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A note on solvability of vector fields in Denjoy-Carleman classes

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Abstract. We deal with solvability in Denjoy-Carleman classes of complex vector fields defined on $\Omega = \mathbb{R} \times S^1$, given by $\mathcal{L} = \partial/\partial t + (a(x,t)+ib(x,t))\partial/\partial x$, $b \not\equiv 0$, near the characteristic set $\Sigma = \{0\} \times S^1$. We assume that $\mathcal{L}$ vanishes of first order along Σ and that a certain invariant associate to $\mathcal{L}$ is an irracional number.

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1 Introduction

Let $\mathcal{M} = \{m_j\}_{j \in \mathbb{Z}_+}$ be a sequence of real numbers satisfying the following assumptions:

(M.1) (initial condition) $m_0 = m_1 = 1$;

(M.2) (logarithmic convexity) $m_j^2 \leq m_{j-1} \cdot m_{j+1}$, $\forall j \geq 1$;

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(M.3) (moderate growth) $\sup_{j,k}\left(\dfrac{m_{j+k}}{m_j\cdot m_k}\right)^{1/(j+k)}\leq H$, for some $H>1$.

Let $\Omega=\mathbb{R}\times S^1$, where S^1 is the unit circle.

We say that a complex-valued function $f\in C^\infty(\Omega)$ is ultradifferentiable of class $\mathcal{M}$ if for each compact set $K\subset\Omega$ there are constants $C,h>0$ such that

$$|D^\alpha f(x)|\leq C\cdot h^{|\alpha|}\cdot m_{|\alpha|}\cdot|\alpha|!,\ \forall x\in K,\ \forall\alpha\in\mathbb{Z}_+^2.$$

We denote $\mathcal{E}_{\mathcal{M}}(\Omega)$ the space of ultradifferentiable functions of class $\mathcal{M}$ in Ω. Such classes of functions are called Denjoy-Carleman classes of Romieu tipe. For more about Denjoy-Carleman classes see, for instance, [8], [14], and [17].

The space $\mathcal{E}_{\mathcal{M}}(\Omega)$ is a ring (with usual operations); also, it is closed for composition and for derivation.

Classical spaces of ultradifferentiable functions are the so-called Gevrey spaces. For $s\geq 1$, the s-Gevrey space is the set of ultradifferentiable functions of class $\mathcal{M}=\{j!^{s-1}\}$ in Ω, usually denoted by $G^s(\Omega)$. Hence, $G^1(\Omega)$ is the space of real-analytic functions on Ω. For more about Gevrey functions see [15].

Let

$$\mathcal{L}=\partial/\partial t+(a(x,t)+ib(x,t))\partial/\partial x,\quad b\not\equiv 0,\tag{1.1}$$

be a complex vector field defined on Ω, where a and b are real-valued functions of class $\mathcal{M}$.

The characteristic set of the structure associated with $\mathcal{L}$, denoted here by $\mathcal{C}(\mathcal{L})$, is the set of all points $(x,t)\in\Omega$ where $\mathcal{L}$ fails to be elliptic, that is:

$$\mathcal{C}(\mathcal{L})=\{(x,t)\in\Omega\,;\,\mathcal{L}_{(x,t)}\text{ and }\overline{\mathcal{L}}_{(x,t)}\text{ are linearly dependent}\}.$$

A characteristic point p is said to be of *finite type* $k\in\mathbb{Z}_+$ if there exists an iterated Lie bracket of $\mathcal{L}$ and $\overline{\mathcal{L}}$ of length k which is nonzero at p, and k is minimal with this property. Otherwise, p is said to be of infinite type.

Let $\Sigma = \{0\} \times S^1$. Throughout this paper we will assume $S^1 \ni t \mapsto (a+ib)(0,t) \equiv 0$, and $b(x,t) \neq 0$ for $x \neq 0$ and for all $t \in S^1$. Hence, $\mathcal{C}(\mathcal{L}) = \Sigma$ and each point in Σ is of infinite type.

Note that under such assumptions $\mathcal{L}$ satisfies the well-known Nirenberg-Treves condition $(\mathcal{P})$ for (smooth) local solvability (see [4] and [13]) and, furthermore, near the characteristic set Σ we may write

$$(a+ib)(x,t) = x^n(a_0(x,t) + ib_0(x,t)), \quad n \geq 1, \quad \text{in} \quad \Omega_\epsilon = (-\epsilon, \epsilon) \times S^1, \tag{1.2}$$

where $a_0, b_0 \in \mathcal{E}_{\mathcal{M}}(\Omega_\epsilon)$, with $0 < \epsilon < 1$.

In this paper we will deal with the solvability of the equation $\mathcal{L}u = f$ in classes of ultradifferentiable functions, in a full neighborhood of Σ, and we will be concerned with the situation where

$$n = 1, \quad \int_0^{2\pi} a_0(0,t)\mathrm{d}t = 0, \quad b_0(0,t) \neq 0 \text{ for all } t \in S^1, \tag{1.3}$$

and the Meziani number λ (an invariant attached to $\mathcal{L}$ (see [12])) satisfies

$$\lambda \doteq \int_0^{2\pi} b_0(0,t)\mathrm{d}t \in \mathbb{R} \setminus \mathbb{Q}. \tag{1.4}$$

It is a follow-up to the paper [1], where the problem was considered in Gevrey classes.

