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The analysis of the harmonic-Spin(7) flow

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Abstract. The group Spin(7) belongs to the list of possible holonomy of an eight-dimensional Riemannian manifold. The weaker notion of Spin(7)-structures plays for manifolds with holonomy Spin(7), the analogue of almost Hermitian for Kähler manifolds. As part of a more general scheme, a notion of harmonicity of Spin(7)-structures is developed with the objective of comparing isometric Spin(7)-structures among themselves. We present here an account of our study in [12] of the harmonic flow of Spin(7)-structures and its analytical properties.

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1 Introduction

At the confluence of holonomy theory and harmonic maps lies the calculus of variations for geometric structures. The catalogue of holonomy groups $\mathrm{Hol}(M,g)$ of an irreducible non-symmetric simply-connected n-dimensional Riemannian manifold is rather brief: $\mathrm{SO}(n)$ (generic), $\mathrm{U}(\frac{n}{2})$

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(Kähler), $\mathrm{SU}(\frac{n}{2})$ (Calabi-Yau), $\mathrm{Sp}(\frac{n}{4})$ (HyperKähler), $\mathrm{Sp}(1)\mathrm{Sp}(\frac{n}{4})$ (Quaternionic Kähler), G_2 $(n = 7)$ and Spin(7) $(n = 8)$. Its main repercussion is on the symmetries of the curvature tensor which then must live in the Lie algebra $\mathfrak{Hol}(M, g)$, so that Calabi-Yau, HyperKähler, G_2 and Spin(7) manifolds are Ricci flat. Our interest is with the last case. As explained in [22], the Clifford algebra Cl_7 is isomorphic to $\mathbb{R}[8] \oplus \mathbb{R}[8]$, so as the group Spin(7) lives in Cl_6, which is isomorphic to one of these two factors, it admits one irreducible representation of dimension eight. Since it must be unitary, Spin(7) can be seen as a 21-dimensional subgroup of SO(8) (cf. [31] for an exposé on its conjugacy classes).

It has been know for quite a while [5, 6, 7] that holonomy in Spin(7) is equivalent to the existence of a parallel 4-form Φ point-wise equal to

$$\begin{aligned}\Phi_p =& dx^{0123} - dx^{0167} - dx^{0527} - dx^{0563} - dx^{0415} - dx^{0426} - dx^{0437} + dx^{4567} \\ & - dx^{4523} - dx^{4163} - dx^{4127} - dx^{2637} - dx^{1537} - dx^{1526},\end{aligned}$$

where $dx^{ijkl} = dx^i \wedge dx^j \wedge dx^k \wedge dx^l$ (though there exist 480 different ways to write down this Euclidean model on $\mathbb{R}^8$).

The group Spin(7) can then be thought of as the group of automorphisms of Φ_p. This geometry comes from the octonions, much like G_2, and has long been suspected to be an impostor waiting to be removed from the list, as happened to Spin(9) with Alekseevsky [1].

The first examples of Riemannian manifolds with Spin(7) holonomy are due to Bryant [8] in 1985 on open subsets of Euclidean spaces and complete examples followed four years later [9] on the spinor bundle of $\mathbb{S}^4$. For compact examples, we had to wait for Joyce in 1996 [19], and a comprehensive account can be found in [20]. Foscolo [15] recently constructed complete non-compact Spin(7)-manifolds with arbitrarily large second Betti number and infinitely many distinct families of asymptotically locally conical Spin(7)-metrics on the same smooth topological M^8. Kovalev [25] adapted in 2003 a conical asymptotical gluing argument to obtain Spin(7)-manifolds from twisted connected sums. More explicit is Salamon's example of the product of $\mathbb{R}^+$ with the nearly-G_2 manifold

SO(5)/SO(3). Spin(7)-Manifolds are hard to find but they are interesting for at least two other reasons:

a) One can define a higher gauge theory of Spin(7)-instantons with the (still fairly remote) hope of defining moduli spaces and invariants, and perhaps a (partial) classification of 8-dimensional manifolds. These "twisted D-T instantons" are vector bundles with a connection A such that their curvature tensor F_A lies in some (irreducible) component Ω^2_{21} (cf. the next section for conventions and notations), equivalently satisfies
$$F_A \wedge \Phi = \star F_A.$$
See [10, 30] for some Spin(7)-instanton constructions.

b) Supersymmetry and string theory have invested a lot in of hope in Spin(7)-manifolds to construct solutions to the gravitino and dilatino equations [18].

However, all these constructions are hard but there exists the softer, more abundant, notion of a Spin(7)-structure.

2 Spin(7)-structures

The best reference for this section, especially pertaining to flows, is Karigiannis' notes [21].

A Spin(7)-structure on an 8-dimensional manifold M is a reduction of the structure group of the frame bundle $\mathrm{Fr}(M)$ to the Lie group $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$. From the point of view of differential geometry, a Spin(7)-structure is a 4-form Φ on M. The existence of such a structure is (equivalent to) a topological condition, cf. [22, Theorem 10.7]: the vanishing of the first and second Stiefel-Whitney classes and, for some orientation
$$p_1^2 - 4p_2 + 8\chi = 0.$$
The space of 4-forms which determine a Spin(7)-structure on M is a subbundle $\mathcal{A}$ of $\Omega^4(M)$, called the bundle of *admissible* 4-forms. This is *not*

a vector subbundle and it is not even an open subbundle, unlike the case for G_2-structures.

A Spin(7)-structure determines a Riemannian metric and an orientation on M in a nonlinear way. Explicit formulas can be found in [21], they are highly involved and it is hard to picture how they could be exploited. But it is crucial to our approach that several Spin(7)-structures will give rise to the same Riemannian metric, much like for the G_2-case. The metric and the orientation determine a Hodge star operator $\star$, and the 4-form is *self-dual*, i.e., $\star\Phi = \Phi$.

Definition 2.1. Let ∇ be the Levi-Civita connection of the metric g_Φ. The pair (M, Φ) is a Spin(7)-*manifold* if $\nabla\Phi = 0$. This is a non-linear partial differential equation for Φ, since ∇ depends on g, which in turn depends non-linearly on Φ. A Spin(7)-manifold has Riemannian holonomy contained in the subgroup $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$. Such a parallel Spin(7)-structure is also called *torsion-free*.

2.1 Decomposition of the space of forms

The existence of a Spin(7)-structure Φ induces a decomposition of the space of differential forms on M into irreducible Spin(7) representations. We have the following orthogonal decomposition, with respect to g_Φ:

$$\Omega^2 = \Omega^2_7 \oplus \Omega^2_{21}, \qquad \Omega^3 = \Omega^3_8 \oplus \Omega^3_{48}, \qquad \Omega^4 = \Omega^4_1 \oplus \Omega^4_7 \oplus \Omega^4_{27} \oplus \Omega^4_{35},$$

where Ω^k_l has pointwise dimension l. Explicitly, Ω^2 and Ω^3 are described as follows:

$$\Omega^2_7 = \{\beta \in \Omega^2 \mid \star(\Phi \wedge \beta) = -3\beta\}, \quad \Omega^2_{21} = \{\beta \in \Omega^2 \mid \star(\Phi \wedge \beta) = \beta\},$$

and

$$\Omega^3_8 = \{X \lrcorner \Phi \mid X \in \Gamma(TM)\}, \quad \Omega^3_{48} = \{\gamma \in \Omega^3 \mid \gamma \wedge \Phi = 0\}.$$

In local coordinates, these spaces of forms are described as, for $\beta \in \Omega^2(M)$,

$$\beta_{ij} \in \Omega^2_7 \iff \beta_{ab}\Phi_{abij} = -6\beta_{ij}, \tag{2.1}$$

$$\beta_{ij} \in \Omega^2_{21} \iff \beta_{ab}\Phi_{abij} = 2\beta_{ij} \tag{2.2}$$

and, for $\gamma \in \Omega^3(M)$,

$$\gamma_{ijk} \in \Omega^3_8 \iff \gamma_{ijk} = X_l \Phi_{ijkl} \quad \text{for some } X \in \Gamma(TM), \tag{2.3}$$

$$\gamma_{ijk} \in \Omega^3_{48} \iff \gamma_{ijk}\Phi_{ijkl} = 0. \tag{2.4}$$

If π_7 and π_{21} are the projection operators on Ω^2, it follows from (2.1) and (2.2) that

$$\pi_7(\beta)_{ij} = \frac{1}{4}\beta_{ij} - \frac{1}{8}\beta_{ab}\Phi_{abij},$$
$$\pi_{21}(\beta)_{ij} = \frac{3}{4}\beta_{ij} + \frac{1}{8}\beta_{ab}\Phi_{abij}.$$

Finally, for $\beta_{ij} \in \Omega^2_{21}$,

$$\beta_{ab}\Phi_{bpqr} = \beta_{pi}\Phi_{iqra} + \beta_{qi}\Phi_{irpa} + \beta_{ri}\Phi_{ipqa},$$

so $\Omega^2_{21} \equiv \mathfrak{so}(7)$ is the Lie algebra of Spin(7).

