tangent space of the generic coarse moduli space of quasi-smooth hypersurfaces of $\mathbb{P}^n_\Sigma$ with divisor class β.*

Proposition 3.6 (Proposition 3.3 in [6]). *There is a morphism*

$$\gamma_p : T_X\mathcal{M}_\beta \otimes H^{p,n-1-p}_{\mathrm{prim}}(X) \to H^{p-1,n-p}_{\mathrm{prim}}(X)$$

such that the diagram

$$\begin{array}{ccc} H^0(\mathcal{O}_{\mathbb{P}^n_\Sigma}(X)) \otimes H^0(\Omega^n_{\mathbb{P}^n_\Sigma}(n-p)X) & \xrightarrow{\cup} & H^0(\Omega^n_{\mathbb{P}^n_\Sigma}(n+1-p)X) \\ \downarrow{\scriptstyle \phi\otimes r_p} & & \downarrow{\scriptstyle r_{p-1}} \\ T_X\mathcal{M}_\beta \otimes H^{p,n-1-p}_{\text{prim}}(X) & \xrightarrow[\gamma_p]{} & H^{p-1,n-p}_{\text{prim}}(X) \end{array}$$

is commutative.

For X defined by the homogeneous polynomial f_0, recall that the Jacobian ring $R(f_0)$ is the quotient of S by the Jacobian ideal of X.

Proposition 3.7 (Proposition 3.4 in [6]). *The morphism γ_p coincides with the multiplication in the ring $R(f_0)$,*

$$R(f_0)_\beta \otimes R(f_0)_{(n-p)\beta-\beta_0} \to R(f_0)_{(n-p+1)\beta-\beta_0}$$

Now we have all the machinery to enunciate a Noether-Lefschetz type theorem.

Theorem 3.8 (Lemma 3.7 in [6]). *If for $n = 2k+1$, the multiplication morphism γ_k is surjective, then for f in the complement of a countable union of closed subschemes of positive codimension one has,*

$$H^{k.k}_{prim}(X_f, \mathbb{Q}) = 0.$$

Corollary 3.9. *Let $\mathbb{P}^3_\Sigma$ be a 3-dimensional simplicial projective toric variety and let X be a very general hypersurface with degree β. If the morphism γ_2 is surjective, then X and $\mathbb{P}^3_\Sigma$ have the same Picard number.*

Remark 3.10. Oda varieties satisfy the surjectivity requirement in the previous Theorem.

4 Codimension bounds for the Noether-Lefschetz components for toric varieties

This section is an overview of [10], so see that paper for more details.

4.1 An explicit Noether-Lefschetz theorem in toric varieties

This section is a natural extension to higher dimensions of the ideas developed in [7, 22] for the case of threefolds. To this end there are two points to consider:

1. Let

$$S = \bigoplus_{\alpha \in \mathrm{Cl}(\Sigma)} S^\alpha$$

be the Cox ring of the toric variety $\mathbb{P}_\Sigma^{2k+1}$ under consideration. In [7, 22] the following assumption was made in the case $k = 1$. Let β and η be ample classes in $\mathrm{Pic}(\mathbb{P}_\Sigma^3)$, with η primitive and 0-regular (in the sense of Castelnuovo regularity), and $\beta - \beta_0 = n\eta$ for some $n \geq 0$, where β_0 is the anticanonical class of $\mathbb{P}_\Sigma^3$. Then one assumes that the multiplication map $S^\beta \otimes S^{n\eta} \to S^{\beta+n\eta}$ is surjective; this implies that a very general quasi-smooth surface of degree β in $\mathbb{P}_\Sigma^3$ has the same Picard number as $\mathbb{P}_\Sigma^3$. In the higher dimensional case, if we assume again the surjectivity of the multiplication map, using Theorem 10.13 and propositionosition 13.7 in [3], and Lemma 3.7 in [7], one proves that the primitive cohomology of degree $2k$ of a very general quasi-smooth hypersurface of degree β is zero. Of course we recover the result of [18] when $k = 1$.