Motived by [1], we will consider the following notion of solvability: given a sequence of real numbers $\mathcal{M} = \{m_j\}_{j \in \mathbb{Z}_+}$ satisfying (M.1)-(M.3), and assuming that coefficients of $\mathcal{L}$ are in $\mathcal{E}_{\mathcal{M}}(\Omega)$, we say that $\mathcal{L}$ is $\mathcal{E}_{\mathcal{M}}$*−solvable* at Σ if for any f belonging to a subspace of finite codimension of $\mathcal{E}_{\mathcal{M}}(\Omega)$ there exists a solution $u \in \mathcal{E}_{\mathcal{M}}$ to the equation $\mathcal{L}u = f$ defined in a neighborhood of Σ.

It was showed in [1] that:

Theorem 1.1. *Let $s \geq 1$. Let $\mathcal{L}$ be a complex vector field defined on Ω in the form (1.1). Assume that the coefficients of $\mathcal{L}$ are in $G^s(\Omega)$ and satisfy (1.2), (1.3), and (1.4). Then, $\mathcal{L}$ is not G^s-solvable at Σ.*

Theorem above is related to results in [3] and [6], where solvability was studied both in the smooth and in the real-analytic cases.

Now, for a sequence $\mathcal{M} = \{m_j\}_{j\in\mathbb{Z}_+}$ consider the following property:

(M.4) there is a constant $A > 1$ such that for all $p \geq 1$,

$$\sum_{j=p}^{\infty} \frac{m_j}{m_{j+1}\cdot(j+1)} \leq A\,\frac{m_p}{m_{p+1}}$$

A sequence of real numbers $\mathcal{M} = \{m_j\}_{j\in\mathbb{Z}_+}$ satisfying (M.1)-(M.4) is said to be *strongly regular*.

It is easy to see that the sequence $\mathcal{M} = \{m_j\}_{j\in\mathbb{Z}_+}$, given by $m_j = (j!)^{s-1}$, is strongly regular.

It follows from [16] that

Proposition 1.2. *Let $\mathcal{M} = \{m_j\}_{j\in\mathbb{Z}_+}$ be a strongly regular sequence. Then, there are real numbers $1 < s < \sigma$ such that $G^s(\Omega) \subset \mathcal{E}_{\mathcal{M}}(\Omega) \subset G^\sigma(\Omega)$.*

Hence, as a consequence of theorem 1.1 and proposition 1.2, we obtain:

Theorem 1.3. *Let $\mathcal{M} = \{m_j\}_{j\in\mathbb{Z}_+}$ be a strongly regular sequence. Let $\mathcal{L}$ be a complex vector field defined on Ω in the form (1.1). Assume that the coefficients of $\mathcal{L}$ are in $\mathcal{E}_{\mathcal{M}}(\Omega)$ and satisfy (1.2), (1.3), and (1.4). Then, $\mathcal{L}$ is not $\mathcal{E}_{\mathcal{M}}$-solvable at Σ.*

By using properties of strongly regular sequences and arguing as in [1], in this present paper we give an alternative proof of the theorem 1.3. For this purpose we extend to ultradifferentiable classes of functions, given by strongly regular sequences, some results previously obtained to Gevrey classes in [1]. We present examples and some useful properties of the class of functions (for more examples and properties see, for instance, [5]). Also, we show that a certain Diophantine condition arised from analytic solvability (see [3] and [6]) revels to be (actually) a necessary condition of $\mathcal{M}$-solvability for any sequence $\mathcal{M}$ satisfying (M.1)-(M.3).

2 Properties, examples and useful results

For a sequence $\mathcal{M} = \{m_j\}_{j\in\mathbb{Z}_+}$ satisfying (M.1)-(M.3) we have the well-known properties:

(a) $\{m_j\}_{j\in\mathbb{Z}_+}$ and $\{(m_j)^{1/j}\}_{j\in\mathbb{Z}_+}$ are non-decreasing sequences;

(b) $m_k \cdot m_{j-k} \leq m_j$, for all $k, j \in \mathbb{Z}_+$, $j \geq k$

It follows from (M.3) that for $j \geq 2$,

$$\left(\frac{m_j}{m_{j-1}m_1}\right)^{1/j} \leq H;$$

hence, for $2 \leq k \leq j$ we obtain

$$m_k^{1/(k-1)} \leq H^{k/(k-1)} \cdot m_{k-1}^{1/(k-1)} \leq H^2 \cdot m_{j-1}^{1/(j-1)} \leq H^2 \cdot m_j^{1/(j-1)}.$$

Therefore, it follows from [9] the following version of inverse mapping theorem:

Theorem 2.1 (Inverse Mapping Theorem)**.** *Let* $\mathcal{M} = \{m_j\}_{j\in\mathbb{Z}_+}$ *be a sequence of real numbers satisfying* (M.1)-(M.3)*. Let* U *be an open subset of* $\mathbb{R}^n, p \in U$ *and* $f \in \mathcal{E}_{\mathcal{M}}(U, \mathbb{R}^n)$*. If* $df(p) : \mathbb{R}^n \to \mathbb{R}^n$ *is an isomorphism, then there exists a neighborhood* V *of* p *and an open subset* $W \subset \mathbb{R}^n$ *such that* $f : V \to W$ *is a diffeomorphism of class* $\mathcal{E}_{\mathcal{M}}$*.*

Example 2.2. Let $\sigma > 0$. The sequence given by $m_j = [\ln(j + e - 1)]^{\sigma j}$, $j \in \mathbb{Z}_+$, satisfies (M.1)-(M.3).

