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A survey on asymptotically autonomous evolution processes

Jacson Simsen

IMC - Universidade Federal de Itajubá
37500-903, Itajubá - MG - Brazil.
jacson@unifei.edu.br

Abstract. The purpose of this survey is to present theoretical results for evolution processes in order to establish convergence in the Hausdorff semi-distance of the component subsets of the pullback attractor of a nonautonomous problem to the global attractor of the corresponding autonomous problem and apply them to Partial Differential Equations.

Keywords: Pullback attractors, evolution processes, global attractors, semigroups, asymptotically autonomous problems, variable exponents.

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1 Introduction

It seems that the earliest results concerning the large time behavior of solutions of asymptotically autonomous ordinary differential equations of the form

$$x' = f(t, x) \tag{1.1}$$

in $\mathbb{R}^n$ with $f(t, x) \to g(x)$ as $t \to \infty$ appeared in the work of Markus [21]. The equation

$$y' = g(y) \tag{1.2}$$

is then called the limit equation of (1.1). It was proved, in the two dimensional case ($n = 2$), that the $\omega-$limit set of a forward bounded solution of (1.1) either contains equilibria of (1.2) or is the union of periodic orbits of (1.2).

Thieme, provided in [27], examples for which the solutions of (1.1) display a large time behavior that differs dramatically from that of the solutions to the limit system (1.2). In the work [28] the author introduced the concept of quasi-autonomous systems and contributed in obtain some more results about $\omega-$limit set of a forward bounded solution of (1.1) relating it to equilibria and periodic orbits of the limit system (1.2). The equation (1.1) is called quasi-autonomous with limit (1.2) if for any compact subset K of X, we have $\int_{s_0}^{\infty} \sup_{x\in K} \|f(s,x) - g(x)\| ds < \infty$ for any $s_0 \in \mathbb{R}$.

The authors of [22] assumed the following two conditions:

(A) $f(t,x) \to g(x),\ t \to \infty,$ uniformly on compact sets of $\mathbb{R}^n$;

(B) g is locally Lipschitz and for each compact subset $K \subseteq \mathbb{R}^n$ there is a function $\mu_K : [0,\infty) \to [0,\infty)$ satisfying $\mu_K(t) \to 0,\ t \to \infty$, and

$$\Big| \int_t^{t+\sigma} [f(s,x) - g(x)] ds \Big| \leq \mu_K(t)$$

for every $(x,\sigma) \in K \times [0,1]$ and $t \geq 0$. By denoting $\phi(t,s,x_0)$ the solution $x(t)$ of (1.1) satisfying $x(s) = x_0$ and $\theta(t,x_0)$ denoting the solution $y(t)$ of (1.2) satisfying $y(0) = x_0$, they proved that

$$\phi(t_j + s_j, s_j, x_j) \to \theta(t,x),\ \ j \to \infty,$$

for any three sequences $t_j \to t,\ \ s_j \to \infty,\ \ x_j \to x$ as $j \to \infty$, with $x, x_j \in X,\ 0 \leq t$ and $s_j \geq t_0$.

The authors of [25] considered asymptotically autonomous ordinary differential equations of the form $x' = f(x) + g(t,x)$ and proved that all the classical solutions tend to zero as $t \to \infty$ provided that f and g are continuous vector-valued functions and $g(t,\cdot)$ approaches zero as $t \to \infty$, uniformly on compact subsets of $\mathbb{R}^n$. Foti considered in [7] asymptotically autonomous ordinary differential inclusions of the form $x' \in F(x)+G(t,x)$ with $G(t,x)$ becoming small in some sense as $t \to \infty$. See also [6] for asymptotically autonomous scalar inclusions.

A second order asymptotically autonomous ordinary differential equation with a potential of the form $x'' = a(t)V'(x)$ with $a(t)$ positive and converging to a constant as $t \to \infty$ was considered in [9] and asymptotically autonomous functional differential systems were considered in [11].

Asymptotically autonomous Partial Differential Equations of semilinear type with the time dependence being only on the external forcing term were considered in the works [1, 2, 5, 8, 19]. The authors of [18] approached a weakly dissipative equation with the time dependence being only on the external forcing term.

Asymptotically autonomous quasilinear PDEs with the main operator depending on time were approached in [13, 14]. After that the authors in [15, 24] had considered asymptotically autonomous Partial Differential Inclusions with the main operator depending on time and the authors of [16] had considered asymptotically autonomous Coupled Systems of Partial Differential Inclusions with the main operators depending on time.