2. In [7, 22] it was also assumed that $H^1(\mathcal{O}_{\mathbb{P}_\Sigma^3}(\beta - \eta)) = H^2(\mathcal{O}_{\mathbb{P}_\Sigma^3}(\beta - 2\eta)) = 0$, which allowed one to conclude that a certain vector bundle was 1-regular with respect to η. Here we assume

$$H^q(\mathcal{O}_{\mathbb{P}_\Sigma^{2k+1}}(\beta - q\eta)) = 0 \qquad \text{for} \quad 1 \leq q \leq 2k \tag{4.1}$$

which is the same regularity for the analogue of that vector bundle.

The next Theorem establishes the lower bound for the codimension of the components of the Noether-Lefschetz locus. Recall that a Gorenstein variety is a variety whose canonical divisor is Cartier.

Theorem 4.1 (Theorem 2.1 in [10])**.** *Let $\mathbb{P}_\Sigma^{2k+1}$ be a Gorenstein projective simplicial toric variety, η a 0-regular primitive ample Cartier class, and β a*

Cartier class such that $k\beta - \beta_0 = n\eta$ $(n > 0)$*, where* β_0 *is the anticanonical class of* $\mathbb{P}^{2k+1}_{\Sigma}$*. Assume that the multiplication morphism* $S^{\beta} \otimes S^{n\eta} \to S^{\beta+n\eta}$ *is surjective, and that* $H^q(\mathcal{O}_{\mathbb{P}^{2k+1}_{\Sigma}}(\beta - q\eta)) = 0$ *for* $q = 1, \ldots, 2k$*; then*

$$n + 1 \leq \operatorname{codim} Z$$

for every irreducible component Z *of the Noether-Lefschetz locus* NL_{β}*.*

4.2 Upper bound for the Codimension of the Noether-Lefschetz Components in Toric Varieties

The explicit Noether-Lefschetz Theorem has provided a lower bound for the codimension of the Noether-Lefschetz components. Hodge theory in toric varieties gives us the upper bound. For a class β as in the previous Section, let $f \in \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^{2k+1}_{\Sigma}}(\beta))$ such that $X_f = \{f = 0\}$ is a quasi-smooth hypersurface. Let $\mathcal{U}_{\beta} \subset \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^{2k+1}_{\Sigma}}(\beta))$ be the open subset parametrizing quasi-smooth hypersurfaces and let $\pi : \chi_{\beta} \to \mathcal{U}_{\beta}$ be its tautological family. One considers the local system $\mathcal{H}^{2k} = R^{2k}\pi_*\mathbb{C} \otimes \mathcal{O}_{\mathcal{U}_{\beta}}$ over $\mathcal{U}_{\beta}$. Let $0 \neq \lambda_f \in H^{k,k}_{\text{prim}}(X_f, \mathbb{Q})$ and let U be a contractible open subset around f. Finally, let $\lambda \in \mathcal{H}^{2k}(U)$ be the section defined by λ_f and let $\bar{\lambda}$ be its image in $(\mathcal{H}^{2k}/F^k\mathcal{H}^{2k})(U)$, where $F^k\mathcal{H}^{2k} = \mathcal{H}^{2k,0} \oplus \mathcal{H}^{2k-1,1} \oplus \cdots \oplus \mathcal{H}^{k,k}$.

Definition 4.2. (Local Noether-Lefschetz locus). $NL^{k,\beta}_{\lambda,U} = \{g \in U \mid \bar{\lambda}_g = 0\}$.

Theorem 4.3. $\operatorname{codim} Z \leq h^{k-1,k+1}(X_f)$ *for every irreducible component* Z *of the Noether-Lefschetz locus* $NL^{k,\beta}_{\lambda,U}$*.*

This section is devoted to presenting this theorem. Classically it is a consequence of Griffiths' transversality, which we extended to the context of projective simplicial toric varieties.

The tautological family $\pi : \mathcal{X}_{\beta} \subset \mathcal{U}_{\beta} \times \mathbb{P}^n_{\Sigma} \to \mathcal{U}_{\beta}$ is of finite type and separated since $\mathcal{X}_{\beta}$ and $\mathcal{U}_{\beta}$ are varieties. By Corollary 5.1 in [29] there exists a Zariski open set $\mathcal{U} \subset \mathcal{U}_{\beta}$ such that $\mathcal{X} = \pi^{-1}(\mathcal{U}) \to \mathcal{U}$ is a locally trivial fibration in the classical topology, i.e., there exists an open cover of $\mathcal{U}$ by contractible open sets such that for every element U of the cover and every point $f_0 \in U$ we have $\mathcal{X}_{|U} \simeq \pi^{-1}(U) \simeq U \times X_0$, where $X_0 = \{f_0 = 0\}$, which implies that $X_f \simeq X_0$ for all $f \in U$ as C^∞-orbifolds; moreover, $H^k(X_f) \simeq H^k(X_0)$. Thanks to the local trivialization and as quasi-smooth hypersurfaces are orbifolds [3], we can put an orbifold structure on $\mathcal{X} = \pi^{-1}(U)$.