To describe Ω^4 in local coordinates, we use the operator $\diamond$ for a (p,q)-tensor ξ and $A \in \mathrm{End}(TM)$:

$$\begin{aligned} \diamond\xi : \quad & \mathrm{End}(TM) \to T^{p,q} \\ & A \mapsto A \diamond \xi := \left.\frac{d}{dt}\right|_{t=0} e^{tA}.\xi. \end{aligned}$$

Now, given $A \in \Gamma(T^*M \otimes TM)$, define

$$A \diamond \Phi = \frac{1}{24}(A_{ip}\Phi_{pjkl} + A_{jp}\Phi_{ipkl} + A_{kp}\Phi_{ijpl} + A_{lp}\Phi_{ijkp})dx^i \wedge dx^j \wedge dx^k \wedge dx^l, \tag{2.5}$$

and hence

$$(A \diamond \Phi)_{ijkl} = A_{ip}\Phi_{pjkl} + A_{jp}\Phi_{ipkl} + A_{kp}\Phi_{ijpl} + A_{lp}\Phi_{ijkp}. \tag{2.6}$$

Recall that $\Gamma(T^*M \otimes TM) = \Omega^0 \oplus S_0 \oplus \Omega^2$, and Ω^2 splits further orthogonally, so

$$\Gamma(T^*M \otimes TM) = \Omega^0 \oplus S_0 \oplus \Omega^2_7 \oplus \Omega^2_{21}.$$

With respect to this splitting, we can write $A = \frac{1}{8}(\operatorname{tr} A)g + A_0 + A_7 + A_{21}$ where A_0 is a symmetric traceless 2-tensor. The diamond contraction (2.6) defines a linear map $A \mapsto A \diamond \Phi$, from $\Omega^0 \oplus S_0 \oplus \Omega^2_7 \oplus \Omega^2_{21}$ to $\Omega^4(M)$. The following proposition is proved in [21, Prop. 2.3].

Proposition 2.2. *The kernel of the map $A \mapsto A \diamond \Phi$ is isomorphic to the subspace Ω^2_{21}. The remaining three summands Ω^0, S_0 and Ω^2_7 are mapped isomorphically onto the subspaces Ω^4_1, Ω^4_{35} and Ω^4_7 respectively.*

To understand Ω^4_{27}, we need another characterization of the space of 4-forms using the Spin(7)-structure. Following [21], we adopt the following:

Definition 2.3. On (M, Φ), define a Φ-equivariant linear operator Λ_Φ on Ω^4 as follows. Let $\sigma \in \Omega^4(M)$ and let $(\sigma \cdot \Phi)_{ijkl} = \sigma_{ijmn}\Phi_{mnkl}$. Then

$$\begin{aligned}(\Lambda_\Phi(\sigma))_{ijkl} =& (\sigma \cdot \Phi)_{ijkl} + (\sigma \cdot \Phi)_{iklj} + (\sigma \cdot \Phi)_{iljk} + (\sigma \cdot \Phi)_{jkil} + (\sigma \cdot \Phi)_{jlki} \\ &+ (\sigma \cdot \Phi)_{klij}.\end{aligned}$$

Proposition 2.4. *The spaces Ω^4_1, Ω^4_7, Ω^4_{27} and Ω^4_{35} are all eigenspaces of Λ_Φ with distinct eigenvalues:*

$$\begin{aligned} \Omega^4_1 &= \{\sigma \in \Omega^4 \mid \Lambda_\Phi(\sigma) = -24\sigma\}, & \Omega^4_7 &= \{\sigma \in \Omega^4 \mid \Lambda_\Phi(\sigma) = -12\sigma\}, \\ \Omega^4_{27} &= \{\sigma \in \Omega^4 \mid \Lambda_\Phi(\sigma) = 4\sigma\}, & \Omega^4_{35} &= \{\sigma \in \Omega^4 \mid \Lambda_\Phi(\sigma) = 0\}. \end{aligned}$$

Moreover, the decomposition of $\Omega^4(M)$ into self-dual and anti-self-dual parts is

$$\Omega^4_+ = \{\sigma \in \Omega^4 \mid \star\sigma = \sigma\} = \Omega^4_1 \oplus \Omega^4_7 \oplus \Omega^4_{27}, \quad \Omega^4_- = \{\sigma \in \Omega^4 \mid \star\sigma = -\sigma\} = \Omega^4_{35}.$$

Before we discuss the torsion of a Spin(7)-structure, we note some contraction identities involving the 4-form Φ. In local coordinates $\{x^1, \cdots, x^8\}$, the 4-form Φ is

$$\Phi = \frac{1}{24}\Phi_{ijkl}\, dx^i \wedge dx^j \wedge dx^k \wedge dx^l$$

where Φ_{ijkl} is totally skew-symmetric. We have the following identities, as always summing on repeated indices, which encapsulate the symmetries of a Spin(7)-structure

$$\begin{aligned}
\Phi_{ijkl}\Phi_{abcl} &= g_{ia}g_{jb}g_{kc} + g_{ib}g_{jc}g_{ka} + g_{ic}g_{ja}g_{kb} \\
&\quad - g_{ia}g_{jc}g_{kb} - g_{ib}g_{ja}g_{kc} - g_{ic}g_{jb}g_{ka} \\
&\quad - g_{ia}\Phi_{jkbc} - g_{ib}\Phi_{jkca} - g_{ic}\Phi_{jkab} \\
&\quad - g_{ja}\Phi_{kibc} - g_{jb}\Phi_{kica} - g_{jc}\Phi_{kiab} \\
&\quad - g_{ka}\Phi_{ijbc} - g_{kb}\Phi_{ijca} - g_{kc}\Phi_{ijab} && (2.7) \\
\Phi_{ijkl}\Phi_{abkl} &= 6g_{ia}g_{jb} - 6g_{ib}g_{ja} - 4\Phi_{ijab} && (2.8) \\
\Phi_{ijkl}\Phi_{ajkl} &= 42g_{ia} && (2.9) \\
\Phi_{ijkl}\Phi_{ijkl} &= 336. && (2.10)
\end{aligned}$$

We also have contraction identities involving $\nabla\Phi$ and Φ

$$\begin{aligned}
(\nabla_m\Phi_{ijkl})\Phi_{abkl} &= -\Phi_{ijkl}(\nabla_m\Phi_{abkl}) - 4\nabla_m\Phi_{ijab} \\
(\nabla_m\Phi_{ijkl})\Phi_{ajkl} &= -\Phi_{ijkl}(\nabla_m\Phi_{ajkl}) \\
(\nabla_m\Phi_{ijkl})\Phi_{ijkl} &= 0.
\end{aligned}$$

We now describe the *torsion* of a Spin(7)-structure. Given $X \in \Gamma(TM)$, we know from [21, Lemma 2.10] that $\nabla_X\Phi$ lies in the subbundle $\Omega^4_7 \subset \Omega^4$.

Definition 2.5. The *torsion tensor* of a Spin(7)-structure Φ is the element of $\Omega^1_8 \otimes \Omega^2_7$ defined by expressing $\nabla\Phi$ in the light of Proposition 2.2:

$$\nabla_m\Phi_{ijkl} = (T_m \diamond \Phi)_{ijkl} = T_{m;ip}\Phi_{pjkl} + T_{m;jp}\Phi_{ipkl} + T_{m;kp}\Phi_{ijpl} + T_{m;lp}\Phi_{ijkp} \tag{2.11}$$

where $T_{m;ab} \in \Omega^2_7$, for each fixed m.

Directly in terms of $\nabla\Phi$, the torsion T is given by

$$T_{m;ab} = \frac{1}{96}(\nabla_m\Phi_{ajkl})\Phi_{bjkl} \tag{2.12}$$

Remark 2.6. We remark that the notation $T_{m;ab}$ *should not* be confused with taking two covariant derivatives of T_m. The torsion tensor T is an element of $\Omega^1_8 \otimes \Omega^2_7$ and thus for each fixed index m, $T_{m;ab} \in \Omega^2_7$, but T is not in Ω^3.

Theorem 2.7. *[14] The* Spin(7)*-structure Φ is torsion-free if, and only if, $d\Phi = 0$. Since $\star\Phi = \Phi$, this is equivalent to $d^*\Phi = 0$.*

Finally, the torsion satisfies a 'Bianchi-type identity'. This was first proved in [21, Theorem 4.2], using the diffeomorphism invariance of the torsion tensor. A different proof can be found in [12, Theorem 3.9], using the Ricci identity

$$\nabla_k \nabla_i X_l - \nabla_i \nabla_k X_l = -R_{kilm} X_m.$$

Theorem 2.8. *The torsion tensor T satisfies the following 'Bianchi-type identity'*

$$\nabla_i T_{j;ab} - \nabla_j T_{i;ab} = 2T_{i;am}T_{j;mb} - 2T_{j;am}T_{i;mb} + \frac{1}{4}R_{jiab} - \frac{1}{8}R_{jimn}\Phi_{mnab}. \tag{2.13}$$

Using the Riemannian Bianchi identity, we see that

$$R_{ijkl}\Phi_{ajkl} = -(R_{jkil} + R_{kijl})\Phi_{ajkl} = -R_{iljk}\Phi_{aljk} - R_{ikjl}\Phi_{akjl},$$

hence

$$R_{ijkl}\Phi_{ajkl} = 0.$$

Using this and contracting (2.13) on j and b gives the expression for the Ricci curvature of a metric induced by a Spin(7)-structure:

$$R_{ij} = 4\nabla_i T_{a;ja} - 4\nabla_a T_{i;ja} - 8T_{i;jb}T_{a;ba} + 8T_{a;jb}T_{i;ba}. \tag{2.14}$$

This also proves that the metric of a torsion-free Spin(7)-structure is Ricci-flat, a result originally due to Bonan [5]. Taking the trace of (2.14) gives the scalar curvature R:

$$R = 4\nabla_i T_{a;ia} - 4\nabla_a T_{i;ia} + 8|T|^2 + 8T_{a;jb}T_{j;ba}.$$

Remark 2.9. 1. A classification of Spin(7)-structures was given by Fernandez in [14] and a formulation in terms of spinors can be found in [27].