Next we will present examples of strongly regular sequences.

Example 2.3. The sequence $\mathcal{M} = \{m_j\}_{j\in\mathbb{Z}_+}$ given by $m_0 = m_1 = 1$ and

$$m_j = j!^\alpha \left[\ln(e + j)\right]^{\beta j}, \text{ for } j \geq 2,$$

where $\alpha > 0$ and $\beta \geq 0$, is strongly regular. Indeed, properties (M.1)-(M.3) are easy to check. We will show (M.4). For $p \geq 2$ we have

$$\sum_{j=p}^{\infty} \frac{m_j}{m_{j+1} \cdot (j+1)} = \sum_{j=p}^{\infty} \frac{j!^\alpha \left[\ln(e + j)\right]^{\beta j}}{(j+1)!^\alpha \left[\ln(e + j + 1)\right]^{\beta(j+1)} (j+1)}$$

$$= \sum_{j=p}^{\infty} \frac{1}{(j+1)^{\alpha+1}\left[\ln(e+j+1)\right]^{\beta}} \cdot \left(\frac{\ln(e+j)}{\ln(e+j+1)}\right)^{\beta j}$$

$$\leq \sum_{j=p}^{\infty} \frac{1}{(j+1)^{\alpha+1}\left[\ln(e+j+1)\right]^{\beta}}$$

$$\leq \int_p^{\infty} \frac{1}{x^{\alpha+1}\left[\ln(e+x)\right]^{\beta}} dx$$

$$= \left[-\frac{1}{\alpha x^{\alpha}\left[\ln(e+x)\right]^{\beta}}\right]_p^{\infty} - \int_p^{\infty} \frac{\beta\left[\ln(e+x)\right]^{-\beta-1}}{\alpha x^{\alpha}(e+x)} dx$$

$$\leq \frac{1}{\alpha p^{\alpha}\left[\ln(e+p)\right]^{\beta}}$$

$$= \frac{1}{\alpha} \cdot \left(\frac{p+1}{p}\right)^{\alpha} \cdot \left(\frac{\ln(e+p+1)}{\ln(e+p)}\right)^{\beta(p+1)} \cdot \frac{p!^{\alpha}\left[\ln(e+p)\right]^{\beta p}}{(p+1)!^{\alpha}\left[\ln(e+p+1)\right]^{\beta(p+1)}}$$

$$\leq \frac{1}{\alpha} \cdot 2^{\alpha} \cdot e^{\beta} \cdot \frac{m_p}{m_{p+1}},$$

since

$$\left(\frac{\ln(e+p+1)}{\ln(e+p)}\right)^{(p+1)} \leq \left(\frac{e+p+1}{e+p}\right)^{(p+1)} \leq \left(1+\frac{1}{1+p}\right)^{(1+p)} \leq e\,.$$

Remark 2.4. For $\beta = 0$ the class $\mathcal{M}$ is the Gevrey classs of order $\alpha + 1$.

Condition (M.4) is called *strong non-quasianalyticity* and implies, in particular, the well-known Denjoy-Carleman condition of *non-quasianalyticity*

(M.4)’ $$\sum_{j=0}^{\infty} \frac{m_j}{m_{j+1}\cdot(j+1)} < \infty\,;$$

hence, $\mathcal{E}_{\mathcal{M}}(\Omega)$ contains non-zero function f which is flat at a point $(x_0, t_0) \in \Omega$.

Recall that a sequence of real numbes $\mathcal{M} = \{m_j\}_{j\in\mathbb{Z}_+}$ satisfying (M.4)’ is said be non-quasianalytic; outherwise, $\mathcal{M}$ is said be quasianalytic.

The next example shows that (M.4) is stronger than (M.4)’.

Example 2.5. Let $\mathcal{M} = \{m_j\}_{j\in\mathbb{Z}_+}$ be given by $m_j = [\ln(j+e-1)]^{\sigma j}$, $j \in \mathbb{Z}_+$. Then, $\mathcal{M}$ is quasianalytic for $0 < \sigma \leq 1$, and non-quasianalytic for $\sigma > 1$. On the other hand, $\mathcal{M}$ is not strongly non-quasianalytic for any $\sigma > 0$.

Let $\mathcal{M} = \{m_j\}_{j\in\mathbb{Z}_+}$ and $\mathcal{N} = \{n_j\}_{j\in\mathbb{Z}_+}$ be sequences of real numbers satisfying (M.1)-(M.3). We say that $\mathcal{M} \preceq \mathcal{N}$ if the following condition is satisfied

$$\sup_{j\in\mathbb{Z}_+}\left(\frac{m_j}{n_j}\right)^{1/j} < \infty.$$

If $\mathcal{M} \preceq \mathcal{N}$ and, also, $\mathcal{N} \preceq \mathcal{M}$, we denote $\mathcal{M} \approx \mathcal{N}$.

It is easy to verify that $\preceq$ is reflexive and transitive; hence, $\approx$ is an equivalence relation.

It follows from [17] that $\mathcal{E}_{\mathcal{M}}(\Omega) \subset \mathcal{E}_{\mathcal{N}}(\Omega)$ if and only if $\mathcal{M} \preceq \mathcal{N}$.