There are already some works in the literature studying the asymptotic autonomy of pullback random attractors, see for example [4, 29].

The theory of evolution processes is an important machinery to study the long time behavior of the global solutions associated with nonautonomous equations. For many of these equations we have the guaranty of the existence of pullback attractors and for some problems the components of the pullback attractor converges towards the global attractor associated to a limiting semigroup. In the Section 3, we will present results that shows the convergence of the pullback attractor to the global attractor if and only if the pullback attractor is forward compact. Other results with different sufficient conditions also highlight this convergence. Results with necessary conditions are also presented. Moreover, we define the limit-set and the lower limit-set of a pullback attractor and we present results that show the relationship between these two and the global attractor. Finally, in Section 4 we present some examples where the abstract results are applied, in particular, in the last example we apply the abstract results to a quasi-linear parabolic equation with variable exponent in which the main operator depends on time. This work is based on the papers [5, 13, 14, 18, 19].

2 Preliminaries

Let (X, d) be a complete metric space. Let us introduce some notations:
$K(X) := \{K \subset X : K$ is a nonempty compact set in X$\}$;
$\mathfrak{B}(X) := \{B \subset X : B$ is a nonempty bounded set in X$\}$;
$\mathbb{R}_+ := [0, \infty)$;
$\uparrow$:= monotonically increasing;
$\downarrow$:= monotonically decreasing.

Definition 2.1. A family of applications $S := \{S(t, s) : X \to X,\ t \geq s \in \mathbb{R}\}$ is called an **evolution process** in X if it satisfies:

(i) $S(s, s) = I_X$ (Identity in X);

(ii) $S(t, s) = S(t, r)S(r, s)$, for all $t \geq r \geq s$.

We will assume that the evolution process S is joint continuous, i.e., the application $[s, +\infty) \times X \ni (t, x) \longmapsto S(t, s)x \in X$ is continuous for any $s \in \mathbb{R}$.

Definition 2.2. An evolution process is called **autonomous** whenever $S(t, s) = S(t - s, 0)$ for all $t \geq s$.

Definition 2.3. [17] A family of operators $T := \{T(t) : t \geq 0\}$ with $T(t) : X \longrightarrow X$ a continuous map for any $t \geq 0$, is called a **semigroup** whenever

(i) $T(0) = I_X$;

(ii) $T(t + s) = T(t)T(s)$, for all $t, s \geq 0$.

Definition 2.4. A semigroup T is called continuous if the following application $\mathbb{R}_+ \times X \ni (t,x) \longmapsto T(t)x \in X$ is continuous.

Remark 2.5. Let $\{S(t-s,0), t \geq s\}$ be an autonomous evolution process. The family of operators $\{T(t) : t \geq 0\}$ given by $T(t) := S(t,0)(t \geq 0)$ defines a semigroup. Reciprocally, if $T := \{T(t) : t \geq 0\}$ is a continuous semigroup, then the family $\{S(t,s) : t \geq s\}$ given by $S(t,s) := T(t-s)$ $(t \geq s)$ defines an evolution process.

Definition 2.6. Let A and M be nonempty subsets of X. We say that A attracts M (through the semigroup T) if for any $\epsilon > 0$, there exists $t(\epsilon, M) \geq 0$ such that $T(t)M \subset \mathcal{O}_\epsilon(A)$ for all $t \geq t(\epsilon, M)$, where $\mathcal{O}_\epsilon(A) := \{x \in X : d(x,A) < \epsilon\}$. We say that A attracts a point $x \in X$ if A attracts the unitary set $\{x\}$.

Definition 2.7. [17] Let A be a nonempty subset of X. If A attracts each point $x \in X$, then A is called a **global attractor of points** (for the semigroup T); If A attracts each set $B \in \mathfrak{B}$, then A is called a **global** $B-$**attractor** .

A time-dependent family of nonempty sets $\mathcal{P} = \{\mathcal{P}(t)\}_{t\in\mathbb{R}}$ in X is said to be a **brochette** over X.