2. Compact simply-connected Riemannian symmetric spaces cannot carry any invariant Spin(7)-structures and the compact simply-connected almost effective homogeneous space with invariant Spin(7)-structures are $\mathrm{SU}(3)/\{e\}$, some torus bundles over $\Big(\mathrm{SU}(2)/\mathrm{U}(1)\Big)^{\times 3}$ and the Calabi-Eckmann $\mathrm{SU}(3)/\mathrm{SU}(2) \times \mathrm{SU}(2)$.

3. Without requiring invariance of the structure, the 8-dimensional compact simply-connected Riemannian symmetric spaces admitting a Spin(7)-structures are $\mathrm{SU}(3)$, $\mathbb{S}^3 \times \mathbb{S}^3 \times \mathbb{S}^2$, $\mathbb{S}^5 \times \mathbb{S}^3$, $\mathbb{HP}^2$, $\mathrm{Gr}_2(\mathbb{C}^4)$ and the Wolf space $\mathrm{G}_2/\mathrm{SO}(4)$ [2].

3 Harmonicity

The ultimate goal in Spin(7)-geometry is to find parallel structures. Not only is it quite a difficult task involving a non-linear equation and hard analysis but topological obstructions also apply.

An alternative strategy to finding the best among all possible Spin(7)-structures is to introduce a variational problem, for example measuring the default of parallelism, and search for minimisers.

This is the junction point between Spin(7)-geometry and harmonic map theory, though the price to pay is we need to fix the metric, i.e. work within the isometric class of Spin(7)-structures.

Definition 3.1. Two Spin(7)-structures Φ_1 and Φ_2 on M are called *isometric* if they induce the same Riemannian metric, that is, if $g_{\Phi_1} = g_{\Phi_2}$. We will denote by $[\![\Phi]\!]$ the space of Spin(7)-structures that are isometric to a given Spin(7)-structure Φ.

Definition 3.2. Let Φ_0 be a fixed initial Spin(7)-structure on M. The *energy functional* E on the set $[\![\Phi_0]\!]$ is

$$E(\Phi) = \frac{1}{2}\int_M |T_\Phi|^2 \operatorname{vol}_{g_{g_\Phi}}, \tag{3.1}$$

where T_Φ is the torsion of Φ.

Once the variational problem has been delineated, the next step is to derive the corresponding Euler-Lagrange equation. We will call critical points of $E|_{[\![\Phi_0]\!]}$ *harmonic* Spin(7)*-structures* and work out the harmonic equation for Spin(7)-structures and the corresponding isometric flow.

The main ingredient is the representation theory properties outlined in the previous section and the recipe is to follow the treatment of the G_2 case in [26, Section 6], only slightly adapted to specific properties of Spin(7)-geometry.

The link with harmonic map theory is the one-one correspondence between Spin(7)-structures Φ and sections σ of an ad-hoc Spin(7)-twistor bundle N, constructed as the Spin(7) quotient of the SO(8) frame bundle of (M^8, g). The fibres are isometric to $\mathbb{RP}^7$ and parametrise isometric Spin(7)-structures on (M^8, g).

To obtain the equation of harmonicity, one must first and foremost identify the tangent space of fibres in order to be able to consider vertical variations and compare (isometric) Spin(7)-structures among themselves.

The first constituent is the connection form f, which identifies the vertical of the tangent bundle of the "twistor space" with $\mathfrak{m}$ the (naturally reductive) complement of $\mathfrak{so}(7)(=\mathfrak{spin}(7))$ in $\mathfrak{so}(8)$. Sections of this space correspond to Spin(7)-structures and restricting ourselves to the vertical part means we only look at variations through Spin(7)-structures.

If $\tilde{\Phi}$ is the universal Spin(7)-structure, a sort of ideal Spin(7)-structure living a couple of fibre bundles above the manifold M (cf. [26] for particulars), then the connection form is characterised by

$$\nabla_A \tilde{\Phi} = f(A).\tilde{\Phi}.$$

Here $f(A)$ is in $\mathfrak{m}$.

We identify $\mathfrak{so}(8)$ with Ω^2 and $\mathfrak{m}$ is then identified with Ω^2_7.

Since $\tilde{\Phi}$ is in Ω^4 (of the appropriate space), the term $f(A).\tilde{\Phi}$ should be understood as the diamond operator of Equation (2.5), which is just the derivation of the natural action of GL(8) by pulling back forms.

To obtain f we need to find an inverse of the $\diamond$ operator and to do this introduce the triple contraction $\lrcorner_3$ between two four-forms (we follow notations and conventions of [21]):

If $\beta = \frac{1}{2}\beta_{ij}dx^i \wedge dx^j$ then $\beta \diamond \Phi \in \Omega^4$ and put $(\beta \diamond \Phi)\lrcorner_3\Phi$ to be the two-form defined by

$$(\beta \diamond \Phi)\lrcorner_3\Phi = \frac{1}{2}((\beta \diamond \Phi)\lrcorner_3\Phi)_{pq}dx^p \wedge dx^q,$$

where

$$((\beta \diamond \Phi)\lrcorner_3\Phi)_{pq} = (\beta \diamond \Phi)_{pijk}\Phi_{qijk}.$$

Because we are interested in the case $\beta = f(A) \in \Omega^2_7$, we can use $\beta_{ab}\Phi_{abij} = -6\beta_{ij}$ to compute that

$$(\beta \diamond \Phi)\lrcorner_3\Phi = 96\beta.$$

Once we have this, the rest follows relatively easily, if one knows where to pick information in [21]:

- The connection form is then given by

$$96f(A) = \nabla_A\tilde{\Phi}\lrcorner_3\tilde{\Phi}$$

  and, since Spin(7)-structures Φ and sections $\sigma : M \to N$ of the twistor space are related by $\Phi = \tilde{\Phi} \circ \sigma$, we can pull back the above formula to obtain

$$f(d\sigma(X)) = \tfrac{1}{96}(\nabla_X\Phi)\lrcorner_3\Phi$$

  which is precisely the torsion $T(X)$ of (2.12) in the space Ω^2_7.

- The (vertical) energy density of the section $\sigma : M \to \mathrm{N}$ is

$$|d^v\sigma|^2 = |T|^2,$$

so the functional we take, the L^2-norm of the torsion, is exactly the Dirichlet energy of σ (at least up to an additive constant due to the contribution of the horizontal part).

- The vertical tension field is

$$I(\tau^v(\sigma)) = \sum_1^8 \nabla_{e_i}(T(e_i)) - T(\nabla_{e_i} e_i) = \operatorname{div} T.$$

- The flow of sections $\sigma_t : M \to N$

$$\frac{d\sigma_t}{dt} = \tau^v(\sigma_t)$$

is equivalent to

$$I(\frac{d\sigma_t}{dt}) = I(\tau^v(\sigma_t)),$$

where I plays the role of an extended f. We know that $I(\tau^v(\sigma_t)) = \operatorname{div} T_t$ and, generalising to $M \times \mathbb{R}$ (or at least on an interval) all the previous objects, we have that

$$I(\frac{d\sigma_t}{dt}) = \tfrac{1}{96}\frac{d\Phi_t}{dt} \lrcorner_3 \Phi_t.$$

On the other hand, since $\operatorname{div} T_t$ is in Ω^2_7

$$\operatorname{div} T_t = \tfrac{1}{96}(\operatorname{div} T_t \diamond \Phi_t) \lrcorner_3 \Phi_t,$$

and $\lrcorner_3 \Phi_t$ is an isomorphism on Ω^2_7 (its kernel is Ω^2_{21}), we have the isometric flow, with initial value:

$$\begin{cases} \frac{d\Phi}{dt} = \operatorname{div} T \diamond \Phi \\ \Phi(0) = \Phi_0. \end{cases} \tag{HF}$$

Remark 3.3.
- The $\operatorname{div} T$ equation is the vertical part of the harmonic map equation of σ, which is known to admit short-time existence, so this property carries over to our heat flow.
- As (the fibres of) the target are isometric to the real seven-dimensional projective space, they have positive sectional curvature, so there can be no certainty about the long-time existence of the flow (cf. [13]).

- Solitons of such flows are studied for a general group H in [12] and [24].

4 Analysis of the flow I

This section develops tools for the analysis of the isometric flow of Spin(7)-structures. Some proofs of the statements in this section have appeared in full in [12] and we refer to them. Others were consequences of more general arguments and here we present their Spin(7) versions, though they are only adaptations of their G_2 counterparts found in [11].

Let $\{\partial_t, e_1, \ldots, e_7\}$ be an orthonormal (geodesic) frame. First, we use the formula

$$\begin{aligned}(R(e_i,e_j)T)(e_a,e_b,e_c) = &- T(R(e_i,e_j)e_a,e_b,e_c) - T(e_a,R(e_i,e_j)e_b,e_c)\\ &- T(e_a,e_b,R(e_i,e_j)e_c)\end{aligned}$$

to derive a formula for the Laplacian of the torsion of a Spin(7)-structure.

Lemma 4.1. *[12, Lemma 4.12] Let* $\Delta = \operatorname{tr}\nabla_{e_i}\nabla_{e_i}$ *be the Laplacian, then*

$$\begin{aligned}(\Delta T)_{m;ab} = &\nabla_m\nabla_i T_{i;ab} - T_{q;ab}R_{imiq} - T_{i;qb}R_{imaq} - T_{i;aq}R_{imbq} + 2\nabla_i T_{i;ap}T_{m;bp}\\ &+ 2T_{i;ap}\nabla_i T_{m;bp} - 2\nabla_i T_{m;ap}T_{i;bp} - 2T_{m;ap}\nabla_i T_{i;pb} + \tfrac{1}{4}\nabla_i R_{miab}\\ &- \tfrac{1}{8}\nabla_i R_{mipq}\Phi_{pqab} - \tfrac{1}{8}R_{mipq}\nabla_i\Phi_{pqab}.\end{aligned}$$

This allows us to compute a local expression for the evolution of the torsion T.

