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Spectrum of elliptic homogeneous differential operators in dimension n on real scales of localized Sobolev spaces

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Abstract. In this note, we present the study of the spectrum for an elliptic homogeneous linear differential operator with constant coefficients of order m in n dimensional case on real scales of $E-$valued localized Sobolev space extending the results in [1]. Our aim is to understand the behavior of the spectrum using the closure of the operator. In particular, we show that there is no complex number in the resolvent set of such operators, which suggest a new way to define spectrum if we want to reproduce the classical theorems of the Spectral Theory in Fréchet spaces.

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1 Introduction

In this work, we present a study of the spectrum of homogeneous linear differential operators with constant coefficients on an open set $\Omega \subset \mathbb{R}^n$ given by

$$a(D) : H_0^{s+m}(\Omega; E) \subset H_{loc}^s(\Omega; E) \longrightarrow H_{loc}^s(\Omega; E), \qquad s \in \mathbb{R},$$

where $H_{loc}^s(\Omega; E)$ is the $E-$valued localized Sobolev space of order s on Ω and $H_0^s(\Omega; E)$ is the closure of $C_c^\infty(\Omega; E)$ with the topology of $H^s(\mathbb{R}^n; E)$ for E a finite dimensional complex vector space. Our main result is the following:

Theorem A. *Let $s \in \mathbb{R}$ and $m \in \mathbb{N}$. Consider $a(D) : H_0^{s+m}(\Omega; E) \subset H_{loc}^s(\Omega; E) \to H_{loc}^s(\Omega; E)$ an elliptic homogeneous linear differential operator of order m with constant coefficients. Then the spectrum does not depend of s and moreover*

$$\sigma(a(D)) = \sigma\left(\overline{a(D)}\right) = \sigma_p\left(\overline{a(D)}\right) = \mathbb{C}.$$

1.1 Preliminary concepts and results

We begin by defining Fréchet spaces, their duals and their topology.

Definition 1.1. Let X be a topological vector space. We say that X is said to be a Fréchet space if it is Hausdorff, complete and its topology is given by a countable family of seminorms $(p_j)_{j \in \mathbb{N}}$.

Some examples of Fréchet spaces are the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, the set of smooth functions $C^\infty(\Omega)$ and the localized Sobolev spaces $H_{loc}^s(\Omega)$ for $s \in \mathbb{R}$, where $\Omega \subset \mathbb{R}^n$ is an open set. We denote by X' the dual of the Fréchet space X equipped with the weak* topology which is also generated by a family of semi-norms.

1.1.1 Sobolev spaces

Definition 1.2. Let $\Omega \subset \mathbb{R}^n$ an open set. The Sobolev space $H^1(\Omega)$ is the set of function $u \in L^2(\Omega)$ such that there exists $g_\alpha \in L^2(\Omega)$ satisfying

$$\int_\Omega u(x)\partial^\alpha \phi(x)dx = -\int_\Omega g_\alpha(x)\phi(x)dx, \tag{1.1}$$

for all $\phi \in C^\infty(\Omega)$ and $|\alpha| = 1$.

The functions g_α, usually denoted by $\partial^\alpha u$, are said to be the weak $\alpha-$derivatives of u. Moreover, the usual topology of $H^1(\Omega)$ is determined by the norm $\|u\|_{H^1(\Omega)} \doteq \sum\limits_{0\leq|\alpha|\leq 1} \|\partial^\alpha u\|_{L^2(\Omega)}$. Given a natural number $m \geq 2$, the Sobolev space $H^m(\Omega)$ is defined inductively as the set of $u \in H^{m-1}(\Omega)$ such that $\partial^\alpha u \in H^{m-1}(\Omega)$ for each $|\alpha| = 1$ and its usual topology is defined by the norm $\|u\|_{H^m(\Omega)} \doteq \sum\limits_{0\leq|\alpha|\leq m} \|\partial^\alpha u\|_{L^2(\Omega)}$ (if $\Omega = \mathbb{R}^n$ we will denote $\|\cdot\|_{H^m(\Omega)}$ by $\|\cdot\|_{H^m}$). We define $H_0^m(\Omega)$ as the closure of $C_c^\infty(\Omega)$ in $H^m(\Omega)$ with the induced topology.

The next theorem, which can be seen in [3], gives an alternative way to describe the space $H^m(\mathbb{R}^n)$ by using the Fourier transform.