Definition 2.8. [18, 19] A brochette $\mathcal{P}$ over X is called

(i) compact (resp. **bounded**) if $\mathcal{P}(t)$ is compact (resp. **bounded**) for each $a, b \in \mathbb{R}$ with $a < b$;

(ii) locally compact if it is compact and $\bigcup_{s\in[a,b]} \mathcal{P}(s)$ is pre-compact;

(iii) forward compact if it is compact and $\bigcup_{s\geq t}\mathcal{P}(s)$ is pre-compact for each $t \in \mathbb{R}$;

(iv) backward compact if it is compact and $\bigcup_{s\leq t}\mathcal{P}(s)$ is pre-compact for each $t \in \mathbb{R}$;

(v) uniformly compact or globally compact if it is compact and $\bigcup_{s\in\mathbb{R}}\mathcal{P}(s)$ is pre-compact;

(vi) decreasing (resp. **increasing**) if $\mathcal{P}(t_1) \supset \mathcal{P}(t_2)$ (resp. $\mathcal{P}(t_1) \subset \mathcal{P}(t_2)$) for $t_1 < t_2$.

Definition 2.9. [14] Let $\{\mathcal{A}(t) : t \in \mathbb{R}\}$ be a brochette over X. We say that this family of sets is **invariant by the evolution process S** if

$$S(t,s)\mathcal{A}(s) = \mathcal{A}(t), \ \forall \ t \geq s.$$

Definition 2.10. [18] We denote by $dist$ the Hausdorff semi-distance in X between the nonempty sets A and B, i.e.,

$$dist(A,B) := \sup_{a\in A} \inf_{b\in B} d(a,b),$$

and by $dist_H$ the Hausdorff distance in X, i.e.,

$$dist_H(A,B) := \max\{dist(A,B), dist(B,A)\}.$$

Definition 2.11. [18] A family $\mathcal{A} := \{\mathcal{A}(t) : t \in \mathbb{R}\}$ of nonempty compact sets in X is called a **pullback attractor** for the evolution process S if

(i) is invariant by the evolution process S;

(ii) it pullback attracts bounded subsets of X, i.e., for each $B \in \mathfrak{B}$, $t \in \mathbb{R}$,

$$\lim_{\tau \longrightarrow +\infty} dist\big(S(t,t-\tau)B, \mathcal{A}(t)\big) = 0.$$

Remark 2.12. In general we are interested on the minimal closed pullback attractor $\mathcal{A}$ which satisfies (i) and (ii), i.e., if there is another invariant family of closed sets $C := \{C(t) : t \in \mathbb{R}\}$ which pullback attracts bounded sets of X, then $\mathcal{A}(t) \subset C(t)$, for all $t \in \mathbb{R}$.

Lemma 2.13. *[19] A pullback attractor for an evolution process must be locally compact.*

Lemma 2.14. *[19] A pullback attractor $\mathcal{A}$ is forward (resp. backward) compact if and only if $\bigcup_{s\geq t_0}\mathcal{A}(s)$ (resp. $\bigcup_{s\leq t_0}\mathcal{A}(s)$) is pre-compact for some $t_0 \in \mathbb{R}$.*

Proposition 2.15. *[12] A pullback attractor is always continuous at any finite time, i.e.,*

$$\lim_{t\longrightarrow t_0} dist_H\big(\mathcal{A}(t), \mathcal{A}(t_0)\big) = 0.$$

The main aim of this work is to present abstract conditions to assure upper semicontinuity or continuity of the pullback attractors at infinity, i.e., when $t \to \infty$.

Definition 2.16. [17] Let $B \in \mathfrak{B}(X)$. We say that B is **invariant by the semigroup T** if $T(t)B = B$, $\forall\, t \geq 0$.

Definition 2.17. [17] A nonempty set $\mathcal{A}_\infty \subseteq X$ is called a **global attractor** for the semigroup T if

(i) it is compact;

(ii) it is invariant by the semigroup T;

(iii) it attracts each bounded subset of X, in other words, $\mathcal{A}_\infty$ is a global $B-$attractor.

Remark 2.18. In general we are interested in the compact set $\mathcal{A}$ which is the minimal closed invariant global $B-$attractor, i.e., if there is another invariant and closed set C which attracts bounded sets of X, then $\mathcal{A} \subset C$. In [10], "global attractor" already mean the maximal compact invariant global $B-$attractor.