Proposition 4.2. *[12, Proposition 4.13] Let* $\{\Phi_t\}$ *be a solution of the har-*

monic Spin(7)*-flow* (HF)*, then its torsion evolves according to the equation*

$$
\begin{aligned}
4\frac{\partial}{\partial t}T_{m;is} = {} & 4(\Delta T)_{m;is} \\
& + \nabla_a T_{m;bc}\Big(4T_{a;bp}\Phi_{pcis} + T_{a;ip}\Phi_{bcps} + T_{a;sp}\Phi_{bcip}\Big) + T_{m;bc}\nabla_a T_{a;bp}\Phi_{pcis} \\
& + 3\nabla_a T_{a;ip}T_{m;ps} + \nabla_a T_{a;sp}T_{m;pi} - 2T_{a;ip}\nabla_a T_{m;sp} + 2\nabla_a T_{m;ip}T_{a;sp} \\
& + T_{m;bc}T_{a;bp}\Big(T_{a;pq}\Phi_{qcis} + T_{a;cq}\Phi_{pqis} + 2T_{a;iq}\Phi_{pcqs} + 2T_{a;sq}\Phi_{pciq}\Big) \\
& + \tfrac{1}{2}T_{m;bc}T_{a;ip}\Big(T_{a;pq}\Phi_{bcqs} + 2T_{a;sq}\Phi_{bcpq}\Big) + \tfrac{1}{2}T_{m;bc}T_{a;sp}T_{a;pq}\Phi_{bciq} \\
& + 4T_{q;is}R_{amaq} - (\nabla_a R_{mais} - \tfrac{1}{2}\nabla_a R_{mapq}\Phi_{pqis}) \\
& + T_{a;qs}R_{amiq} + T_{a;iq}R_{amsq} + \tfrac{1}{8}R_{mapq}\nabla_a\Phi_{pqis} - T_{a;qc}R_{ambq}\Phi_{bcis} \\
& - \tfrac{1}{16}R_{mapq}\nabla_a\Phi_{pqbc}\Phi_{bcis}.
\end{aligned}
$$

But the real information is the evolution of the norm of the torsion.

Proposition 4.3. *[12, Proposition 4.14] If $\{\Phi_t\}$ is a solution of the harmonic* Spin(7)*-flow* (HF)*, then the evolution equation for $|T|^2$ is*

$$
\begin{aligned}
2\frac{\partial}{\partial t}|T|^2 = {} & 2\Delta|T|^2 - 4|\nabla T|^2 + 16T_{a;bp}T_{m;bc}T_{a;pq}T_{m;qc} + 16T_{a;bp}T_{m;bc}T_{a;cq}T_{m;pq} \\
& + 16T_{a;qs}T_{m;is}R_{amiq} + 4T_{q;is}T_{m;is}R_{amaq} - 4T_{m;is}\nabla_a R_{mais}.
\end{aligned}
$$

Both the doubling-time estimate and Shi-type estimates can be derived from general properties of the harmonic flow of H-structures (cf. [24]) and do not feature in [12] but proofs specific to Spin(7) can be written.

Lemma 4.4. *[12, Corollary 4.9] There exists $\delta > 0$ such that*

$$\mathcal{T}(t) \leq 2\mathcal{T}(0)$$

for all $0 \leq t \leq \delta$, where

$$\mathcal{T}(t) = \sup_M |T(x,t)|$$

Proof. We follow the arguments of [11, Proposition 3.2] and adapt them to the group Spin(7).

Wlog, we can assume that $|T| > 1$. Then

$$\begin{aligned}\frac{\partial}{\partial t}|T|^2 &= 2\langle \Delta T, T\rangle + 2\langle \nabla T * T * \Phi, T\rangle + 2\langle T * T * T, T\rangle \\ &\quad + 2\langle T * \nabla R * R * \Phi, T\rangle \\ &= \Delta |T|^2 - 2|\nabla T|^2 + 2\langle \nabla T * T * \Phi, T\rangle + 2\langle T * T * T, T\rangle \\ &\quad + 2\langle T * \nabla R * R * \Phi, T\rangle \\ &\leq \Delta |T|^2 - 2|\nabla T|^2 + C|\nabla T||T|^2 + C|T|^4 + C|T|^2 + C|T|\end{aligned}$$

because of the bounded geometry.

Use Young Inequality $ab \leq \frac{1}{2\epsilon}a^2 + \frac{\epsilon}{2}b^2$ to get rid of the term $|\nabla T||T|^2$:

$$\frac{\partial}{\partial t}|T|^2 \leq \Delta |T|^2 + (-2 + \tfrac{C}{2\epsilon})|\nabla T|^2 + C(1 + \tfrac{\epsilon}{2})|T|^4 + C|T|^2 + C|T|,$$

with ϵ large enough to ensure that $(-2 + \frac{C}{2\epsilon}) < 0$.

Then, using $|T| > 1$, we obtain a formula similar to [11, (3.10)]

$$\frac{\partial}{\partial t}|T|^2 \leq \Delta |T|^2 + (-2 + \tfrac{C}{2\epsilon})|\nabla T|^2 + C(1 + \tfrac{\epsilon}{2})|T|^4 + C|T|^2, \tag{4.1}$$

and argue as in [11, page 22] with the maximal principle to get the DTE. $\square$

Shi-type estimates are crucial at several steps of our various arguments in the next section. They essentially control higher-derivatives from a bounds on the (norm of the) torsion and the geometry of the manifold. A much more general version of these Shi-type estimates can be found in [12].

Lemma 4.5 (Shi-type estimates)**.** *[12, Corollary 4.10] There exist constants C_m such that if ($\forall j \in \mathbb{N}$)*

$$|T| \leq K \text{ and } |\nabla^j R| \leq B_j K^{2+j}$$

on $M \times [0, 1/K^2]$ then ($\forall m \in \mathbb{N}$)

$$|\nabla^m T| \leq C_m t^{-\frac{m}{2}} K.$$

Remark 4.6. Note that this version has a conclusion valid over an interval slightly larger than in [12], up to $1/K^2$ instead of $1/K^4$, but this has no bearing on the issue.

Proof. We closely follow the proof by induction in [11], mutatis mutandis, and only indicate the key steps and differences. We use the symbol $*$ to denote various tensor contractions, the precise form of which is unimportant.

The base case of the induction:

We start with the evolution equation for ∇T:

$$\begin{aligned}
&\frac{\partial}{\partial t}\nabla T = \Delta\nabla T + \nabla(\nabla T * T * \Phi) + \nabla T * T * T * \Phi + T * T * T * \nabla\Phi \\
&+ \nabla T * R + T * \nabla R + \nabla^2 R * \Phi + \nabla R * \nabla\Phi + \nabla R * \nabla\Phi * \Phi + R * \nabla^2\Phi * \Phi \\
&+ R * \nabla\Phi * \nabla\Phi + \nabla T * R * \Phi + T * \nabla R * \Phi + T * R * \nabla\Phi,
\end{aligned}$$

therefore

$$\begin{aligned}
&\frac{\partial}{\partial t}|\nabla T|^2 = \Delta|\nabla T|^2 - 2|\nabla^2 T|^2 \\
&+ 2\langle \nabla T, \nabla(\nabla T * T * \Phi) + \nabla T * T * T * \Phi + T * T * T * \nabla\Phi + \nabla T * R \\
&+ T * \nabla R + \nabla^2 R * \Phi + \nabla R * \nabla\Phi + \nabla R * \nabla\Phi * \Phi + R * \nabla^2\Phi * \Phi \\
&+ R * \nabla\Phi * \nabla\Phi + \nabla T * R * \Phi + T * \nabla R * \Phi + T * R * \nabla\Phi\rangle,
\end{aligned}$$

therefore

$$\begin{aligned}
\frac{\partial}{\partial t}|\nabla T|^2 \leq{}& \Delta|\nabla T|^2 - 2|\nabla^2 T|^2 + 2\langle \nabla T, \nabla(\nabla T * T * \Phi)\rangle \\
&+ C|\nabla T|^2|T|^2 + C|\nabla T||T|^4 + C|\nabla T|^2|R| + C|\nabla T||T||\nabla R| \\
&+ C|\nabla T||\nabla^2 R| + C|\nabla T||T|^2|R|.
\end{aligned}$$

Since, by assumption, $|R| \leq B_0 K^2$, $|\nabla R| \leq B_1 K^3$, $|\nabla^2 R| \leq B_2 K^4$ and $|T| \leq K$ we have

$$\frac{\partial}{\partial t}|\nabla T|^2 \leq \Delta|\nabla T|^2 - 2|\nabla^2 T|^2 + 2\langle \nabla T, \nabla(\nabla T * T * \Phi)\rangle + CK^2|\nabla T|^2 + CK^4|\nabla T|.$$

As

$$\langle \nabla T, \nabla(\nabla T * T * \Phi)\rangle \leq CK|\nabla T||\nabla^2 T| + C|\nabla T|^3 + CK^2|\nabla T|^2,$$

with Young Inequality we have

$$2CK|\nabla T||\nabla^2 T| \leq \tfrac{CK^2}{\epsilon}|\nabla T|^2 + C\epsilon|\nabla^2 T|^2,$$

and

$$\frac{\partial}{\partial t}|\nabla T|^2 \leq \Delta|\nabla T|^2 - (2 - C\epsilon)|\nabla^2 T|^2 + CK^2|\nabla T|^2 + CK^4|\nabla T| + C|\nabla T|^3. \tag{4.2}$$

The problem lies with the $|\nabla T|^3$ term.