Theorem 1.3. *For each $m \in \mathbb{N}$, the Sobolev space $H^m(\mathbb{R}^n)$ is characterized by the set of $u \in \mathcal{S}'(\mathbb{R}^n)$ such that $\hat{u}$ is a measurable function with $(1+|\xi|^2)^{m/2}\hat{u} \in L^2(\mathbb{R}^n)$. Furthermore, the norm $\|u\|_m \doteq \big\|(1+|\cdot|^2)^{m/2}\hat{u}\big\|_{L^2(\mathbb{R}^n)}$ is equivalent to $\|\cdot\|_{H^m}$.*

This result suggests a way to define Sobolev spaces for real values:

Definition 1.4. Let $s \in \mathbb{R}$. We say a tempered distribution $u \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $H^s(\mathbb{R}^n)$ if $(1+|\cdot|^2)^{s/2}\hat{u} \in L^2(\mathbb{R}^n)$.

Given an open set $\Omega \subset \mathbb{R}^n$ and $s \in \mathbb{R}$ we define the localized Sobolev space of order s on Ω as

$$H^s_{loc}(\Omega) = \big\{u \in \mathcal{D}'(\Omega); \phi u \in H^s(\mathbb{R}^n),\ \forall\ \phi \in C_c^\infty(\Omega)\big\}.$$

The space $H^s_{loc}(\Omega)$ is a Fréchet space and its semi-norms are given by $p_j(u) \doteq \|\phi_j u\|_{H^s}$, where $\phi_j = 1$ in $\overline{\Omega_j}$, $\phi_j \in C^\infty_c(\Omega_{j+1})$ and $\Omega_j \subset \Omega$ is a sequence of open sets that exhaust Ω, precisely $\Omega = \cup_{j\in\mathbb{N}}\Omega_j$, $\overline{\Omega_j} \subset \Omega_{j+1}$ and $\overline{\Omega_j}$ is compact for every $j \in \mathbb{N}$. Finally we denote $H^s_0(\Omega)$ as the closure of $C^\infty_c(\Omega)$ with the topology of $H^s(\mathbb{R}^n)$ and clearly $H^s_0(\Omega) \subset H^s_{loc}(\Omega)$. Naturally we can define $H^s_0(\Omega; E)$ and $H^s_{loc}(\Omega; E)$ for E a finite dimensional complex vector space extending for each component.

1.1.2 Elliptic homogeneous differential operators

Let $a(D) : C^\infty(\Omega; E) \to C^\infty(\Omega; F)$ a homogeneous linear differential operator of order m with constant coefficients given by $a(D) = \sum_{|\alpha|=m} a_\alpha \partial^\alpha$, with $a_\alpha \in \mathcal{L}(E; F)$ where E and F are finite dimensional complex vector spaces. We denote by $a(\xi) : E \to F$ the symbol of the operator $a(D)$ by

$$a(\xi) = \sum_{|\alpha|=m} a_\alpha \xi^\alpha, \quad \xi \in \mathbb{R}^n.$$

We say that $a(D)$ is elliptic if the symbol $a(\xi)$ is injective for $\xi \neq 0$.

Remark 1.5. Observe that if $a(D)$as before is elliptic follows by definition that there exists a multi-index $|\beta| = m$ such that $a_\beta \in \mathcal{L}(E; F)$ is injective. Indeed, by definition, the symbol $a(\xi)$ is injective for each $\xi \neq 0$, so choosing $\xi := e_j \neq 0$ the j-canonical vector in $\mathbb{R}^n$ and $\beta_j := m \cdot e_j$ for any $1 \leq j \leq n$ we have $a(e_j) = a_{\beta_j} : E \to F$ injective.

Next we present an important definition in this work extended for vector values.

Definition 1.6. ([4]) Given $m \in \mathbb{R}$, a linear operator $A : C^\infty_c(\Omega; E) \to C^\infty(\Omega; F)$ is said to be *an operator of order m on real scales of vector valued localized Sobolev space* if A extends to a linear operator $A_s : H^{s+m}_0(\Omega; E) \subset H^{s+m}_{loc}(\Omega; E) \to H^s_{loc}(\Omega; F)$ for every $s \in \mathbb{R}$.