Example 2.6. $G^1(\Omega) \subset \mathcal{E}_{\mathcal{M}}(\Omega)$ for any sequence $\mathcal{M}$ satisfying (M.1)-(M.3).

Example 2.7. If $1 \leq s \leq \sigma$ then $G^s(\Omega) \subset G^\sigma(\Omega)$.

Example 2.8. Let $s \geq 1$ and let $\mathcal{M}$ be given in example 2.3, with $\alpha = s-1$. Then, $G^s(\Omega) \subset \mathcal{E}_{\mathcal{M}}(\Omega)$.

Example 2.9. Let $\mathcal{M} = \{m_j\}_{j\in\mathbb{Z}_+}$ and $\mathcal{M}^\sigma = \{(m_j)^\sigma\}_{j\in\mathbb{Z}_+}$, with $0 < \sigma < 1$. Then, $\mathcal{E}_{\mathcal{M}^\sigma}(\Omega) \subset \mathcal{E}_{\mathcal{M}}(\Omega)$. Indeed, for $0 < \sigma < 1$ we have

$$\sup_{j\in\mathbb{Z}_+}\left(\frac{m_j^\sigma}{m_j}\right)^{1/j} \leq 1.$$

For strong non-quasianalitic sequences we have the following version of Borel's theorem (see [14] - Theorem 2.1 (a)(ii); also, [17]):

Theorem 2.10. *Let $\mathcal{M}$ be a strongly regular sequence. Let $\{\alpha_J\}_{J\in\mathbb{Z}_+^2}$ be a sequence in $\mathbb{C}$, satisfying*

$$|\alpha_J| \leq B^{|J|+1} \cdot m_{|J|} \cdot J!, \tag{2.1}$$

for some constant $B > 0$. Then, there is $f \in \mathcal{E}_{\mathcal{M}}(\mathbb{R}^2)$ such that $\partial^J f(0,0) = \alpha_J$.

Note that the sequence $\mathcal{M} = \{m_j\}_{j\in\mathbb{Z}_+}$ given in example 2.3 satisfies the following property: $\mathcal{M}^\sigma = \{(m_n)^\sigma\}_{n\in\mathbb{Z}_+}$ is strongly regular for any $\sigma > 0$. Actually, as can be seen below, this property is satisfied for any strongly regular sequence.

Lemma 2.11. *[16, Lema 1.3.4] Let $\mathcal{M} = \{m_j\}_{j\in\mathbb{Z}_+}$ be a strongly regular sequence. Then, for any real number $\sigma > 0$, the sequence $\mathcal{M}^\sigma = \{(m_j)^\sigma\}_{j\in\mathbb{Z}_+}$ is strongly regular.*

3 Solvability results

Let $\mathcal{M} = \{m_j\}_{j\in\mathbb{Z}_+}$ be a sequence of real numbers satisfying (M.1)-(M.3).

Let $\mathcal{L}$ be given by (1.1). Assume that the coefficients of $\mathcal{L}$ are in $\mathcal{E}_{\mathcal{M}}(\Omega)$ and satisfy (1.2)-(1.4).

Let $f \in \mathcal{E}_{\mathcal{M}}(\Omega)$ and assume that there is $u \in \mathcal{E}_{\mathcal{N}}(\Omega)$, with $\mathcal{M} \preceq \mathcal{N}$, solution to $\mathcal{L}u = f$ in a neighborhood of Σ. Then,

$$\int_0^{2\pi} f(0,t)dt = \int_0^{2\pi} \mathcal{L}u(0,t)dt = \int_0^{2\pi} \frac{\partial u}{\partial t}(0,t)dt = 0\,. \tag{3.1}$$

Since the invariant λ given by (1.4) is an irrational number, (3.1) is the unique compatility condition for the equation $\mathcal{L}u = f$ (see, for instance, [7]).

Define

$$\mathcal{F}_{\mathcal{M}} = \{f \in \mathcal{E}_{\mathcal{M}}\,;\, f \text{ satisfies } (3.1)\}\,.$$

For $\mathcal{M} \preceq \mathcal{N}$, we say that $\mathcal{L}$ is $(\mathcal{N},\mathcal{M})$-solvable at Σ if for any $f \in \mathcal{F}_{\mathcal{M}}$ there exist $u \in \mathcal{E}_{\mathcal{N}}(\Omega)$ solution to $\mathcal{L}u = f$ in a neighborhood of Σ.

The same arguments used in the proof of the necessity of Proposition 3.2 of [6] can be used to prove that:

Proposition 3.1. *Let $\mathcal{M}$ and $\mathcal{N}$ be sequences of real numbers satisfying* (M.1)-(M.3). *Assume that $\mathcal{M} \preceq \mathcal{N}$. Let $\mathcal{L}$ be a complex vector field defined on Ω in the form* (1.1). *Assume that the coefficients of $\mathcal{L}$ are in $\mathcal{E}_{\mathcal{M}}(\Omega)$*

and satisfy (1.2)-(1.4). *If* $\mathcal{L}$ *is* $(\mathcal{N},\mathcal{M})$*-solvable at* Σ *then* $\mathcal{L}$ *is equivalent, via a* $\mathcal{N}$ *diffeomorphism, to a non-vanishing multiple of*

$$\mathsf{L}_\lambda = \partial/\partial t + i\lambda x\partial/\partial x\,. \tag{3.2}$$

.