Proposition 2.19. *The maximal compact invariant global $B-$attractor coincides with the compact set which is the minimal closed and invariant global $B-$attractor.*

Proof. Let us suppose first that there exists maximal compact invariant global $B-$attractor and let us call it $\mathcal{A}$. Let C be an arbitrary closed and invariant set which is a global $B-$attractor. Then, C attracts the bounded set $\mathcal{A}$. Hence $\omega(\mathrm{A}) \subset C$. Since $\mathcal{A}$ is an invariant closed set, we have $\mathcal{A} = \omega(\mathrm{A})$. So, $\mathcal{A} \subset C$. Therefore, $\mathcal{A}$ is the compact minimal closed and invariant global $B-$attractor.

On the other hand, if we suppose that there exists the compact minimal closed and invariant global $B-$attractor and let us call it $\mathcal{M}$. Let D be an arbitrary compact and invariant global $B-$attractor. Then, $D = \omega(D) \subset \mathcal{M}$. Therefore, $\mathcal{M}$ is the maximal compact invariant global $B-$attractor. $\square$

3 Asymptotically autonomous processes: generic results

The reader can find results which establish conditions for the existence of the pullback attractor for an evolution process in [2, 3, 12]. For results which establish conditions for the existence of the global attractor for a semigroup we refer the reader to [10, 17, 20, 26]. We will consider in this section an evolution process S and a semigroup T and we will assume that S has a pullback attractor $\mathcal{A}$ and that T possess a global attractor $\mathcal{A}_\infty$ in X.

3.1 Abstract results

Definition 3.1. [18] We say that S is **asymptotically autonomous** to T if

$$\lim_{\tau \longrightarrow +\infty} d(S(\tau+t,\tau)x_\tau, T(t)x_0) = 0, \ \forall \ t \geq 0, \tag{3.1}$$

whenever $x_\tau \longrightarrow x_0$ as $\tau \longrightarrow +\infty$.

We say that S is **uniformly asymptotically autonomous** to T if the convergence in (3.1) is uniform on $t \geq 0$, i.e.,

$$\lim_{\tau \longrightarrow +\infty} \sup_{t\geq 0} d(S(\tau+t,\tau)x_\tau, T(t)x_0) = 0.$$

Definition 3.2. [18] We say that S is **weakly asymptotically autonomous** to T if for each $t \geq 0$,

$$\lim_{\tau \longrightarrow +\infty} d(S(\tau+t,\tau)x_\tau, T(t)x_0) = 0, \tag{3.2}$$

whenever $x_\tau \in \mathcal{A}(\tau)$ and $x_\tau \longrightarrow x_0$ as $\tau \longrightarrow +\infty$.

The next theorem was proved in [18] and it reduces the condition of uniformity of Theorem 3.2 in [13]. Moreover, it gives a necessary and sufficient condition.

Theorem 3.3. *[18] Suppose S is weakly asymptotically autonomous to T. Then the upper semicontinuity holds, i.e,*

$$\lim_{\tau \longrightarrow +\infty} dist(\mathcal{A}(\tau), \mathcal{A}_\infty) = 0 \tag{3.3}$$

if and only if $\mathcal{A}$ is forward compact.

3.1.1 Sufficient conditions for forward convergence

In this section, some results will be presented with sufficient conditions to guarantee the convergence of the pullback attractor to the global attractor. Here, we present results that show the convergence of the pullback attractor to the global attractor using forward boundedness instead of forward compactness. For certain EDP's it is not possible to show that the pullback attractor is forward compact.

Definition 3.4. [5] A family $\{E(t)\}_{t\in\mathbb{R}}$ of nonempty sets is said to be

(i) forward bounded, if there exists a bounded set B such that

$$\bigcup_{t\geq 0} E(t) \subset B;$$

(ii) backwards bounded, if there exists a bounded set K such that

$$\bigcup_{t\leq 0} E(t) \subset K.$$

Remark 3.5. Note that the property of being asymptotically autonomous

$$\lim_{t\longrightarrow +\infty} d\big(S(t+T_0,t)x, T(T_0)x\big) = 0, \forall\, T_0 > 0,$$

can be rewritten as the following equivalent form

$$\lim_{t\longrightarrow +\infty} d\big(S(t,t-T_0)x, T(T_0)x\big) = 0, \forall\, T_0 > 0.$$