Examples of operators of order m on real scales of localized Sobolev space are given by the class of pseudodifferential operators in the Hörmander class $p(x, D) \in OpS^m_{1,0}(\Omega)$ (see [4, Theorem 5.19]) defined by

$$a(x, D)u(x) = \int e^{2\pi i x\cdot\xi} a(x, \xi)\hat{u}(\xi)d\xi, \quad u \in \mathcal{S}'(\mathbb{R}^n), \tag{1.2}$$

where $\hat{u}$ is the Fourier transform of u (also denoted by $\mathcal{F}u$) associated to the class of symbols $a = a(x, \xi) \in S^m_{1,0}(\Omega \times \mathbb{R}^n)$ given by a smooth function satisfying the following estimates

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^{m-|\beta|}, \ \ \alpha, \beta \in \mathbb{Z}^n_+$$

with $\langle \xi \rangle := (1+|\xi|^2)^{1/2}$. Clearly a homogeneous linear differential operator $a(D)$ with order m defined previously belongs to the class $OpS^m_{1,0}(\Omega)$.

1.1.3 Closed operators and spectrum

Consider a Fréchet space X and a linear operator $A : D(A) \subset X \to X$. Analogous to Banach spaces we present the following definitions:

Definition 1.7. The graph of A is the set

$$G(A) = \{(u, Au) : u \in D(A)\} \subset X \times X.$$

The operator A is said to be a closed operator if its graph $G(A) \subset X \times X$ is a closed set.

Definition 1.8. We say that A is a closable, if there exists a closed linear operator $\overline{A} : D\left(\overline{A}\right) \subset X \to X$, with $D(A) \subset D\left(\overline{A}\right)$ and $Au = \overline{A}u$, for each $u \in D(A)$.

Definition 1.9. Let X be a complex Fréchet space and $A : D(A) \subset X \longrightarrow X$ be a linear operator. The resolvent set of A, denoted by $\rho(A)$, is the set of all $\lambda \in \mathbb{C}$ such that:

(a) The operator $\lambda - A : D(A) \subset X \longrightarrow X$ is injective.

(b) The range of $\lambda - A : D(A) \subset X \longrightarrow X$ is dense in X.

(c) The inverse $(\lambda - A)^{-1} : R(\lambda - A) \subset X \longrightarrow X$ is continuous.

If $\lambda \in \rho(A)$, the operator $(\lambda - A)^{-1} : R(\lambda - A) \subset X \longrightarrow X$ is called the resolvent of A on λ. We denote the spectrum of A by $\sigma(A) := \mathbb{C} \setminus \rho(A)$. For a closed operator A, we classify the spectrum in three types:

(a) **Point Spectrum:** the set of $\lambda \in \mathbb{C}$ such that $\lambda - A$ is not injective denoted by $\sigma_p(A)$;

(b) **Residual Spectrum:** the set of $\lambda \in \mathbb{C}$ such that $\lambda - A$ is injective with $\overline{R(\lambda - A)} \neq X$ denoted by $\sigma_r(A)$;

(c) **Continuous Spectrum:** the set of $\lambda \in \mathbb{C}$ such that $\lambda - A$ is injective, $\overline{R(\lambda - A)} = X$ but $(\lambda - A)^{-1} : R(\lambda - A) \to X$ is not continuous denoted by $\sigma_c(A)$.

Note that $\sigma(A) = \sigma_p(A) \cup \sigma_r(A) \cup \sigma_c(A)$.

The next result presented in [5] allows us to study the spectrum of a closable operator A by means of the spectrum of its closure $\overline{A}$

Theorem 1.10. *Consider X a Fréchet space. If $A : D(A) \subset X \longrightarrow X$ is closable then*

(i) $\rho(A) = \{\lambda \in \mathbb{C} : \lambda - A : D(A) \longrightarrow X$ *is bijective*$\}$;

(ii) $\sigma(A) = \sigma\left(\overline{A}\right)$, *where* $\overline{A} : D\left(\overline{A}\right) \subset X \longrightarrow X$ *is its closure.*

2 Closure of an elliptic homogeneous differential operator

The goal of this section is to calculate the closure of an elliptic homogeneous linear differential operator with constant coefficients $a(D)$ of order $m \geq 1$ on $H^s_{loc}(\Omega; E)$. We start constructing a convenient method of approximation (see [3] by motivation) in the scalar case.

Let $\Omega \subset \mathbb{R}^n$ be an open set and $s \in \mathbb{R}$. Given a function $u \in H^s_{loc}(\Omega)$, consider the natural extension

$$u_e(x) = \begin{cases} u(x), & \text{if } x \in \Omega \\ 0, & \text{if } x \in \mathbb{R}^n \setminus \Omega. \end{cases}$$

Now let $(\Omega_j)_{j\in\mathbb{N}}$ a sequence of open bounded sets with $\Omega = \bigcup_{j\in\mathbb{N}} \Omega_j$, $\overline{\Omega_j} \subset \Omega_{j+1}$ and $d(\Omega_j, \mathbb{R}^n \setminus \Omega) \geq 2/j$. Consider $g_j(x) = \chi_{\Omega_j}(x) \cdot u_e(x)$, where χ_{Ω_j} is the characteristic function of Ω_j, and $u_j = \phi_j \star g_j$, where $\phi_j(x) = j^n\phi(jx)$ with $\phi \in C_c^\infty(B_1(0))$, $\phi \geq 0$ and $\int_{\mathbb{R}^n} \phi(x)dx = 1$. Clearly $g_j \in H^s(\mathbb{R}^n)$ and $u_j \in C_c^\infty(\Omega)$.