Proof. In order to keep this work as self-contained as possible we will include here the arguments used in [6].

Let $Z(x,t) = x^{1/\lambda}e^{it}$ be defined in Ω. For $x \neq 0$ we have $L_\lambda Z = 0$, and $dZ \neq 0$; that is, Z is a first integral of L_λ in $\Omega \setminus \Sigma$.

We will look for a diffeomorfism in the form

$$x \mapsto xX(x,t), \quad t \mapsto -t + T(x,t) \tag{3.3}$$

such that

$$W(x,t) \doteq (xX)^{1/\lambda}e^{i(-t+T)}$$

satisfies $\mathcal{L}W = 0$.

We may write

$$(a+ib)(x,t) = x(a_0(x,t) + ib_0(x,t))\,, \quad \text{in} \quad \Omega_\epsilon = (-\epsilon,\epsilon)\times S^1,$$

where $a_0, b_0 \in \mathcal{E}_\mathcal{M}(\Omega_\epsilon)$, with $0 < \epsilon < 1$.

For $x \in (-\epsilon,\epsilon)$ a simple calculation shows that

$$\mathcal{L}\{\lambda^{-1}\ln x - it\} = -i + \lambda^{-1}(a_0 + ib_0)(x,t) \doteq f(x,t).$$

Moreover, by using our assumptions we have

$$\int_0^{2\pi} f(0,t)dt = -2\pi i + 2\pi i = 0\,; \tag{3.4}$$

in particular, the function f belongs to $\mathcal{E}_\mathcal{M}(\Omega_\epsilon)$ and satisfies (3.1). Hence, by our solvability assumption, there is $u \in \mathcal{E}_\mathcal{N}(\Omega)$ such that

$$\mathcal{L}u = -(-i + \lambda^{-1}(a_0 + ib_0)(x,t))$$

in $\Omega_\delta = (-\delta,\delta)\times S^1$, for some $0 < \delta \leq \epsilon$.

For $(x,t) \in \Omega_\delta$ define

$$W(x,t) \doteq \exp\{\lambda^{-1} \ln x - it + u(x,t)\} = x^{1/\lambda} e^{-it+u(x,t)}\,;$$

be simple calculations we obtain $\mathcal{L}W = 0$.

Finally, define the functions

$$X(x,t) \doteq \exp\{\lambda \mathsf{Re} u(x,t)\}, \quad \text{and} \quad T(x,t) \doteq \mathsf{Im} u(x,t).$$

Note that

$$W(x,t) = (x e^{\lambda \mathsf{Re} u})^{1/\lambda} e^{i(-t+\mathsf{Im} u)} = x^{1/\lambda} e^{-it+u} = (xX)^{1/\lambda} e^{i(-t+T)}\,.$$

Now, for $(x,t) \in \Omega_\delta$ define

$$F(x,t) \doteq (xX(x,t), -t + T(x,t)).$$

The determinant of the jacobian matrix of F for points $(0,t) \in \Sigma$ is given by

$$\begin{aligned} \det(J_F)(0,t) &= e^{\lambda \mathsf{Re} u(0,t)} \left(-1 + \mathsf{Im} \frac{\partial u}{\partial t}(0,t) \right) \\ &= e^{\lambda \mathsf{Re} u(0,t)} \left(-1 + (1 - \lambda^{-1} b_0(0,t)) \right) \\ &= -\lambda^{-1} e^{\lambda \mathsf{Re} u(0,t)} b_0(0,t) \neq 0\,. \end{aligned}$$

Hence, by theorem 2.1, F defines a local diffeomorfism of class $\mathcal{N}$ for any $(0,t) \in \Sigma$. Moreover, $F_{|_\Sigma}$ is injective, since

$$\frac{\partial(-t+T)}{\partial t}(0,t) = -1 + (1 - \lambda^{-1} b_0(0,t)) < 0.$$

We claim that there is $0 < \delta' < \delta$ such that $F_{|_{(-\delta',\delta') \times S^1}}$ is injective. Indeed, assume by contradiction that for each $j \in \mathbb{Z}_+$, with $j \geq \delta^{-1}$, there are $p_j, q_j \in \left(-\frac{1}{j}, \frac{1}{j}\right) \times S^1$, with $p_j \neq q_j$ and $F(p_j) = F(q_j)$. In particular, $\{p_j\}, \{q_j\}$ are sequences in the compact set $[-\delta, \delta] \times S^1$. Hence, there are subsequences $\{p_{j_k}\} \subset \{p_j\}$ and $\{q_{j_k}\} \subset \{q_j\}$ such that $p_{j_k} \to (0,t')$ and $q_{j_k} \to (0,t'')$. Furthermore, $F(0,t') = \lim F(p_{j_k}) = \lim F(q_{j_k}) = F(0,t'')$ and, by the injectivity of $F_{|_\Sigma}$, we have $t' = t''$. Let U be a neighborhood of $(0,t')$ for which $F_{|_U}$ is a diffeomorfism. We have $p_{j_k}, q_{j_k} \in U$, for k suficiently large. Hence, $F(p_{j_k}) = F(q_{j_k})$ and, consequently, $p_{j_k} = q_{j_k}$, which is a contradiction.