In local coordinates the expression of $4(\nabla T * T * \Phi)$ is

$$\begin{aligned}4(\nabla T * T * \Phi)_{m;is} = {} & \nabla_a T_{m;bc}\Big(4T_{a;bp}\Phi_{pcis} + T_{a;ip}\Phi_{bcps} + T_{a;sp}\Phi_{bcip}\Big) \\ & + T_{m;bc}\nabla_a T_{a;bp}\Phi_{pcis} + 3\nabla_a T_{a;ip}T_{m;ps} + \nabla_a T_{a;sp}T_{m;pi} - 2T_{a;ip}\nabla_a T_{m;sp} \\ & + 2\nabla_a T_{m;ip}T_{a;sp},\end{aligned}$$

so the terms making up $|\nabla T|^3$ are

$$\begin{aligned}
& i)4\nabla_a T_{m;bc}\nabla_k T_{a;bp}\Phi_{pcis}\nabla_k T_{m;is};\\
& ii)\nabla_a T_{m;bc}\nabla_k T_{a;ip}\Phi_{bcps}\nabla_k T_{m;is};\\
& iii)\nabla_a T_{m;bc}\nabla_k T_{a;sp}\Phi_{bcip}\nabla_k T_{m;is};\\
& iv)\nabla_k T_{m;bc}\nabla_a T_{a;bp}\Phi_{pcis}\nabla_k T_{m;is};\\
& v)3\nabla_a T_{a;ip}\nabla_k T_{m;ps}\nabla_k T_{m;is};\\
& vi)\nabla_a T_{a;sp}\nabla_k T_{m;pi}\nabla_k T_{m;is};\\
& vii) - 2\nabla_k T_{a;ip}\nabla_a T_{m;sp}\nabla_k T_{m;is};\\
& viii)2\nabla_a T_{m;ip}\nabla_k T_{a;sp}\nabla_k T_{m;is}.
\end{aligned}$$

Using skew-symmetry and the Bianchi-type identity of Φ, the terms ii) and vii) can be re-written:

$$
\begin{aligned}
ii)\nabla_a T_{m;bc}\nabla_k T_{a;ip}\Phi_{bcps}\nabla_k T_{m;is} &= \tfrac{1}{2}\nabla_k T_{a;ip}\nabla_k T_{m;is}\Phi_{bcps}(\nabla_a T_{m;bc}-\nabla_m T_{a;bc})\\
&= \nabla T * \nabla T * T * T * \Phi + \nabla T * \nabla T * R * \Phi * \Phi;\\
vii)-2\nabla_k T_{a;ip}\nabla_a T_{m;sp}\nabla_k T_{m;is} &= -\nabla_k T_{a;ip}\nabla_k T_{m;is}(\nabla_a T_{m;sp}-\nabla_m T_{a;sp})\\
&= \nabla T * \nabla T * T * T + \nabla T * \nabla T * R * \Phi.
\end{aligned}
$$

Exchanging i and s we have that the term iii) equals ii) and the term viii) equals the term vii), while vi) is the opposite of v).

Since $T_{m;is}$ is in Ω^2_7, the term iv) can be re-written

$$
\begin{aligned}
iv)\nabla_k T_{m;bc}\nabla_a T_{a;bp}\Phi_{pcis}\nabla_k T_{m;is} &= \nabla_k T_{m;bc}\nabla_a T_{a;bp}(\nabla_k(\Phi_{pcis}T_{m;is}) - \nabla_k\Phi_{pcis}T_{m;is})\\
&= \nabla_k T_{m;bc}\nabla_a T_{a;bp}(-6\nabla_k(T_{m;pc}) - \nabla_k\Phi_{pcis}T_{m;is})\\
&= -\nabla_k T_{m;bc}\nabla_a T_{a;bp}\nabla_k\Phi_{pcis}T_{m;is}\\
&= \nabla T * \nabla T * T * \nabla\Phi.
\end{aligned}
$$

We do a similar thing to the first term (forgetting the factor 4):

$$
\begin{aligned}
i)\nabla_a T_{m;bc}\nabla_k T_{a;bp}\Phi_{pcis}\nabla_k T_{m;is} &= \nabla_a T_{m;bc}\nabla_k T_{a;bp}(\nabla_k(\Phi_{pcis}T_{m;is}) - \nabla_k\Phi_{pcis}T_{m;is})\\
&= -6\nabla_a T_{m;bc}\nabla_k T_{a;bp}\nabla_k T_{m;pc} + \nabla T * \nabla T * T * T,
\end{aligned}
$$

and exchanging a and m and then b and c, we have

$$
2\nabla_a T_{m;bc}\nabla_k T_{a;bp}\nabla_k T_{m;pc} = \nabla_k T_{a;bp}\nabla_k T_{m;pc}(\nabla_a T_{m;bc} - \nabla_m T_{a;bc}),
$$

so we can use the Bianchi-type equality again.

In conclusion, the terms leading to the problematic term $|\nabla T|^3$ can be re-written in terms of the type:

$$
\begin{aligned}
&\nabla T * \nabla T * T * T + \nabla T * \nabla T * R * \Phi + \nabla T * \nabla T * T * T * \Phi\\
&+ \nabla T * \nabla T * R * \Phi * \Phi + \nabla T * \nabla T * T * \nabla\Phi,
\end{aligned}
$$

and

$$
|\nabla T|^3 \leq CK^2|\nabla T|^2,
$$

so, for a suitable ϵ, Inequality (4.2) becomes

$$\frac{\partial}{\partial t}|\nabla T|^2 \leq \Delta|\nabla T|^2 + CK^2|\nabla T|^2 + CK^4|\nabla T|,$$

which is exactly equation (3.22) in [11].

For the function $f = t|\nabla T|^2 + \beta|T|^2$, combining results for $\frac{\partial}{\partial t}|\nabla T|^2$ and $\frac{\partial}{\partial t}|T|^2$, keeping in mind that $t \leq 1/K^2$ and choosing β large enough, this implies that

$$\begin{aligned}\frac{\partial}{\partial t}f &= |\nabla T|^2 + t\frac{\partial}{\partial t}|\nabla T|^2 + \beta\frac{\partial}{\partial t}|T|^2 \\ &\leq \Delta f + C\beta K^4.\end{aligned}$$

As $f(x,0) = \beta|T|^2 \leq \beta K^2$ then $\sup_M f(x,t) \leq CK^2 + C\beta t K^4 \leq CK^2$ hence $t|\nabla T|^2 \leq CK^2$.

The m-step of the induction:

Assume that $|\nabla^j T| \leq C_j K t^{-\frac{j}{2}}$ for $j = 1, \ldots, m-1$.

The evolution equation for $|\nabla^m T|^2$ is

$$\begin{aligned}\frac{\partial}{\partial t}\nabla^m T &= \Delta\nabla^m T + \sum_{i=0}^{m}\nabla^{m-i}T * \nabla^i R + \nabla^m(\nabla T * T * \Phi) \\ &+ \sum_{a+b+c+d=m}\nabla^a T * \nabla^b T * \nabla^c T * \nabla^d \Phi + \sum_{i=0}^{m}\nabla^i T * \nabla^{m-i}R \\ &+ \sum_{i=0}^{m}\nabla^{m+1-i}R * \nabla^i\Phi + \sum_{a+b+c=m}\nabla^a R * \nabla^{b+1}\Phi * \nabla^c\Phi \\ &+ \sum_{a+b+c=m}\nabla^a T * \nabla^b R * \nabla^c \Phi,\end{aligned}$$

therefore

$$\begin{aligned}&\frac{\partial}{\partial t}|\nabla^m T|^2 = \Delta|\nabla^m T|^2 - 2|\nabla^{m+1}T|^2 + \sum \nabla^m T * \nabla^{m-i}T * \nabla^i R \\ &+ \nabla^m T * \nabla^m(\nabla T * T * \Phi) + \sum \nabla^m T * \nabla^a T * \nabla^b T * \nabla^c T * \nabla^d \Phi \\ &+ \sum \nabla^m T * \nabla^i T * \nabla^{m-i} R + \sum \nabla^m T * \nabla^{m+1-i} R * \nabla^i \Phi \\ &+ \sum \nabla^m T * \nabla^a R * \nabla^{b+1}\Phi * \nabla^c \Phi + \sum \nabla^m T * \nabla^a T * \nabla^b R * \nabla^c \Phi.\end{aligned}$$

Third term: separating the $i=0$ from the others, we get

$$|\sum \nabla^m T * \nabla^{m-i} T * \nabla^i R| \leq CK^2 |\nabla^m T|^2 + CK^3 t^{-\frac{m}{2}} |\nabla^m T|.$$

By induction we show that $|\nabla^i \Phi| \leq C \sum_{j=1}^{i} K^j t^{-\frac{j-i}{2}}$.
Fifth term: Separating the cases where a or b equals m and using $K^2 T \leq 1$ we have

$$|\sum \nabla^m T * \nabla^a T * \nabla^b T * \nabla^c T * \nabla^d \Phi| \leq CK^2 |\nabla^m T|^2 + CK^3 |\nabla^m T| t^{-\frac{m}{2}}.$$

Seventh term: Using $K^2 T \leq 1$ we have

$$|\sum \nabla^m T * \nabla^{m+1-i} R * \nabla^i \Phi| \leq CK^3 |\nabla^m T| t^{-\frac{m}{2}}.$$

Sixth term: Separate the case $i=m$ from the others and use $K^2 T \leq 1$ to obtain

$$|\sum \nabla^m T * \nabla^i T * \nabla^{m-i} R| \leq CK^2 |\nabla^m T|^2 + CK^3 |\nabla^m T| t^{-\frac{m}{2}}.$$