The Cartan-Lie formula. For every k, let $\mathcal{H}^k$ be the complex vector bundle on $\mathcal{U}_\beta$ associated to the local system $R^k\pi_*\mathbb{C}$. Let Ω be a Zariski k-form on the orbifold $\mathcal{X}$ such that $\Omega_f = \Omega_{|X_f}$ is closed for every $f \in U$; we can associate with it a local section ω of the vector bundle $\mathcal{H}^k$ by letting

$$\omega(f) = [\Omega_f] \in H^k(X_f, \mathbb{C}).$$

Definition 4.4. The interior product $\iota_v(\alpha)$ for a tangent vector v and a differential form α is the $(k-1)$-form $\iota_v(\alpha)(v_1, \ldots, v_{k-1}) := \alpha(v, v_1, \ldots v_k)$.

The following result computes the toric Gauss-Manin connection $\nabla : \mathcal{H}^k \to \mathcal{H}^k \otimes \Omega_{\mathcal{U}_\beta}$ in the direction w restricted to X_0.

Proposition 4.5 (Cartan-Lie Formula). *If $w \in T_{\mathcal{U},X_0}$ and $v \in \Gamma(T_{\mathcal{X}|X_0})$ are such that $\pi_{*,x}(v) = w$ for all $x \in X_0$, one has*

$$\nabla_w(\omega) = \left[\iota_v(d\Omega)_{|X_0}\right] \tag{4.2}$$

Again we take U a contractible open set trivializing $\mathcal{X}_{\mathcal{U}|U} \simeq U \times X_0$.

Definition 4.6. The period map

$$\mathcal{P}^{p,k} : \mathcal{U} \to \mathrm{Grass}(b^{p,k}, H^k(X, \mathbb{C}))$$

is the map $f \mapsto F^pH^k(X_f, \mathbb{C})$, where $F^pH^k(X_f, \mathbb{C})$ is the Hodge filtration of $H^k(X_f, \mathbb{C}) \simeq H^k(X_0, \mathbb{C})$.

Here $b^{p,k} = \dim F^pH^k(X_f, \mathbb{C})$. Note that $\mathcal{P}^{p,k}$ is a map of complex manifolds.

Proposition 4.7. *The period map $\mathcal{P}^{p,k}$ is holomorphic.*

Remark 4.8. There is an intrinsic relation between the differential

$$d\mathcal{P}_f^{p,k}(w)\colon F^pH^k(X_f) \to H^k(X_0)/F^pH^k(X_f)$$

and the covariant derivative $\nabla_w\colon \mathcal{H}^k \to \mathcal{H}^k$, namely, given $\sigma \in F^pH^k(X_f)$ one can construct a local section of $\mathcal{H}^k$ over U

$$\begin{array}{rccc} \tilde{\sigma}\colon & U & \to & H^k(X_u) \\ & f' & \mapsto & \tilde{\sigma}(f') \in F^pH(X_{f'}) \end{array}$$

such that $\tilde{\sigma}(f) = \sigma$. Hence,

$$d\mathcal{P}_f^{p,k}(w)(\sigma) = \nabla_w\tilde{\sigma} \bmod F^pH^k(X_f).$$

Remark 4.9. The Hodge decomposition

$$\mathcal{H}^k = \bigoplus_{p+q=k} \mathcal{H}^{p,q}$$

of the bundle $\mathcal{H}^k$ is not holomorphic, but the bundles of the Hodge filtration

$$F^p\mathcal{H}^k = \bigoplus_{p=0}^{k} \mathcal{H}^{k-p,p}$$

are. This can be shown using the period map. Indeed by the very definition of the period map (see also [30], Section 10.2.1 for the smooth case) one has

$$F^p\mathcal{H}^k \simeq (\mathcal{P}^{p,k})^*\mathcal{T}_{p,k},$$

where $\mathcal{T}_{p,k}$ is the tautological bundle on the Grassmannian $\mathrm{Grass}(b_p, H^k(X_0,\mathbb{C}))$. Thus the bundles $F^p\mathcal{H}^k$ are indeed holomorphic.