Theorem 2.1. *Let $s \in \mathbb{R}$ and $u \in H^s_{loc}(\Omega)$. For each $j \in \mathbb{N}$ the sequence $u_j \doteq \phi_j \star (\chi_{\Omega_j} u_e) \in C_c^\infty(\Omega)$ converges to u in $H^s_{loc}(\Omega)$.*

In order to prove the Theorem 2.1 we will need some ingredients. The first is a version of Minkowski's inequality for integrals in H^s norm.

Lemma 2.2. *Suppose that $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ is a measurable function and let $s \geq 0$. If $f(\cdot, y) \in H^s(\mathbb{R}^n)$ for a.e. $y \in \mathbb{R}^n$ and the function $y \mapsto \|f(\cdot, y)\|_{H^s(\mathbb{R}^n)}$ is in $L^1(\mathbb{R}^n)$ then $f(x, \cdot) \in L^1(\mathbb{R}^n)$ for a.e. $x \in \mathbb{R}^n$, the function $x \mapsto \int f(x, y)dy$ is in $H^s(\mathbb{R}^n)$ and moreover*

$$\left\| \int f(\cdot, y)dy \right\|_{H^s} \leq \int \|f(\cdot, y)\|_{H^s} dy.$$

Proof. Given a measurable function $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ satisfying the hypotheses of the theorem we may define

$$g(\xi, y) = (1 + |\xi|^2)^{s/2} \int_{\mathbb{R}^n} e^{-2\pi i x\xi} f(x, y)dx,$$

which is well defined for a.e. $y \in \mathbb{R}^n$. It follows that $g(\cdot, y) \in L^2(\mathbb{R}^n)$ for a.e. $y \in \mathbb{R}^n$ and $\|g(\cdot, y)\|_{L^2} = \|f(\cdot, y)\|_{H^s}$ is in $L^1(\mathbb{R}^n)$, so by the Minkowski's inequality for integrals

$$\xi \mapsto \int_{\mathbb{R}^n} g(\xi, y)dy := \int_{\mathbb{R}^n} (1 + |\xi|^2)^{s/2} \left(\int_{\mathbb{R}^n} e^{2\pi i x\xi} f(x, y)dx \right) dy$$

is in $L^2(\mathbb{R}^n)$ for a.e. $\xi \in \mathbb{R}^n$ and $\left\| \int g(\cdot, y)dy \right\|_{L^2} \leq \int \|g(\cdot, y)\|_{L^2}$.

Now observe that $x \mapsto \int_{\mathbb{R}^n} f(x,y)dy$ is in $L^2(\mathbb{R}^n)$ a.e. $x \in \mathbb{R}^n$, because $f(\cdot,y) \in H^s(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ for $s \geq 0$, then $(1+|\xi|)^{s/2}\mathcal{F}\left[\int_{\mathbb{R}^n} f(\cdot,y)dy\right](\xi) = \int_{\mathbb{R}^n} g(\xi,y)dy$, thus

$$\left\|\int f(\cdot,y)dy\right\|_{H^s} = \left\|\int g(\cdot,y)dy\right\|_{L^2} \leq \int \|g(\cdot,y)\|_{L^2}dy = \int \|f(\cdot,y)\|_{H^s}dy$$

as we wanted to prove.

□

For $u \in \mathcal{D}'(\mathbb{R}^n)$ and $y \in \mathbb{R}^n$ we define $\tau_y u \in \mathcal{D}'(\mathbb{R}^n)$ by $\langle \tau_y u, \phi\rangle := \langle u, \tau_{-y}\phi\rangle$, for all $\phi \in C_c^\infty(\mathbb{R}^n)$ where $\tau_{-y}\phi(x) = \phi(x-y)$.