Theorem 3.6. *[13] Let $\mathcal{A}$ be a pullback attractor for the evolution process S in X and $\mathcal{A}_\infty$ be a global attractor for the semigroup T in X. Suppose* *that for each $\epsilon > 0$ there exist $\tau_0 = \tau_0(\epsilon)$ and a bounded set $\mathcal{B}(\tau_0)$ in X such that*

$$\sup_{\psi \in \mathcal{A}(\tau_0)} d\big(S(t,\tau_0)\psi, T(t-\tau_0)\psi\big) < \epsilon, \ \forall\ t \geq \tau_0, \tag{3.4}$$

$$\bigcup_{t \geq \tau_0} \mathcal{A}(t) \subset \mathcal{B}(\tau_0). \tag{3.5}$$

Then

$$\lim_{t \longrightarrow +\infty} dist\big(\mathcal{A}(t), \mathcal{A}_\infty\big) = 0.$$

Proof. Let $\epsilon > 0$ be given and let $\tau_1 = \tau_0(\epsilon/3)$. Since the global attractor $\mathcal{A}_\infty$ of the semigroup T attracts bounded sets of X, there exists a positive $t_1 = t_1(\epsilon/3, \mathcal{B}(\tau_1)) > \tau_1$ such that

$$\mathrm{dist}\,(T(t-\tau_1)\mathcal{B}(\tau_1), \mathcal{A}_\infty)\ <\ \frac{\epsilon}{3}$$

for $t \geq t_1$.

Then, by (3.4) with $\epsilon/3$ and τ_1 instead of ϵ and τ_0,

$$\begin{aligned}
\mathrm{dist}\,(\mathcal{A}(t), \mathcal{A}_\infty) &= \mathrm{dist}\,(S(t,\tau_1)\mathcal{A}(\tau_1), \mathcal{A}_\infty) \\
&= \sup_{\psi \in \mathcal{A}(\tau_1)} \mathrm{dist}\,(S(t,\tau_1)\psi, \mathcal{A}_\infty) \\
&\leq \sup_{\psi \in \mathcal{A}(\tau_1)} \mathrm{dist}\,(S(t,\tau_1)\psi, T(t-\tau_1)\psi) \\
&\qquad\qquad + \sup_{\psi \in \mathcal{A}(\tau_1)} \mathrm{dist}\,(T(t-\tau_1)\psi, \mathcal{A}_\infty) \\
&\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} < \epsilon,
\end{aligned}$$

for all $t \geq t_1$. □

Corollary 3.7. *[5] Suppose that S is an evolution process which has a pullback attractor $\mathcal{A}$ and T is a semigroup which possesses a global attractor $\mathcal{A}_\infty$. If*

(i) *$\mathcal{A}$ is forward bounded, i.e., there exists a bounded set B such that*

$$\bigcup_{t \geq 0} \mathcal{A}(t) \subset B;$$

(ii) *the following condition holds*

$$\lim_{t\longrightarrow\infty}\sup_{x\in B} d\big(S(t+T_0,t)x, T(T_0)x\big) = 0, \forall\ T_0 > 0. \tag{3.6}$$

Then

$$\lim_{t\longrightarrow\infty} dist\big(\mathcal{A}(t), \mathcal{A}_\infty\big) = 0. \tag{3.7}$$

Proof. Just note that condition (i) in this corollary implies (3.5) of Theorem 3.6 and (3.6) is stronger than (3.4). $\square$

The next proposition provide conditions to obtain convergence with the Hausdorff distance instead of semi-distance. Let us start with the more general situation of a pullback attractor converging to another pullback attractor as $t \longrightarrow +\infty$.

Theorem 3.8. *[23] Suppose that $\mathcal{A}$ and $\mathfrak{A} = \{\mathfrak{A}(t)\}_{t\in\mathbb{R}}$ are pullback attractors for the evolution processes S and S_∞, respectively. If*

(i) *$\mathcal{A}$ is forward bounded, i.e., there exists a bounded set B such that*

$$\bigcup_{t\geq 0} \mathcal{A}(t) \subset B;$$

(ii) *the following convergence holds*

$$\sup_{x\in B,\tau\in\mathbb{R}_+} d\big(S(t,t-\tau)x, S_\infty(t,t-\tau)x\big) \longrightarrow 0 \quad as\ t \longrightarrow +\infty. \tag{3.8}$$

Then

$$\lim_{t\longrightarrow+\infty} dist\big(\mathcal{A}(t), \mathfrak{A}(t)\big) = 0. \tag{3.9}$$