Proposition 4.10 (Griffiths Transversality)**.**

$$\nabla F^p\mathcal{H}^k \subset F^{p-1}\mathcal{H}^k$$

Proof. By the Cartan-Lie formula and the above remark

$$d\mathcal{P}^{p,k}_w(\sigma) = \left[\iota_v d\Omega_{|X_0}\right] \mod F^p H^k(X_f).$$

The fact that $\mathcal{P}^{p,k}$ is holomorphic implies that $\iota_v d\Omega_{|X_0} \in F^p H^k(X_f)$ if v is of type $(0,1)$, so that if v is of type $(1,0)$ we get $\iota_v d\Omega_{|X_0} \in F^{p-1} H^k(X_f)$. □

Theorem 4.11. *Each $NL^{k,\beta}_{\lambda,U} \subset \mathcal{U}$ can be defined locally by $h^{k-1,k+1}$ holomorphic equations, where $h^{k-1,k+1} = \operatorname{rk} F^{k-1}\mathcal{H}^{2k}/F^k\mathcal{H}^{2k}$.*

Proof. Once Griffiths transversality has been generalized, the proof goes as in the classical case, see Lemma 3.1 in [30] and section 5.3 in [31]. □

This proves Theorem 4.3.

5 A Noether-Lefschetz type theorem for quasi-smooth intersection subvarieties

This Section is a natural extension of Section 3 to quasi-smooth intersection subvarieties.

5.1 A Lefschetz type theorem

Definition 5.1. X is a codimension s quasi-smooth intersection if

$$V(f_1, \dots, f_s) \cap U(\Sigma)$$

is either empty or a smooth intersection subvariety of codimension s in $U(\Sigma)$.

Theorem 5.2 (Proposition 1.4 in [23])**.** *Let $X \subset \mathbb{P}^n_\Sigma$ be a closed subset, defined by homogeneous polynomials $f_1, \dots f_s \in B_\Sigma$. Then the natural map $i^* : H^i(\mathbb{P}^n_\Sigma) \to H^i(X)$ is an isomorphism for $i < n - s$ and an injection for $i = n - s$. In particular, this is true if the hypersurfaces cut by the polynomials f_i are ample.*

Thanks to the previous theorem we can give an extension of primitive cohomology for quasi-smooth hypersurfaces to quasi-smooth intersection subavarieties.

Definition 5.3. The primitive cohomology group $H^{n-s}_{\text{prim}}(X)$ is the quotient

$$H^{n-s}(X,\mathbb{C})/i^*\left(H^{n-s}(\mathbb{P}^n_\Sigma)\right)$$

5.2 Cayley propositionosition

The next propositionosition we called the Cayley propositionosition.

Proposition 5.4 (Proposition 2.3 in [10])**.** *Let $X = X_1 \cap \cdots \cap X_s$ be a quasi-smooth intersection subvariety in $\mathbb{P}^n_\Sigma$ cut off by homogeneous polynomials $f_1 \dots f_s$. Then for $p \neq \frac{n+s-1}{2}, \frac{n+s-3}{2}$*

$$H^{p-1,n+s-1-p}_{\text{prim}}(Y) \simeq H^{p-s,n-p}_{\text{prim}}(X).$$

Corollary 5.5. *If $n+s = 2(k+1)$,*

$$H^{k+1-s,k+1-s}_{\text{prim}}(X) \simeq H^{k,k}_{\text{prim}}(Y)$$

Remark 5.6. The above isomorphisms are also true with rational coefficients since $H^\bullet(X,\mathbb{C}) = H^\bullet(X,\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$

See the beginning of Section 7.1 in [30] for more details.

5.3 Another Noether-Lefschetz type theorem

The following theorem is a natural extension of Theorem 3.8.