Lemma 2.3. *The translation $\tau_y : H^s(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$ is an isometry. Moreover $\|u - \tau_y u\|_{H^s} \to 0$ as $y \to 0$.*

Proof. If $u \in \mathcal{S}'(\mathbb{R}^n)$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$ notice that $(\tau_y u)^\wedge$ satisfies

$$\langle (\tau_y u)^\wedge, \psi\rangle = \langle u, \tau_{-y}\hat{\psi}\rangle = \langle u, (e^{-2\pi ixy}\psi)^\wedge\rangle = \langle e^{-2\pi ixy}\hat{u}, \psi\rangle,$$

so $(\tau_y u)^\wedge = e^{-2\pi ixy}\hat{u}$. Finally given $u \in H^s(\mathbb{R}^n)$ we have that

$$(1+|\xi|)^{s/2}|(\tau_y u)^\wedge(\xi)| = (1+|\xi|)^{s/2}|e^{-2\pi ixy}\hat{u}(\xi)| = (1+|\xi|)^{s/2}|\hat{u}(\xi)|$$

which implies that $\|(\tau_y u)^\wedge\|_{H^s} = \|u\|_{H^s}$, i.e., τ_y is an isometry. The second part follows by Dominated Convergence Theorem. □

Given $\varphi_\ell \in C_c^\infty(\Omega)$ from the seminorms of $H^s_{loc}(\Omega)$, now we are going to calculate $\|\varphi_\ell(u_j - u)\|_{H^s}$. Let $s \geq 0$ therefore it follows that

$$\varphi_\ell(u_j - u)(x) = \int_{B_{1/j}} \phi_j(y)[\varphi_\ell(u - \tau_{-y}u)](x)dy, \tag{2.1}$$

for $x \in \Omega_{\ell+1}$, $j \geq l+1$ and j sufficiently large. Indeed, first note that

$$(\varphi_\ell u_j)(x) = \varphi_\ell(x)[\phi_j \star (\chi_{\Omega_j}u)](x) = \varphi_\ell(x)\int_{B_{1/j}(0)} \phi_j(y)(\chi_{\Omega_j}u)(x-y)dy.$$

For $x \in \Omega_{\ell+1}$, since supp $\varphi_\ell \subset \Omega_{\ell+1}$ and $y \in B_{1/j}(0)$, there exists $j_0 \in \mathbb{N}$ such that for all $j \geq \max\{j_0, l+1\}$ we have that $x - y \in \Omega_{\ell+1} +$

$B_{1/j}(0) \subset \Omega_j$ so $\chi_{\Omega_j}(x-y) = 1$ and consequently $\varphi_\ell(x)(\chi_{\Omega_j} u)(x-y) = (\varphi_\ell \tau_{-y} u)(x)$. So we conclude that

$$(\varphi_\ell u_j)(x) = \int_{B_{1/j}} \phi_j(y)(\varphi_\ell \tau_{-y} u)(x) dy. \tag{2.2}$$

Since $H^s(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ for $s \geq 0$ and $\int_{B_{1/j}} \phi_j(y)dy = 1$, then we may write

$$(\varphi_\ell u)(x) = \int_{B_{1/j}} \phi_j(y)(\varphi_\ell u)(x) dy. \tag{2.3}$$

Combining (2.2) and (2.3) we obtain the identity (2.1).

Equipped with the above tools we can extend the results [1, Lemma 2.11 and Theorem 2.12] and prove the convergence of the above sequence for all $s \in \mathbb{R}$.

Proof of Theorem 2.1. First suppose $s \geq 0$ and the control

$$\|\varphi_\ell(u - \tau_{-y}u)\|_{H^s} \leq \|\varphi_\ell u - \tau_{-y}(\varphi_\ell u)\|_{H^s} + \|(\varphi_\ell - \tau_{-y}\varphi_\ell)(\tau_{-y}u)\|_{H^s}, \tag{2.4}$$

since $[\tau_{-y}(\varphi_\ell u)](x) = (\tau_{-y}\varphi_\ell)(x)(\tau_{-y}u)(x)$. Note that supp $(\tau_{-y}\varphi_\ell) \subset \Omega_{\ell+1} + B_{1/j}(0) \subset \Omega_{\ell_0}$ for some $l_0 \in \mathbb{N}$, then

$$(\varphi_\ell - \tau_{-y}\varphi_\ell)(\tau_{-y}u) = (\varphi_\ell - \tau_{-y}\varphi_\ell)(\varphi_{\ell_0+1}\tau_{-y}u).$$