Therefore, $F_{|_{(-\delta',\delta')\times S^1}} : (-\delta',\delta')\times S^1 \to F((-\delta',\delta')\times S^1)$ is a diffeomorfism of class $\mathcal{N}$, which concludes the proof. □

Before we present our solvability result, we prove the following technical lemma:

Lemma 3.2. *Let $\mathcal{M} = \{m_j\}_{j\in\mathbb{Z}_+}$ be a strongly regular sequence and let $a > 0$. Then, there is a strongly regular sequence $\mathcal{N} = \{n_j\}_{j\in\mathbb{Z}_+}$, with $\mathcal{N} \preceq \mathcal{M}$, such that for any $f \in \mathcal{E}_{\mathcal{N}}(\mathbb{R})$, flat at $x = 0$, the function $g(x) = f(x^a)$ belongs to $\mathcal{E}_{\mathcal{M}}(\mathbb{R})$.*

Proof. The function g is smooth since f is flat at $x = 0$. Also, we can assume without loss of generality that $0 < a < 1$, since otherwise we can write $a = m\tilde{a}$, for some $m \in \mathbb{Z}_+$ and $\tilde{a} \in (0,1)$, and $g(x) = f(x^a) = \tilde{f}(x^{\tilde{a}})$, where $\tilde{f}(y) = f(y^m)$ defines a $\mathcal{M}$ function in $\mathbb{R}$ which is certainly flat at $x = 0$ as well.

By using Faà di Bruno's formula we obtain

$$g^{(\ell)}(x) = \sum_{k_1+2k_2+\ldots+\ell k_\ell=\ell} \frac{\ell!}{k_1!\cdots k_\ell!} f^{(k)}(x^a) \prod_{j=1}^{\ell} \left[\frac{(x^a)^{(j)}}{j!}\right]^{k_j},$$

where $k = k_1 + \ldots + k_\ell$.

Since $(x^a)^{(j)} = a(a-1)\ldots(a-(j-1))x^{a-j}$ we have

$$\begin{aligned} |g^{(\ell)}(x)| &\leq \sum_{k_1+2k_2+\ldots+\ell k_\ell=\ell} \frac{\ell!}{k_1!\cdots k_\ell!} |f^{(k)}(x^a)| \prod_{j=1}^{\ell} \left[\frac{(j-1)!(|x|^{a-j})}{j!}\right]^{k_j} \\ &\leq \sum_{k_1+2k_2+\ldots+\ell k_\ell=\ell} \frac{\ell!}{k_1!\cdots k_\ell!} |f^{(k)}(x^a)||x|^{ka-\ell}. \end{aligned}$$

Let N be the positive integer number satisfying

$$\frac{\ell}{a} \leq N + k < \frac{\ell}{a} + 1\,.$$

Since $f \in \mathcal{E}_{\mathcal{N}}(\mathbb{R})$ is flat at $y = 0$ we can write

$$f^{(k)}(y) = y^N F(y),$$

where

$$|F(y)| \le C R^{N+k} n_{N+k} \frac{(N+k)!}{N!}\,.$$

Hence,

$$\begin{aligned}
|g^{(\ell)}(x)| \le & \sum_{k_1+2k_2+\ldots+\ell k_\ell=\ell} \frac{\ell!}{k_1!\cdots k_\ell!} C R^{N+k} n_{N+k} \frac{(N+k)!}{N!} \\
\le & \sum_{k_1+2k_2+\ldots+\ell k_\ell=\ell} \frac{\ell!}{k_1!\cdots k_\ell!} C R^{N+k} n_{N+k} \frac{2^{N+k} N! k!}{N!} \\
\le & \sum_{k_1+2k_2+\ldots+\ell k_\ell=\ell} \frac{\ell!}{k_1!\cdots k_\ell!} C R^{\ell/a+1} n_{N+k} 2^{\ell/a+1} k! \\
= & \, 2C[(2R)^{1/a}]^{\ell} \ell! \sum_{k_1+2k_2+\ldots+\ell k_\ell=\ell} \frac{k!}{k_1!\cdots k_\ell!}\, n_{N+k}\,.
\end{aligned}$$

Let q be the positive integer satisfying

$$1 < \frac{1}{a} \le q < \frac{1}{a} + 1.$$

Note that $N + k \le q\ell$ and, hence,

$$n_{N+k} \le n_{q\ell};$$

moreover, it follows from (M.3) that

$$n_{q\ell} \le H^{(2+\ldots+q)\ell} n_\ell^q\,.$$

Hence, by taking the sequence $\mathcal{N}$ given by $n_j = m_j^{1/q}$ we obtain

$$|g^{(\ell)}(x)| \le 2C[(2R)^{1/a} H^{(2+\ldots+q)/q}]^{\ell} \ell! m_\ell \sum_{k_1+2k_2+\ldots+\ell k_\ell=\ell} \frac{k!}{k_1!\cdots k_\ell!}\,.$$

Finally, recalling that (see [10] - lemma 1.4.1; also, [2] - lemma 2.2)

$$\sum_{k_1+2k_2+\ldots+\ell k_\ell=\ell} \frac{k!}{k_1!\cdots k_\ell!} = 2^{\ell-1}$$

we obtain

$$|g^{(\ell)}(x)| \le C[2(2R)^{1/a}H^{(2+\ldots+q)/q}]^{\ell}\ell! m_\ell \,.$$

□

The next result is an adaptation of [3], Theorem 3.1.