Eighth term: Separate the $b=m$ term from the others, use $\nabla^{i+1}\Phi = \nabla^i T + \text{lot}$ and $K^2 T \leq 1$ to obtain

$$|\sum \nabla^m T * \nabla^a R * \nabla^{b+1} \Phi * \nabla^c \Phi| \leq CK^2 |\nabla^m T|^2 + CK^2 |\nabla^m T| t^{-\frac{m+1}{2}}.$$

Ninth term: Separate the $a=m$ term from the others and use $K^2 T \leq 1$ to obtain

$$|\sum \nabla^m T * \nabla^a T * \nabla^b R * \nabla^c \Phi| \leq CK^2 |\nabla^m T|^2 + CK^3 |\nabla^m T| t^{-\frac{m}{2}}.$$

Fourth term: Using $\nabla(T * \Phi) = \nabla T * \Phi + T * T * \Phi$,

$$\begin{aligned}
&|\nabla^m T * \nabla^m(\nabla T * T * \Phi)| \\
&\leq |\nabla^m T * \nabla^{m+1} T * T * \Phi)| + |\nabla^m T * \nabla^m T * \nabla(T * \Phi)| \\
&+ |\nabla^m T * \sum_{i=2}^{m-1} \nabla^{m+1-i} T * \nabla^i (T * \Phi)| + |\nabla^m T * \nabla T * \nabla^m (T * \Phi)| \\
&\leq CK|\nabla^m T||\nabla^{m+1} T| + C|\nabla^m T|^2 (Kt^{-\frac{1}{2}} + K^2) \\
&+ C|\nabla^m T| \sum_{i=2}^{m-1} Kt^{-\frac{m+1-i}{2}} \sum_{j=0}^{i} Kt^{-\frac{i-j}{2}} \sum_{k=1}^{j} K^k t^{\frac{k-j}{2}} \\
&+ C|\nabla^m T| Kt^{-\frac{1}{2}} (|\nabla^m T| + \sum_{j=1}^{m} Kt^{-\frac{m-i}{2}} \sum_{k=1}^{i} K^k t^{\frac{k-i}{2}} \\
&\leq CK|\nabla^m T||\nabla^{m+1} T| + C|\nabla^m T|^2 (Kt^{-\frac{1}{2}} + K^2) + CK^2 |\nabla^m T| t^{-\frac{m+1}{2}}.
\end{aligned}$$

In conclusion

$$\begin{aligned}
&\frac{\partial}{\partial t} |\nabla^m T|^2 \leq \Delta |\nabla^m T|^2 - 2|\nabla^{m+1} T|^2 + CK^2 |\nabla^m T|^2 + CK^3 t^{-\frac{m}{2}} |\nabla^m T| \\
&+ CK^2 t^{-\frac{m+1}{2}} |\nabla^m T| + CK|\nabla^m T||\nabla^{m+1} T| + CKt^{-\frac{1}{2}} |\nabla^m T|^2,
\end{aligned}$$

which is exactly Equation (3.32) in [11].

As the rest of the proof completely relies on this equation, it can be read in [11]. □

5 Analysis of the flow II

Let (M, g) be a complete Riemannian manifold. For $x_0 \in M$, let u be the fundamental solution of the backward heat equation, starting with the delta function at x_0 [16]:

$$\Big(\frac{\partial}{\partial t} + \Delta\Big) u = 0, \quad \lim_{t \to t_0} u = \delta_{x_0}$$

and set $u = \frac{e^{-f}}{\big(4\pi(t_0 - t)\big)^4}$.

For a solution $\{\Phi(t)\}_{t\in[0,t_0)}$ of the harmonic Spin(7)-flow on (M,g), we define the function

$$\Theta_{(x_0,t_0)}(\Phi(t)) = (t_0-t)\int_M |T_{\Phi(t)}|^2 u \ \mathrm{vol_g}\,. \tag{5.1}$$

We start off with a derivation of the function $\Theta\circ\Phi$ and give a little bit more details of the proof than in [12, Lemma 5.1].

Lemma 5.1.

$$\begin{aligned}
\frac{d}{dt}\Theta &= -2(t_0-t)\int_M |\operatorname{div} T - f \lrcorner T|^2 u \ \mathrm{vol_g}\\
&\quad - 2(t_0-t)\int_M \big(\nabla_m\nabla_l u - \frac{\nabla_m u \nabla_l u}{u} + \frac{u g_{ml}}{2(t_0-t)}\big) T_{m;is}T_{l;is} \ \mathrm{vol_g}\\
&\quad - (t_0-t)\int_M uR_{mlis}(2T_{l;ir}T_{m;rs} - 2T_{m;ir}T_{l;rs} + \tfrac{1}{4}R_{mlis} - \tfrac{1}{8}R_{mlab}\Phi_{abis}) \ \mathrm{vol_g}\\
&\quad - 2(t_0-t)\int_M T_{m;is}u\nabla_l R_{mlis} \ \mathrm{vol_g}\,.
\end{aligned}$$

Proof. By direct computation, we have

$$\begin{aligned}
\frac{d}{dt}\Theta &= \int_M (t_0-t)u\frac{\partial}{\partial t}|T|^2 - |T|^2 u + (t_0-t)|T|^2\frac{\partial}{\partial t}u\\
&= \int_M (t_0-t)u\frac{\partial}{\partial t}|T|^2 - |T|^2 u - (t_0-t)|T|^2\Delta u\\
&= \int_M 2(t_0-t)uT_{m;is}\frac{\partial}{\partial t}T_{m;is} - |T|^2 u - (t_0-t)|T|^2\Delta u\\
&= \int_M 2(t_0-t)uT_{m;is}\Big(\nabla_r T_{r;ip}T_{m;ps} - \nabla_r T_{r;sp}T_{m;pi} + \pi_7(\nabla_m(\nabla_r T_{r;is}))\Big)\\
&\quad - |T|^2 u - (t_0-t)|T|^2\Delta u,
\end{aligned}$$

but

1. $T_{m;is}\nabla_r T_{r;ip}T_{m;ps} = 0$ because $T_{m;is}T_{m;ps}$ is symmetric in (i,p) and $\nabla_r T_{r;ip}$ is skew-symmetric in (i,p);

2. $T_{m;is}\nabla_r T_{r;sp}T_{m;pi} = 0$ because $T_{m;is}T_{m;pi}$ is symmetric in (s,p) and $\nabla_r T_{r;sp}$ is skew-symmetric in (s,p);

3. $T_{m;is} \in \Lambda^2_7$,

therefore

$$\frac{d}{dt}\Theta = \int_M 2(t_0-t)uT_{m;is}(\nabla_m(\nabla_r T_{r;is})) - |T|^2u - (t_0-t)|T|^2\Delta u.$$

Integrating by parts and using the Bianchi identity, we have

$$\begin{aligned}
\frac{d}{dt}\Theta &= \int_M -2(t_0-t)u\nabla_m T_{m;is}\nabla_r T_{r;is} - 2(t_0-t)\nabla_m u T_{m;is}\nabla_r T_{r;is} - |T|^2u \\
&+ 2(t_0-t)T_{m;is}\nabla_l T_{m;is}\nabla_l u \\
&= \int_M -2(t_0-t)\Big(|\operatorname{div} T|^2 u + \nabla_m u T_{m;is}\nabla_r T_{r;is}\Big) - |T|^2u \\
&\quad + 2(t_0-t)T_{m;is}\nabla_l T_{m;is}\nabla_l u \\
&= \int_M -2(t_0-t)\Big(|\operatorname{div} T|^2 u + \nabla_m u T_{m;is}\nabla_r T_{r;is}\Big) - |T|^2u \\
&+ 2(t_0-t)T_{m;is}\nabla_l u\Big(\nabla_m T_{l;is} + 2T_{l;ir}T_{m;rs} - 2T_{m;ir}T_{l;rs} + \tfrac{1}{4}R_{mlis} \\
&\quad - \tfrac{1}{8}R_{mlab}\Phi_{abis}\Big) \\
&= \int_M -2(t_0-t)\Big(|\operatorname{div} T|^2 u + \nabla_m u T_{m;is}\nabla_r T_{r;is}\Big) - |T|^2u \\
&\quad + 2(t_0-t)\Big(T_{m;is}\nabla_l u\nabla_m T_{l;is} + 2T_{m;is}\nabla_l u T_{l;ir}T_{m;rs} - 2T_{m;is}\nabla_l u T_{m;ir}T_{l;rs} \\
&\quad + T_{m;is}\nabla_l u\tfrac{1}{4}R_{mlis} - \tfrac{1}{8}T_{m;is}\nabla_l u R_{mlab}\Phi_{abis}\Big),
\end{aligned}$$

but, as previously,

1. $T_{m;is}T_{l;ir}T_{m;rs} = 0$;

2. $T_{m;is}T_{m;ir}T_{l;rs} = 0$;