Since $\varphi_{\ell_0} = 1$ in Ω_{ℓ_0} then

$$\begin{aligned} \|(\varphi_\ell - \tau_{-y}\varphi_\ell)(\tau_{-y}u)\|_{H^s} &= \|(\varphi_\ell - \tau_{-y}\varphi_\ell)(\varphi_{\ell_0}\tau_{-y}u)\|_{H^s} \\ &\leq C_s \|(\varphi_\ell - \tau_{-y}\varphi_\ell)\|_{H^s} \, \|(\varphi_{\ell_0}\tau_{-y}u)\|_{H^s}, \end{aligned}$$

for some $C_s > 0$ (note that $\varphi_\ell - \tau_{-y}\varphi_\ell \in C_c^\infty(\Omega)$ then there is not the restriction $s > n/2$ for the product in H^s). Thus $\varphi_{\ell_0}\tau_{-y}u = \tau_{-y}\left[(\tau_y\varphi_{\ell_0})\, u\right]$ which implies that $\|\varphi_{\ell_0}\tau_{-y}u\|_{H^s} = \|\left(\tau_y\varphi_{\ell_0}\right) u\|_{H^s}$.

Furthermore, by the exaustion of Ω there exists $l_1 \in \mathbb{N}$ such that supp $[\tau_y\varphi_{\ell_0}] \subset \Omega_{\ell_0+1} + B_{1/j}(0) \subset \Omega_{\ell_1}$ for $|y| < 1/j$ and $j \geq l_1$ which implies $\tau_y\varphi_{\ell_0} = (\tau_y\varphi_{\ell_0})\varphi_{\ell_1}$ and

$$\begin{aligned} \|\left(\tau_y\varphi_{\ell_0}\right) u\|_{H^s} &= \|\left(\tau_y\varphi_{\ell_0}\right)\varphi_{\ell_1} u\|_{H^s} \leq \\ &\leq C_s\|\tau_y\varphi_{\ell_0}\|_{H^s}\|\varphi_{\ell_1}u\|_{H^s} = C_s\|\varphi_{\ell_0}\|_{H^s}\|\varphi_{\ell_1}u\|_{H^s}. \end{aligned}$$

In other words, there exists $K_{s,l_0} > 0$ with $\|(\tau_y \varphi_{\ell_0})u\|_{H^s} \leq K_{s,l_0}\|\varphi_{\ell_1}u\|_{H^s}$.

Now, by the Lemma 2.3, for every $\epsilon > 0$ there exists $j_0 \in \mathbb{N}$ such that for every $j \geq j_0$ we have $\|(\varphi_\ell - \tau_{-y}\varphi_\ell)\|_{H^s} \leq \epsilon(2K_s\|\varphi_{\ell_1}u\|_{H^s})^{-1}$ and

$$\|\varphi_\ell u - \tau_{-y}(\varphi_\ell u)\|_{H^s(\mathbb{R}^n)} \leq \epsilon/2 \text{ for every } y \in B_{1/j}(0).$$

Analogously $\|\varphi_\ell u - \tau_{-y}(\varphi_\ell u)\|_{H^s} \leq \epsilon/2$ and then from (2.4) we have

$$\|\varphi_\ell(u - \tau_{-y}u)\|_{H^s} \leq \epsilon \tag{2.5}$$

for $y \in B_{1/j}(0)$ and $j \geq j_0$.

In order to conclude the theorem follows from the identity (2.1) and Lemma 2.2 with $f(x,y) \doteq \phi_j(y)[\varphi_\ell(u - \tau_{-y}u)](x)$ that

$$\|\varphi_\ell(u_j - u)\|_{H^s} = \left\|\int f(\cdot,y)dy\right\|_{H^s} \leq \int \|f(\cdot,y)\|_{H^s}dy$$

Observe that $f(\cdot,y) \in H^s(\mathbb{R}^n)$ for each $y \in B_{1/j}(0)$, because

$$\|f(\cdot,y)\|_{H^s} \leq \phi_j(y)\|\varphi_\ell(u - \tau_{-y}u)\|_{H^s} \leq \epsilon\phi_j(y)$$

and then $y \mapsto \chi_{B_{1/j}(0)}\|f(\cdot,y)\|_{H^s}$ is in $L^1(\mathbb{R}^n)$ since $\int \phi_j(y)dy = 1$ and $j \geq j_0$. So $\|\varphi_\ell(u_j - u)\|_{H^s} \leq \epsilon$ for $j \geq j_0$ por each $\ell \in \mathbb{N}$, thus $u_j \to u$ in the topology of $H^s_{loc}(\Omega)$ for $s \geq 0$.