If, moreover, $\mathfrak{A}$ is also forward bounded, then the pullback attractors $\mathcal{A}$ and $\mathfrak{A}$ are ***asymptotically equivalent in future time****, i.e.,*

$$\lim_{t\longrightarrow+\infty} dist_H\big(\mathcal{A}(t), \mathfrak{A}(t)\big) = 0. \tag{3.10}$$

Proof. Let us prove (3.9) by contradiction. Suppose that (3.9) is not true. Then there exist $\delta > 0$ and a sequence $0 < t_n \uparrow +\infty$ such that

$$dist\big(\mathcal{A}(t_n), \mathfrak{A}(t_n)\big) \geq \delta, \forall\ n \in \mathbb{N}.$$

Since the pullback attractor $\mathcal{A}$ is compact, for each $n \in \mathbb{N}$, we can choose $x_n \in \mathcal{A}(t_n)$ such that

$$dist\big(x_n, \mathfrak{A}(t_n)\big) = dist\big(\mathcal{A}(t_n), \mathfrak{A}(t_n)\big) \geq \delta, \forall\ n \in \mathbb{N}. \tag{3.11}$$

By the invariance of the pullback attractor $\mathcal{A}$, for each $m, n \in \mathbb{N}$, we can rewrite x_n as

$$x_n = S(t_n, t_n - m)b_{n,m},$$

with $b_{n,m} \in \mathcal{A}(t_n - m) \subset B$. Thus, by condition (3.8), there is $N = N(\delta) > 0$ such that for all $m \in \mathbb{N}$,

$$\begin{aligned} &d\big(x_N, S_\infty(t_N, t_N - m)b_{N,m}\big) \\ &\quad = d\big(S(t_N, t_N - m)b_{N,m}, S_\infty(t_N, t_N - m)b_{N,m}\big) \\ &\quad \leq \sup_{x \in B, \tau \in \mathbb{R}_+} d\big(S(t_N, t_N - \tau)x, S_\infty(t_N, t_N - \tau)x\big) \\ &\quad < \frac{\delta}{2}. \end{aligned} \tag{3.12}$$

Moreover, as $\{b_{n,m}\} \subset B$ is attracted by $\mathfrak{A}$ through the evolution process S_∞, there is $M = M(N, \delta) > 0$ such that

$$\begin{aligned} &dist\big(S_\infty(t_N, t_N - m)b_{N,m}, \mathfrak{A}(t_N)\big) \\ &\quad \leq dist\big(S_\infty(t_N, t_N - m)B, \mathfrak{A}(t_N)\big) < \delta/2, \forall\ m \geq M. \end{aligned} \tag{3.13}$$

Hence, from (3.12) and (3.13) it follows that, for all $m \geq M$,

$$\begin{aligned} dist\big(x_N, \mathfrak{A}(t_N)\big) &\leq d\big(x_N, S_\infty(t_N, t_N - m)b_{N,m}\big) \\ &\qquad + dist\big(S_\infty(t_N, t_N - m)b_{N,m}, \mathfrak{A}(t_N)\big) \\ &< \frac{\delta}{2} + \frac{\delta}{2} = \delta, \end{aligned}$$

what contradicts (3.11).

Therefore,

$$\lim_{t \longrightarrow +\infty} dist\big(\mathcal{A}(t), \mathfrak{A}(t)\big) = 0.$$

Now, let us prove the second part. In the case of $\mathfrak{A}$ also being forward bounded, we change $\mathcal{A}$ by $\mathfrak{A}$ in the proof above and we obtain:

$$\lim_{t \longrightarrow +\infty} dist\big(\mathfrak{A}(t), \mathcal{A}(t)\big) = 0.$$

So, (3.10) holds, i.e., the pullback attractors $\mathcal{A}$ and $\mathfrak{A}$ are asymptotically equivalent in future time.