Theorem 5.7 (Theorem 2.5 in [11])**.** *Let $\mathbb{P}^n_\Sigma$ be an Oda projective simplicial toric variety. Then for a very general intersection subvariety X cut off by $f_1, \dots f_s$ such that $n+s = 2(k+1)$ and $\sum_{i=1}^s \deg(f_i) - \beta_0$ is nef, one has that*

$$H^{k+1-s,k+1-s}(X,\mathbb{Q}) \simeq i^*\left(H^{k+1-s,k+1-s}(\mathbb{P}^n_\Sigma,\mathbb{Q})\right).$$

6 Some Open Problems

6.1 Oda and Hodge conditions

The assumption in the Noether-Lefschetz type theorem, Theorem 3.8, is the surjectivity of the multiplication map

$$R(f)_{\beta} \otimes R(f)_{k\beta-\beta_0} \to R(f)_{(k+1)\beta-\beta_0},$$

this assumption is named by Bruzzo and Grassi in [8] as the Hodge condition since the theorem tells us that on a very general hyper-surface with degree β the Hodge conjecture holds, i.e., every rational (k,k)-cohomology class is algebraic.

The Oda condition is a condition on the toric variety and it can be expressed in terms of its Cox ring S, that is, the multiplication map

$$S^{\beta} \otimes S^{k\beta-\beta_0} \to S^{(k+1)\beta-\beta_0}$$

is surjective. It is clear that Oda varieties satisfy the Oda condition and that the Oda condition implies the Hodge condition but whether these two conditions are equivalent is an open problem, see Section 6 in [8] for more details.

6.2 Constructing Noether-Lefschetz components of a given codimension

Combining the codimension bounds along Section 4 we have that every irreducible component in the Noether-Lefschetz locus with codimension c satisfies:

$$n+1 \leq c \leq h^{k-1,k+1}(X). \tag{6.1}$$

In [14] Ciliberto and Lopez proved for the 3-dimensional projective space the existence of irreducible components of NL_d for suitable values of d and c. The existence of these components is another open problem when $\mathbb{P}^{2k+1}_{\Sigma}$ is not $\mathbb{P}^3$.

6.3 Components of the Noether-Lefschetz locus with maximal codimension and their density

For the projective space $\mathbb{P}^3$ and more generally for projective normal 3-folds, the study of the components of the Noether-Lefschetz locus with maximal codimension has been studied by many authors [19, 14, 9, 25]. The components of the Noether-Lefschetz locus with maximal codimension are called general components since they are dense in the Classical and in the Zariski topology.

The upper bound in (6.1) depends on X, an open problem is the independence of the given hypersurface in the Noether-Lefschetz locus for the upper bound codimension, a fact which is true for toric 3-folds, see the propositionosition 4.6 in [7].

Also, it is expected but yet to be proved that the density propositionerty of the components with maximal codimension is also true for $\mathbb{P}^{2k+1}_{\Sigma}$ for $k > 1$.

6.4 A prediction of the Hodge conjecture

The Hodge conjecture predicts that the Local Noether-Lefschetz locus is algebraic, a fact proved in 1995 by Cattani, Deligne and Kaplan in [13] for the classical projective space. In Voisin's words:

"This is a remarkable piece of evidence for the Hodge conjecture".

In 2020 Bakker, Klingler and Tsimerman presented a new proof of the algebraicity of the local Noether-Lefschetz locus in [2] using model theory results.

The algebraicity of this locus is an open problem for $\mathbb{P}^{2k+1}_{\Sigma}$ different from the projective space.

6.5 An extended Noether-Lefschetz locus

Having a Noether-Lefschetz type theorem for quasi-smooth intersection subvarieties, Theorem 5.7, allows us to extend the Noether-Lefschetz locus, namely:

Definition 6.1. The Noether-Lefschetz locus $NL_{\beta_1,\dots,\beta_s}$ of quasi-smooth intersection subvarieties is the locus of s-tuples $(f_1,\dots,f_s) \in |\beta_1|\times\cdots\times|\beta_s|$ such that $X = X_{f_1} \cap \cdots \cap X_{f_s}$ is a quasi-smooth intersection with

$$H^{k+1-s,k+1-s}(X,\mathbb{Q}) \neq i^*(H^{k+1-s,k+1-s}(\mathbb{P}^{2k+1}_{\Sigma},\mathbb{Q})).$$

For these geometrical objects with $s > 1$, all the above-mentioned results and open problems in all previous sections are awaiting to be studied and explored.

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