3. $T_{m;is}(\frac{1}{4}R_{mlis} - \frac{1}{8}R_{mlab}\Phi_{abis}) = T_{m;is}R_{mlis}$,

so

$$
\begin{aligned}
\frac{d}{dt}\Theta = &\int_M -2(t_0-t)\Big(|\operatorname{div}T|^2u+\nabla_m uT_{m;is}\nabla_r T_{r;is}\Big)-|T|^2u\\
&\quad +2(t_0-t)\Big(T_{m;is}\nabla_l u\nabla_m T_{l;is}+T_{m;is}\nabla_l uR_{mlis}\Big)\\
=&\int_M -2(t_0-t)\Big(|\operatorname{div}T|^2u+\nabla_m uT_{m;is}\nabla_r T_{r;is}\Big)-|T|^2u\\
&-2(t_0-t)\int_M \nabla_m T_{m;is}\nabla_l uT_{l;is}+T_{m;is}\nabla_m\nabla_l uT_{l;is}\\
&-2(t_0-t)\int_M \nabla_l T_{m;is}uR_{mlis}+T_{m;is}u\nabla_l R_{mlis}\\
=&\int_M -2(t_0-t)\Big(|\operatorname{div}T|^2u+2\nabla_m uT_{m;is}\nabla_r T_{r;is}\Big)-|T|^2u\\
&-2(t_0-t)\int_M T_{m;is}\nabla_m\nabla_l uT_{l;is}\\
&-(t_0-t)\int_M uR_{mlis}(\nabla_l T_{m;is}-\nabla_m T_{l;is})\\
&-2(t_0-t)\int_M T_{m;is}u\nabla_l R_{mlis}\\
=&\int_M -2(t_0-t)\Big(|\operatorname{div}T|^2u+2\nabla_m uT_{m;is}\nabla_r T_{r;is}\Big)-|T|^2u\\
&-2(t_0-t)\int_M T_{m;is}\nabla_m\nabla_l uT_{l;is}\\
&-(t_0-t)\int_M uR_{mlis}(2T_{l;ir}T_{m;rs}-2T_{m;ir}T_{l;rs}+\tfrac{1}{4}R_{mlis}-\tfrac{1}{8}R_{mlab}\Phi_{abis})\\
&-2(t_0-t)\int_M T_{m;is}u\nabla_l R_{mlis}\\
=&\int_M -2(t_0-t)\Big(|\operatorname{div}T|^2u-2\langle\operatorname{div}T,\nabla f\lrcorner T\rangle u\Big)\\
&-2(t_0-t)\int_M (\nabla_m\nabla_l u+\frac{ug_{ml}}{2(t_0-t)})T_{m;is}T_{l;is}\\
&-(t_0-t)\int_M uR_{mlis}(2T_{l;ir}T_{m;rs}-2T_{m;ir}T_{l;rs}+\tfrac{1}{4}R_{mlis}-\tfrac{1}{8}R_{mlab}\Phi_{abis})\\
&-2(t_0-t)\int_M T_{m;is}u\nabla_l R_{mlis}\\
=&\int_M -2(t_0-t)\Big(|\operatorname{div}T|^2u-2\langle\operatorname{div}T,\nabla f\lrcorner T\rangle u+|f\lrcorner T|^2u\Big)\\
&-2(t_0-t)\int_M (\nabla_m\nabla_l u-\frac{\nabla_m u\nabla_l u}{u}+\frac{ug_{ml}}{2(t_0-t)})T_{m;is}T_{l;is}
\end{aligned}
$$

$$- (t_0 - t) \int_M u R_{mlis} (2T_{l;ir} T_{m;rs} - 2T_{m;ir} T_{l;rs} + \tfrac{1}{4} R_{mlis} - \tfrac{1}{8} R_{mlab} \Phi_{abis})$$
$$- 2(t_0 - t) \int_M T_{m;is} u \nabla_l R_{mlis},$$

and we obtain the formula we wish for. □

Theorem 5.2 (almost monotonicity formula)**.** *[12, Theorem 5.2] Let $\{\Phi(t)\}$ be a solution of the harmonic* Spin(7)*-flow* (HF) *on* (M^8, g)*.*

1. *If M is compact, then, for any $0 < \tau_1 < \tau_2 < t_0$, there exist K_1, $K_2 > 0$ depending only on the geometry of (M, g) such that*
$$\Theta(\Phi(\tau_2)) \leq K_1 \Theta(\Phi(\tau_1)) + K_2(\tau_1 - \tau_2)(E(0) + 1).$$

2. *When $(M, g) = (\mathbb{R}^8, g_{\mathrm{Eucl}})$, then, for any $x_0 \in \mathbb{R}^8$ and $0 \leq \tau_1 < \tau_2$ we have*
$$\Theta(\Phi(\tau_2)) \leq \Theta(\Phi(\tau_1)).$$

Proof. We sketch the proof following [11, Theorem 5.3].

1. The following equation is a direct adaptation of [11, Lemma 5.2], using the Spin(7)-Bianchi identity (2.13):
$$\begin{aligned}\frac{d}{dt}\Theta_{(x_0,t_0)}(\Phi(t)) &= -2(t_0 - t) \int_M |\operatorname{div} T - \nabla f \lrcorner T|^2 u \\ &- 2(t_0 - t) \int_M \big(\nabla_m \nabla_l u - \frac{\nabla_m u \nabla_l u}{u} + \frac{u g_{ml}}{2(t_0 - t)}\big) T_{m;is} T_{l;is} \\ &- (t_0 - t) \int_M u R_{mlis} (2T_{l;ir} T_{m;rs} - 2T_{m;ir} T_{l;rs} + \tfrac{1}{4} R_{mlis} \\ &- \tfrac{1}{8} R_{mlab} \Phi_{abis}) - 2(t_0 - t) \int_M T_{m;is} u \nabla_l R_{mlis}.\end{aligned} \tag{5.2}$$

2. The third and fourth terms of Lemma (5.1) are bounded by
$$C(1 + \Theta(\Phi(t))),$$
due to the bounded geometry of (M, g), Young's inequality and $\int_M u = 1$.

For the second term of Lemma (5.1), use [17] and the decreasing of $E(\Phi(t))$ along the harmonic Spin(7)-flow to bound it by

$$C\big(E(\Phi(0)) + \log\frac{B}{(t_0-t)^4}\Theta(\Phi(t))\big),$$

so that

$$\begin{aligned}\frac{d}{dt}\Theta(\Phi(t)) \leq & -2(t_0-t)\int_M |\operatorname{div} T - \nabla f \lrcorner T|^2 u \\ & + C_1\Big(1+\log\big(\frac{B}{(t_0-t)^4}\big)\Big)\Theta(\Phi(t)) + C_2(1+E(\Phi(0))).\end{aligned}$$

To control the logarithmic term, let $\xi(t)$ be any function satisfying

$$\xi'(t) = 1 + \log\frac{B}{(t_0-t)^4}.$$

The claim is then obtained by integration over $[t_0-1, t_0[$ of

$$\frac{d}{dt}\Big[e^{-C_1\xi(t)}\Theta(\Phi(t))\Big] \leq K\big(E(\Phi(0))+1\big).$$

3. On $(M^8, g) = (\mathbb{R}^8, g_{\mathrm{Eucl}})$, the backward heat kernel is

$$u(x,t) = \frac{1}{(4\pi(t_0-t))^4}\exp\left\{-\frac{|x-x_0|^2}{4(t_0-t)}\right\}$$

so indeed $\frac{d}{dt}\Theta(\Phi(t)) \leq 0$.

$\square$

There exists a more direct and cost-effective to obtain a decreasing quantity from $|T|^2$, though its importance remains uncertain.

Lemma 5.3 (a simpler monotonicity formula)**.** *Put $\epsilon(t) = |T|^2$ and consider the function*

$$Z(t) = (t_{\max}-t)\int_M \epsilon k \ \mathrm{vol_g}, \quad 0 \leq t < t_{\max},$$

where k is any (positive) solution of the backward heat equation $\partial_t k = -\Delta k$ on $M_{t_{\max}}$. Then

$$Z(t) \leq Z(0)e^{Ct}$$

for $0 \leq t \leq \delta$ (from DTE).

Proof. Since

$$\partial_t Z = -\int_M \epsilon k + (t_{\max} - t)\int_M k\partial_t\epsilon + \epsilon\partial_t k.$$

By self-adjointness of the Laplacian and the 'reaction-diffusion' Bochner formula of Equation (4.1), the second integral satisfies the following upper bound:

$$\begin{aligned}\int_M k\partial_t\epsilon + \epsilon\partial_t k &= \int_M k\partial_t\epsilon - \epsilon\Delta k = \int_M k\partial_t\epsilon - k\Delta\epsilon \\ &= \int_M k(\partial_t\epsilon - \Delta\epsilon) \\ &\leq \int_M k(C_1\epsilon + C_2\epsilon^2),\end{aligned}$$

and therefore

$$\partial_t Z \leq C_1 Z(t) + (t_{\max} - t)\int_M k\epsilon(C_2\epsilon), \quad 0 \leq t < t_{\max}.$$

Then by DTE, we have

$$C_2\epsilon(x,t) \leq C_2\mathcal{T}(t) \leq 2C_2\mathcal{T}(0) = C_0$$

so

$$\begin{aligned}\partial_t Z &\leq C_1 Z(t) + (t_{\max} - t)\int_M k\epsilon(C_2\epsilon) \\ &\leq C_1 Z(t) + C_0(t_{\max} - t)\int_M k\epsilon \\ &\leq CZ(t)\end{aligned}$$

so

$$Z(t) \leq Z(0)e^{Ct}.$$

□

Definition 5.4. Let (M^8, Φ, g) be a compact manifold with a Spin(7)-structure. Let $u_{(x,t)}(y,s) = u^g_{(x,t)}(y,s)$ be the backward heat kernel, starting from $\delta(x,t)$ as $s \to t$. For $\sigma > 0$ we define

$$\lambda(\Phi,\sigma) = \max_{(x,t)\in M\times(0,\sigma]}\left\{t\int_M |T_\Phi|^2(y)u_{(x,t)}(y,0)\ \mathrm{vol_g}\right\}. \qquad (5.3)$$

One should think of σ as the "scale" at which we are analyzing the flow. Since M is compact, the maximum in (5.3) is achieved.

We can now state the ε-regularity theorem for the harmonic Spin(7)-flow.