$\square$

Corollary 3.9. *Suppose that S is an evolution process with a pullback attractor $\mathcal{A}$ and that T is a semigroup with a global attractor $\mathcal{A}_\infty$. If*

(i) $\mathcal{A}$ is forward bounded, i.e., there is a bounded set B such that

$$\bigcup_{t\geq 0} \mathcal{A}(t) \subset B;$$

(ii) the following condition holds

$$\sup_{x\in B, \tau\in\mathbb{R}_+} d\big(S(t,t-\tau)x, T(\tau)x\big) \longrightarrow 0, \ \text{as } t \longrightarrow +\infty,$$

then the global attractor $\mathcal{A}_\infty$ is the ω-limit set of the pullback attractor $\mathcal{A}$, i.e.,

$$\lim_{t\longrightarrow+\infty} dist_H\big(\mathcal{A}(t), \mathcal{A}_\infty\big) = 0.$$

Proof. Define $S_\infty(t,s)x := T(t-s)x$ for $t \geq s$ and $x \in X$. Then, S_∞ is an evolution process with a pullback attractor $\mathfrak{A} \equiv \mathcal{A}_\infty$. Therefore, by Theorem 3.8 the result follows. $\square$

3.1.2 Sufficient conditions for backwards convergence

Here, we will see the results of convergence of the pullback attractor to the global attractor as $t \to -\infty$ by using backwards compactness and boundedness of the pullback attractor. The proofs are similar of the results in the previous subsection.

The next theorem is similar to Theorem 3.3 but now with the condition of the pullback attractor $\mathcal{A}$ being backward compact.

Proposition 3.10. *[5] Let S be an evolution process with a pullback attractor $\mathcal{A}$ and T a semigroup with a global attractor $\mathcal{A}_\infty$. Suppose that*

(i) for any $\{x_t\}$ with $x_t \in \mathcal{A}(t)$ and $\lim_{t\longrightarrow-\infty} d(x_t, x_0) = 0$,

$$\lim_{t\longrightarrow-\infty} d\big(S(t,t-T_0)x_t, T(T_0)x_0\big) = 0, \forall\, T_0 \in \mathbb{R}_+; \tag{3.14}$$

(ii) *$\mathcal{A}$ is backward compact.*

Then

$$\lim_{t\longrightarrow -\infty} dist\big(\mathcal{A}(t),\mathcal{A}_\infty\big)=0. \tag{3.15}$$

Proposition 3.11. *[5] Suppose that $\mathcal{A}$ and $\mathfrak{A}=\{\mathfrak{A}(t)\}_{t\in\mathbb{R}}$ are pullback attractors for the evolution processes S and S_∞, respectively. If*

(i) *$\mathcal{A}$ is backward bounded, i.e., there is a bounded set B such that*

$$\bigcup_{t\le 0}\mathcal{A}(t)\subset B;$$

(ii) *the following convergence holds*

$$\sup_{x\in B,\tau\in\mathbb{R}_+} d\big(S(t,t-\tau)x,S_\infty(t,t-\tau)x\big)\longrightarrow 0,\ \ \textit{as } t\longrightarrow -\infty. \tag{3.16}$$

Then

$$\lim_{t\longrightarrow -\infty} dist\big(\mathcal{A}(t),\mathfrak{A}(t)\big)=0. \tag{3.17}$$

If, moreover, $\mathfrak{A}$ is backward bounded, then the pullback attractors $\mathcal{A}$ and $\mathfrak{A}$ are ***asymptotically equivalent in past time****, i.e.,*

$$\lim_{t\longrightarrow -\infty} dist_H\big(\mathcal{A}(t),\mathfrak{A}(t)\big)=0. \tag{3.18}$$

Corollary 3.12. *[5] Suppose that S is an evolution process with a pullback attractor $\mathcal{A}$ and that T is a semigroup with a global attractor $\mathcal{A}_\infty$. If*

(i) *$\mathcal{A}$ is backward bounded, i.e, there is a bounded set B such that*

$$\bigcup_{t\le 0}\mathcal{A}(t)\subset B;$$

(ii) *the following convergence holds*

$$\sup_{x\in B,\tau\in\mathbb{R}_+} d\big(S(t,t-\tau)x,T(\tau)x\big)\longrightarrow 0,\ \ \textit{as } t\longrightarrow -\infty. \tag{3.19}$$

Then, the global attractor $\mathcal{A}_\infty$ is the α-limit set of the pullback attractor $\mathcal{A}$, i.e.,

$$\lim_{t\longrightarrow -\infty} dist_H\big(\mathcal{A}(t),\mathcal{A}_\infty\big)=0.$$

3.1.3 Necessary conditions

Here we shall present two propositions with necessary conditions to assure convergence of the pullback attractor to the global attractor.