Theorem 3.3. *Let L_λ be given by (3.2) and defined on Ω, where $\lambda \in \mathbb{R}_+ \setminus \mathbb{Q}$. Let $\mathcal{M} = \{m_j\}_{j\in\mathbb{Z}_+}$ be a strongly regular sequence. Then there exists a function $f \in \mathcal{E}_{\mathcal{M}}(\Omega)$, satisfying*

$$\int_0^{2\pi} f(0,t)dt = 0\,,$$

such that $\mathsf{L}_\lambda u = f$ has no smooth solutions defined near Σ.

Proof. It is enough to construct f near Σ. Again we denote $\Omega_\epsilon = (-\epsilon,\epsilon)\times S^1$, where $\epsilon > 0$; also,

$$\Omega_\epsilon^+ = (0,\epsilon)\times S^1, \quad \Omega_\epsilon^- = (-\epsilon,0)\times S^1.$$

Let $Z(x,t) = |x|^{1/\lambda}e^{it}$, which defines a continuous function on $\Omega = \mathbb{R}\times S^1$ that is real-analytic away from Σ. Simple computations show that on $\Omega\setminus\Sigma$ we have

$$\frac{\partial Z}{\partial x} = \frac{1}{\lambda x}Z \neq 0, \quad \frac{\partial Z}{\partial t} = iZ\,;$$

in particular $\mathsf{L}_\lambda Z = 0$. That is, Z is a first-integral for L_λ on $\Omega\setminus\Sigma$. Moreover, $Z(\Sigma) = \{0\}$ and $Z(\overline{\Omega_\epsilon^+}) = Z(\overline{\Omega_\epsilon^-}) = \overline{D(0,\epsilon^{1/\lambda})}$. Furthermore, $\bar{Z}(x,t) = |x|^{1/\lambda}e^{-it}$; hence, $\mathsf{L}_\lambda\bar{Z} = -2i\bar{Z}$ on $\Omega\setminus\Sigma$.

Therefore, the pushforward of the equations

$$\mathsf{L}_\lambda u = f \quad \text{on} \quad \Omega_\epsilon^\pm$$

via the first integral Z lead us

$$-2i\bar{z}\frac{\partial\tilde{u}^\pm}{\partial\bar{z}} = \tilde{f}^\pm \quad \text{on} \quad D(0,\epsilon^{1/\lambda})\setminus\{0\}\,,$$

where for each $g \in C^\infty(\Omega_\epsilon^\pm)$ we denote $\tilde{g}^\pm = g\circ Z^{-1} \in C^\infty(D(0,\epsilon^{1/\lambda})\setminus\{0\})$.

Now let $\mathcal{N} = \{n_j\}_{j\in\mathbb{Z}_+}$ be as in Lemma 3.2 (relative to $\mathcal{M}$ and $a = \lambda^{-1}$) and take the the formal power series in $z \in \mathbb{C}$ given by

$$\sum_{j=0}^{\infty} n_j j! z^j\,, \tag{3.5}$$

which diverges everywhere off $z = 0$. It follows from theorem 2.10 that there exists $g \in \mathcal{E}_{\mathcal{N}}(\mathbb{C})$ whose Taylor series at $z = 0$ is given by (3.5); hence, $\partial g/\partial\bar{z} \in \mathcal{E}_{\mathcal{N}}(\mathbb{C})$ is flat at $z = 0$. We let

$$f(x,t) = \begin{cases} -2ix^\lambda e^{-it}\frac{\partial g}{\partial \bar{z}}(Z(x,t)), & \text{for } x > 0 \\ 0, & \text{for } x \le 0 \end{cases},$$

which defines a function f on Ω. That is,

$$f = \begin{cases} -2i\bar{Z}\frac{\partial g}{\partial \bar{z}}(Z), & \text{on } (0,\infty)\times S^1 \\ 0, & \text{on } (-\infty,0]\times S^1 \end{cases}.$$

By construction, f is flat along $\Sigma = \{0\}\times S^1 = Z^{-1}(0)$; in particular, $f \in C^\infty(\Omega)$. As a consequence of lemma 3.2 $f \in \mathcal{E}_{\mathcal{M}}(\Omega)$.

We claim that $\mathsf{L}_\lambda u = f$ admits no smooth solutions near Σ. Indeed, assume by contradiction that there is a such solution u. Then, for $z \in D(0,\epsilon^{1/\lambda}) \setminus \{0\}$, we obtain

$$-2i\bar{z}\frac{\partial \tilde{u}^+}{\partial \bar{z}}(z) = -2i\bar{z}\frac{\partial g}{\partial \bar{z}}(z) \quad \text{and} \quad -2i\bar{z}\frac{\partial \tilde{u}^-}{\partial \bar{z}}(z) = 0\,.$$

Thus there are holomorphic functions h^+, h^- on $D(0,\epsilon^{1/\lambda})$ such that $\tilde{u}^+ = g + h^+$ and $\tilde{u}^- = h^-$ in $D(0,\epsilon^{1/\lambda}) \setminus \{0\}$; hence,

$$u(x,t) = \begin{cases} g(Z(x,t)) + h^+(Z(x,t)), & \text{for } x > 0 \\ h^-(Z(x,t)), & \text{for } x < 0 \end{cases}.$$

Therefore, we may conclude that the Taylor series of g at $z = 0$ would match that of $h^- - h^+$, which contradicts the fact that the former series diverges by construction while the latter does not by analyticity. $\square$

Therefore, we can conclude that for any strongly regular sequence $\mathcal{M} = \{m_j\}_{j\in\mathbb{Z}_+}$, a complex vector field $\mathcal{L}$ satisfying the hypotheses of Proposition 3.1 is never $\mathcal{M}$-solvable at Σ, no matter the nature of the irrational invariant λ.