Now we are moving on to $s < 0$. Since $H^s(\mathbb{R}^n)$ is isomorphic to the dual space of $H^{-s}(\mathbb{R}^n)$ for $s > 0$, the proof follows by duality. If $g \in H^{-s}_{loc}(\Omega)$ then $g_j \doteq \phi_j \star (\chi_{\Omega_j} g_e)$ converges to g in $H^{-s}_{loc}(\Omega)$ when

$$\langle \varphi_\ell g, u\rangle = \lim_{j\to\infty} \langle \varphi_\ell g_j, u\rangle, \qquad \forall\, u \in H^s(\mathbb{R}^n) \text{ and } \ell \in \mathbb{N}.$$

Note that, for each $\ell \in \mathbb{N}$ and $u \in H^s(\mathbb{R}^n)$ we have that

$$\langle \varphi_\ell g_j, u\rangle = \langle g_j, \varphi_\ell u\rangle = \left\langle g, \chi_{\Omega_j}\left[\tilde{\phi}_j \star (\varphi_\ell u)\right]\right\rangle$$

where $\tilde{\phi}_j(x) := \phi_j(-x), x \in \mathbb{R}^n$ and $j \in \mathbb{N}$. Moreover, for j sufficient large supp $[\tilde{\phi}_j \star (\varphi_\ell u)] \subset \Omega_{\ell+1} + B_{1/j}(0) \subset \Omega_{\ell+2} \subset \Omega_j$ and then

$\chi_{\Omega_j}\left[\tilde{\phi}_j \star (\varphi_\ell u)\right] = \tilde{\phi}_j \star (\varphi_\ell u)$. Since $\tilde{\phi}_j \star v \xrightarrow{H^s} v$ and taking $v = \varphi_\ell u$ we have that $\tilde{\phi}_j \star (\varphi_\ell u) \xrightarrow{H^s} \varphi_\ell u$ and consequently

$$\langle \varphi_\ell g_j, u\rangle = \left\langle g, \tilde{\phi}_j \star (\varphi_\ell u)\right\rangle \longrightarrow \langle g, \varphi_\ell u\rangle = \langle \varphi_\ell g, u\rangle$$

as $j \to \infty$, as desired. $\square$

Now we use the previous theorem to calculate the closure of an elliptic homogeneous differential operator.

Theorem 2.4. *Let $a(D) : H_0^{s+m}(\Omega; E) \subset H^s_{loc}(\Omega; E) \to H^s_{loc}(\Omega; F)$ be an elliptic homogeneous linear differential operator with constant coefficients of order m with $s \in \mathbb{R}$ and $m \in \mathbb{N}$. Then its closure is given by*

$$\overline{a(D)} : H^{s+m}_{loc}(\Omega; E) \subset H^s_{loc}(\Omega; E) \longrightarrow H^s_{loc}(\Omega; F)$$

with $\overline{a(D)}$ and $a(D)$ are given by the same formula.

Proof. Let $\overline{a(D)} : D\left[\overline{a(D)}\right] \subset H^s_{loc}(\Omega; E) \longrightarrow H^s_{loc}(\Omega; F)$ be the closure of $a(D)$ where

$$D\left[\overline{a(D)}\right] = \Big\{u \in H^s_{loc}(\Omega; E);\ \exists\, (u_j)_{j\in\mathbb{N}} \subset H_0^{s+m}(\Omega; E) \text{ and}$$
$$f \in H^s_{loc}(\Omega; F) \text{ s.t. } u_j \xrightarrow{H^s_{loc}} u \text{ and } a(D)u_j \xrightarrow{H^s_{loc}} f\Big\}.$$

We claim that $D\left[\overline{a(D)}\right] = H^{s+m}_{loc}(\Omega; E)$. If $u \in D\left[\overline{a(D)}\right]$ then there exists a sequence $(u_j)_{j\in\mathbb{N}} \subset H_0^{s+m}$ with $u_j \xrightarrow{H^s_{loc}} u$ and $a(D)u_j \xrightarrow{H^s_{loc}} f$ for some $f \in H^s_{loc}(\Omega; F)$. Thus, $a(D)u = f$ in $\mathcal{D}'(\Omega; F)$ and as $f \in H^s_{loc}(\Omega; F)$ follows from [2, Theorem 6.33], under ellipticity of $a(D)$, which implies that $u \in H^{s+m}_{loc}(\Omega; E)$.