Theorem 5.5 (ε-regularity). *[12, Theorem 5.5] Let (M^8, g) be compact and $E_0 > 0$. There exist ε, $\bar{\rho} > 0$ such that, for every $\rho \in (0, \bar{\rho}]$, there exist $r \in (0, \rho)$ and $C < \infty$ such that the following holds:*

Suppose $\{\Phi(t)\}_{t\in[0,t_0)}$ is a solution of the harmonic Spin(7)*-flow* (HF)*, with induced metric g, satisfying $E(\Phi(0)) \leq E_0$. Whenever*

$$\Theta_{(x_0,t_0)}(\Phi(t_0 - \rho^2)) < \varepsilon, \quad \text{for some } x_0 \in M,$$

then, setting $\Lambda_r(x,t) = \min\left(1 - r^{-1}d_g(x_0, x), \sqrt{1 - r^{-2}(t_0 - t)}\right)$*, we have*

$$\Lambda_r(x,t)|T_\Phi(x,t)| \leq \frac{C}{r}, \quad \forall\ (x,t) \in B(x_0, r) \times [t_0 - r^2, t_0].$$

An immediate corollary of the ε-regularity theorem is the following result, which states that if the entropy of the initial Spin(7)-structure is small then the torsion is controlled at all times. Again, the proof is similar to [11, Cor. 5.8].

Corollary 5.6 (small initial entropy controls torsion). *[12, Corollary 5.6] Let $\{\Phi(t)\}$ be a solution of the harmonic* Spin(7)*-flow* (HF) *on compact (M, g), starting at Φ_0. For every $\sigma > 0$, there exist $\varepsilon, t_0 > 0$ and $C < \infty$ such that, if Φ_0 induces g and its entropy* (5.3) *satisfies*

$$\lambda(\Phi_0, \sigma) < \varepsilon,$$

then

$$\max_M |T_{\Phi(t)}| \leq \frac{C}{\sqrt{t}}.$$

Theorem 5.7 (small initial torsion gives long-time existence). *[12, Theorem 5.9] Let (M, Φ_0, g) be a compact* Spin(7)*-structure manifold. For* *every $\delta > 0$, there exists $\varepsilon(\delta, g) > 0$ such that, if $|T_{\Phi_0}| < \varepsilon$, then a harmonic* Spin(7)*-flow* (HF) *starting at Φ_0 exists for all time and converges subsequentially smoothly to a* Spin(7)*-structure Φ_∞ such that*

$$\operatorname{div} T_{\Phi_\infty} = 0, \quad |T_{\Phi_\infty}| < \delta.$$

Sketch of proof. 1) If $|T_{\Phi_0}| < \varepsilon_0$ then by the (DTE) there exists $\delta > 0$ such that

$$t_* := \max\{t \geq 0 \ : \ |T_{\Phi(t)}| \leq 2\varepsilon_0\} > \delta. \tag{5.4}$$

2) If $t_* < \infty$ then the Shi-type estimates on $]t_* - \delta, t_*[$ would imply

$$|\nabla T_{\Phi(t_*)}| < c_0. \tag{5.5}$$

3) But our flow is the negative gradient so $E(\Phi(t_*)) \leq E(\Phi_0)$ so we can invoke the interpolation lemma (which is a static result): If $|\nabla T| \leq C$ and no collapsing, i.e.

$$\mathrm{vol_g}(B(x,r)) \geq v_0 r^8, \text{for } 0 < r \leq 1,$$

for some constant $v_0(M, g) > 0$, then, for every $\varepsilon > 0$, there exists $\delta(\varepsilon, C, v_0) \geq 0$ such that, if $E(\Phi) < \delta$ then $|T| < \varepsilon$.
4) Conclude taking $\varepsilon < \min(\varepsilon_0, \gamma_{2\varepsilon_0}$ so that $|\nabla T_{\Phi(t_*)}| < \varepsilon_0$ implies $|T(t_*)| < 2\varepsilon_0$ which contradicts the maximality of t_* and forces $t_* = +\infty$.
5) If Λ is the first eigenvalue of the Laplacian on 2-forms we can easily show that (cf. [12, Lemma 5.7])

$$\frac{d^2}{dt^2} E(\Phi(t)) \geq \int_M (\Lambda - 3|T|^2) |\operatorname{div} T|^2,$$

so if $|T|^2 \leq \frac{\Lambda}{6}$

$$\frac{d}{dt}\int_M |\operatorname{div} T_{\Phi(t)}|^2 = -\frac{d^2}{dt^2} E(\Phi(t)) \leq -\frac{\Lambda}{2}\int_M |\operatorname{div} T_{\Phi(t)}|^2.$$

If we take $\varepsilon < \min(\varepsilon_0, \gamma_{2\varepsilon_0}, \gamma_{\sqrt{\frac{\Lambda}{6}}})$ then we obtain the decay estimate

$$\int_M |\operatorname{div} T_{\Phi(t)}|^2 \leq e^{-\frac{\Lambda t}{2}} \int_M |\operatorname{div} T_{\Phi(0)}|^2, \quad \forall\, t \geq 0. \tag{5.6}$$

6) Take $s_1 < s_2$ and integrate to obtain

$$\begin{aligned}\int_M |\Phi(s_2) - \Phi(s_1)| &\leq \int_M \int_{s_1}^{s_2} |\partial_t \Phi(s)|\, ds = \int_{s_1}^{s_2} \int_M |\operatorname{div} T_{\Phi(s)}| ds \\ &\leq c \int_{s_1}^{s_2} \left(\int_M |\operatorname{div} T_{\Phi(s)}|^2 \right)^{\frac{1}{2}} ds \\ &\leq c \int_{s_1}^{s_2} e^{-\frac{\Lambda s}{4}} ds.\end{aligned} \tag{5.7}$$

7) $\Phi(t)$ converges in L^1 to Φ_∞.
8) The uniform bound on T combined with Shi-type estimates gives estimates on all $|\nabla^m T|$ and smooth convergence to Φ_∞.
9) The exponential decay of the integrals implies that $\operatorname{div} T_{\Phi_\infty} = 0$, and by the interpolation lemma, we also achieve that $|T_{\Phi_\infty}| < \delta$. □

Theorem 5.8 (small entropy gives long-time existence)**.** *[12, Theorem 5.10] On a compact* Spin(7)*-structure manifold* (M, Φ_0, g)*, there exist constants* $C_k(M, g) < +\infty$*, such that the following holds. For each* $\varepsilon > 0$ *and* $\sigma > 0$*, there exists* $\lambda_\varepsilon(g, \sigma) > 0$ *such that, if the entropy* (5.3) *satisfies*

$$\lambda(\Phi_0, \sigma) < \lambda_\varepsilon, \tag{5.8}$$

then the torsion becomes eventually pointwise small along the harmonic Spin(7)*-flow* (HF) *starting at* Φ_0*. Therefore the flow exists for all time and subsequentially converges to a* Spin(7)*-structure* Φ_∞ *such that*

$$\operatorname{div} T_{\Phi_\infty} = 0, \quad |T_{\Phi_\infty}| < \varepsilon \quad \textit{and} \quad |\nabla^k T_{\Phi_\infty}| < C_k, \ \forall\ k \geq 1.$$

Sketch of proof. 1) λ_ε small enough implies $|T| \leq \frac{C}{\sqrt{t}}$ for all $t \leq \tau$.
2) Shi-type estimates imply $|\nabla T(\tau)| < C'$.
3) Interpolation lemma: $\forall \epsilon > 0$, for small enough λ_ε, $|T(\tau)| < \epsilon$.
4) Small $|T(\tau)|$ implies long-time existence.
5) We conclude as with the previous result. □

Let ε and $\bar{\rho}$ be the quantities from the ε-regularity Theorem 5.5. We define the *singular set* of the flow by

$$S = \{x \in M \ : \ \Theta_{(x,\tau)}(\Phi(\tau - \rho^2)) \geq \varepsilon, \text{ for all } \rho \in (0, \bar{\rho}]\}. \tag{5.9}$$

The following lemma explains why S is called the singular set of the flow.

Lemma 5.9. *The harmonic* Spin(7)*-flow* $\{\Phi(t)\}_{t\in[0,\tau)}$ *restricted to* $M \setminus S$ *converges as* $t \to \tau$*, smoothly and uniformly away from* S*, to a smooth harmonic* Spin(7)*-structure* $\Phi(\tau)$ *on* $M \setminus S$*. Moreover, for every* $x \in S$*, there is a sequence* $(x_i, t_i) \to (x, \tau)$ *such that*

$$\lim_i |T_\Phi(x_i, t_i)| = \infty.$$

Thus, S *is indeed the singular set of the flow.*

Theorem 5.10 (Hausdorff measure of the singularity set)**.** *[12, Theorem D]*

$$E(\Phi_0) = \frac{1}{2}\int_M |T_{\Phi_0}|^2 \,\mathrm{vol_g} \le E_0. \tag{5.10}$$

Suppose that the maximal smooth harmonic Spin(7)*-flow* $\{\Phi(t)\}_{t\in[0,\tau)}$ *starting at* Φ_0 *blows up at time* $\tau < +\infty$*. Then, as* $t \to \tau$*,* (HF) *converges smoothly to a* Spin(7)*-structure* $\mathrm{Spin}(7)_\tau$ *away from a closed set* S*, with finite* 6*-dimensional Hausdorff measure satisfying*

$$\mathcal{H}^6(S) \le CE_0,$$

for some constant $C < \infty$ *depending on* g*. In particular, the Hausdorff dimension of* S *is at most 6.*

Sketch of proof. The proof relies on the following computation:

$$\begin{aligned}
\varepsilon\mathcal{H}^6(S) = \int_S \varepsilon d\mathcal{H}^6(x) &\le \int_S \Theta(\Phi(\tau-\rho^2))\, d\mathcal{H}^6(x)\\
&\le \int_S\int_M \rho^2|T|^2 u\, d\mathcal{H}^6(x)\\
&\le \int_M \rho^2|T|^2 u\\
&\le C\int_M |T|^2\\
&\le CE_0.
\end{aligned}$$

□

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