So, a natural question appears: in the case where $\mathcal{M}$ is not a strongly regular sequence, what can we say about $\mathcal{E}_{\mathcal{M}}$-solvability at Σ of $\mathcal{L}$?

An irrational number λ satisfies the Diophantine condition (DC) if there exists $C > 0$ such that

$$|k\lambda + j| \geq C^{j+1}, \quad \forall j \in \mathbb{Z}_+,\ k \in \mathbb{Z}. \tag{DC}$$

Concerning the analytic solvability it was proved in [6], Proposition 3.2:

Proposition 3.4. *Let $\mathcal{L}$ be given by* (1.1)*. Assume that the coefficients of $\mathcal{L}$ satisfy* (1.2)-(1.4)*. Then $\mathcal{L}$ is analytically solvable at Σ if and only if $\mathcal{L}$ is real-analytically equivalent to a non-vanishing multiple of L_λ and λ satisfies condition* (DC)*.*

The next result reveals the true nature of condition (DC): it is a necessary condition to $\mathcal{E}_{\mathcal{M}}$-solvability at Σ of $\mathcal{L}$, no matter the nature of the sequence $\mathcal{M}$.

Proposition 3.5. *Let $\mathcal{M}$ be a sequence of real numbers satisfying* (M.1)-(M.3)*. Suppose that $\lambda \in \mathbb{R}_+ \setminus \mathbb{Q}$ does not satisfy* (DC)*. Then there is a real analytic function f near Σ satisfying* (3.1) *for which the equation $\mathsf{L}_\lambda u = f$ does not have solution $u \in \mathcal{E}_{\mathcal{M}}$, in any neighborhood of Σ.*

Proof. Let $a = 1/\lambda$ and let $\mathsf{T}_a = a\mathsf{L}_\lambda$. If λ does not satisfy (DC) then the same occurs with a ([3], Lemma 2.3) and consequently for each $\ell \in \mathbb{Z}_+$ there are $j_\ell, k_\ell \in \mathbb{Z}_+$ such that

$$|j_\ell - ak_\ell| < \ell^{-(j_\ell+1)}. \tag{3.6}$$

It follows that $j_\ell/k_\ell \to a$, $j_\ell \nearrow \infty$, and there are positive constants c_1 and c_2 such that

$$c_1 j_\ell \leq k_\ell \leq c_2 j_\ell\,.$$

Define

$$f(x,t) = \sum_{\ell \geq 0} x^{j_\ell} e^{ik_\ell t}.$$

As in [3], $f \in G^1(\Sigma)$. If u is a C^∞ solution of $\mathsf{T}_a u = f$ in a neighborhood of Σ then simple computations would give

$$u(x,t) = \sum_{\ell \geq 0} \frac{i}{j_\ell - k_\ell a} x^{j_\ell} e^{ik_\ell t} + K, \quad \text{for some } K \in \mathbb{C}.$$

Now, if u were a $\mathcal{E}_{\mathcal{M}}([-\delta,\delta])$ function then we would have the following decay of its partial Fourier coefficients with respect to t (see, for instance, [11]) there are constants $C, h > 0$:

$$\frac{|x|^{j_\ell}}{|j_\ell - k_\ell a|} = |\hat{u}(x,k_\ell)| < C \cdot \inf_{j \in \mathbb{Z}_+} \left(\frac{m_j \cdot j!}{h^j \cdot (1+k_\ell)^j} \right),$$

for $\ell \in \mathbb{Z}_+$, and for $x \in [-\delta,\delta]$. Hence, it follows from (3.6) that

$$\ell^{j_\ell+1}|x|^{j_\ell} < C \cdot \inf_{j \in \mathbb{Z}_+} \left(\frac{m_j \cdot j!}{h^j \cdot (1+k_\ell)^j} \right).$$

and, consequently, when $|x|\ell \geq 1$,

$$\sup_{j \in \mathbb{Z}_+} \left(\frac{h^j \cdot (1+k_\ell)^l}{m_j \cdot j!} \right) < \ell(\ell|x|)^{j_\ell} \cdot \sup_{j \in \mathbb{Z}_+} \left(\frac{h^j \cdot (1+k_\ell)^j}{m_j \cdot j!} \right) < C.$$

Since $k_\ell \nearrow \infty$ (see (iii) above) we can take ℓ such that $1 + k_\ell > \dfrac{1+C}{h}$; hence,

$$1 + C = h \cdot \left(\frac{1+C}{h} \right) < \sup_{j \in \mathbb{Z}_+} \left(\frac{h^j \cdot (1+k_\ell)^j}{m_j \cdot j!} \right) < C,$$

which is a contradiction. Therefore, u is not $\mathcal{E}_{\mathcal{M}}$ in any neighborhood of Σ. □

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