Conversely, if $u \in H^{s+m}_{loc}(\Omega; E)$ we have that $f := a(D)u \in H^s_{loc}(\Omega; F)$. Let $u_j = \varphi_j \star (\chi_{\Omega_j} u_e)$ for $j \in \mathbb{N}$ as Theorem 2.1 applied for each component of u, then

$$u_j \xrightarrow{H^{s+m}_{loc}} u \implies \partial^\alpha u_j \xrightarrow{H^s_{loc}} \partial^\alpha u \implies a_\alpha \partial^\alpha u_j \xrightarrow{H^s_{loc}} a_\alpha \partial^\alpha u$$

for $|\alpha| = m$. Thus, $a(D)u_j \xrightarrow{H^s_{loc}} a(D)u \in H^s_{loc}(\Omega; F)$ and the conclusion follows. $\square$

3 Proof of Theorem A

Let $d := \dim E$ and $a_\beta \in \mathcal{L}(E;E)$ invertible for $\beta := m\cdot e_1$ for simplicity (see Remark 1.5). Consider the decomposition

$$a(D) = b(D) + \sum_{\alpha\neq\beta} a_\alpha \partial^\alpha$$

where $b(D) = a_\beta \frac{\partial^m}{\partial x_1^m}$ is a homogeneous differential operator with constant coefficients of order m only in the variable x_1.

We claim that for every $\lambda \in \mathbb{C}$, the one dimensional differential operator $\lambda I_E - b(D)$ has a solution in $C^\infty(\mathbb{R};E)\setminus\{0\}$. Since a_β is a complex matrix invertible, there exists a basis of E such that a_β is upper triangular. After change of coordinates, we may assume

$$a_\beta = \begin{bmatrix} c_{11} & c_{12} & c_{13} & \dots & c_{1d} \\ 0 & c_{22} & c_{23} & \dots & c_{2d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & c_{dd} \end{bmatrix}$$

with $c_{jk} \in \mathbb{C}$ for $1 \leq j,k \leq d$ and $\Pi_{j=1}^d c_{jj} \neq 0$. Thus a solution for $b(D)u = \lambda u$ can be rewritten as

$$\begin{aligned} c_{11}\frac{\partial^m u_1}{\partial x_1^m} + c_{12}\frac{\partial^m u_2}{\partial x_1^m} + \ldots + c_{1d}\frac{\partial^m u_d}{\partial x_1^m} &= \lambda u_1 \\ c_{22}\frac{\partial^m u_2}{\partial x_1^m} + \ldots + c_{2d}\frac{\partial^m u_d}{\partial x_1^m} &= \lambda u_2 \\ &\vdots \\ c_{dd}\frac{\partial^m u_d}{\partial x_1^m} &= \lambda u_d, \end{aligned}$$

where $u(x) = (u_1(x), \ldots, u_d(x))$. Let $f(s) = e^{\xi_0 s}$ and $F(s) = (f(s), 0, \ldots, 0)$ where $F : \mathbb{R} \to \mathbb{C}^d$ and ξ_0 is a complex root of $p(t) = c_{11}t^m - \lambda$. Clearly $b(D)F = \lambda F$.

Defining $u(x_1, \ldots, x_n) := F(x_1) \in C^\infty(\Omega; E) \setminus \{0\}$ we have that $\partial^\alpha u = 0$ for $\alpha \neq \beta$ so

$$a(D)u = b(D)u + \sum_{\alpha \neq \beta} a_\alpha \partial^\alpha u = b(D)F = \lambda F = \lambda u,$$

i.e. $[\lambda I_E - a(D)]\, u = 0$ for $u \in C^\infty(\Omega; E) \setminus \{0\}$ and then $\lambda \in \sigma_p\left(\overline{a(D)}\right)$. Therefore

$$\mathbb{C} \subset \sigma_p\left(\overline{a(D)}\right) \implies \sigma\left(\overline{a(D)}\right) = \sigma_p\left(\overline{a(D)}\right) = \mathbb{C}.$$

□

Combining the Theorem 2.4 and the Theorem A, analogous in [1] for dimension one, we may conclude the following:

Corollary 3.1. *Let $s \in \mathbb{R}$. The Laplace operator $\Delta : H_0^{s+2}(\Omega) \subset H^s_{loc}(\Omega) \longrightarrow H^s_{loc}(\Omega)$ and its closure $\overline{\Delta} : H^{s+2}_{loc}(\Omega) \subset H^s_{loc}(\Omega) \longrightarrow H^s_{loc}(\Omega)$, have resolvent set empty and their spectra are the whole plane, in the other words*

$$\sigma(\Delta) = \sigma\left(\overline{\Delta}\right) = \sigma_p\left(\overline{\Delta}\right) = \mathbb{C}.$$

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