Definition 3.13. [5] A family $\mathfrak{E} = \{E_t\}_{t\in\mathbb{R}}$ of nonempty compact sets is called **locally uniformly compact**, if for any bounded interval $I \subset \mathbb{R}$ the union $\bigcup_{t\in I} E_t$ is pre-compact.

Proposition 3.14. *[5] Suppose that $\{E_t\}_{t\in\mathbb{R}}$ is a locally uniformly compact family of nonempty compact sets in X. Then there is a nonempty compact set E such that $\lim\limits_{t\longrightarrow\infty} dist(E_t, E) = 0$ if and only if $\{E_t\}_{t\in\mathbb{R}}$ is forward compact.*

Proposition 3.15. *[23] Suppose that $\{E_t\}_{t\in\mathbb{R}}$ is a locally uniformly compact family of nonempty compact sets in X. Then there exists a nonempty compact set E such that $\lim\limits_{t\longrightarrow-\infty} dist(E_t, E) = 0$ if and only if the family $\{E_t\}_{t\in\mathbb{R}}$ is backward compact.*

Proof. $(\Rightarrow)$ We want to prove that $\bigcup_{t\leq 0} E_t$ is pre-compact. Take an arbitrarily sequence $\{x_n\} \subset \bigcup_{t\leq 0} E_t$. Then, we need to show that $\{x_n\}$ possess a convergent subsequence. Since $\{x_n\} \subset \bigcup_{t\leq 0} E_t$, there is a sequence $\{t_n\} \subset (-\infty, 0]$ such that $x_n \in E_{t_n}$ for all $n \in \mathbb{N}$. We have two cases to consider:

- *First case*: there exists a finite quantity $E_1, E_2, \cdots, E_k$ such that

$$\{x_n\} \subset \bigcup_{j=1}^{k} E_j.$$

Note that $\{x_n\}$ possess a convergent subsequence once $\bigcup_{j=1}^{k} E_j$ is compact.

- *Second case*: there is an infinity of indexes l's such that $x_l \in E_{t_l}$, for all $l \in \mathbb{N}$, with $x_l \neq x_j$ and $E_{t_l} \neq E_{t_j}$ when $l \neq j$.

Consider a decreasing subsequence $\{x_{l_k}\}$ of the sequence $\{x_l\}$.

We have two possibilities:

- *Possibility 1*: $t_{l_k} \downarrow -\infty$.

Since $\{x_{l_k}\} \subset \bigcup_{k=1}^{\infty} E_{t_{l_k}}$, with $x_{l_k} \in E_{t_{l_k}}$, for all $k \in \mathbb{N}$, we have:

$$d(x_{l_k}, E) \leq dist(E_{t_{l_k}}, E) \longrightarrow 0 \text{ as } k \longrightarrow +\infty.$$

Thus, $\lim_{k \longrightarrow +\infty} d(x_{l_k}, E) = 0$. Then, for each given $j \in \mathbb{N}$ there exists $k_0(j)$ such that $x_{l_{k_j}} \in \mathcal{O}_{1/j}(E)$ for $l_{k_j} \geq k_0(j)$. Moreover, there exists $w_{k_j} \in E$ such that $d(x_{l_{k_j}}, w_{k_j}) < 1/j$. Once E is a compact set, we can consider $w_{k_j} \longrightarrow x \in E$ as $j \longrightarrow \infty$.

Claim: $x_{l_{k_j}} \longrightarrow x$.

Indeed,

$$\begin{aligned} d(x_{l_{k_j}}, x) &\leq d(x_{l_{k_j}}, w_{k_j}) + d(w_{k_j}, x) \\ &< \frac{1}{j} + d(w_{k_j}, x). \end{aligned}$$

Hence, $\lim_{j \longrightarrow +\infty} d(x_{l_{k_j}}, x) = 0$.

So, the sequence $\{x_l\}$ possess a convergent subsequence $\{x_{l_{k_j}}\}$. Since $\{x_l\}$ is a subsequence of the original sequence $\{x_n\}$, we had prove that the sequence $\{x_n\}$ has a convergent subsequence.

- *Possibility 2*: $t_{l_k} \downarrow a$ with $-\infty < a \leq 0$.

We have $\{x_{l_k}\} \subset \bigcup_{k=1}^{\infty} E_{t_{l_k}}$, with $x_{l_k} \in E_{t_{l_k}}$ for all $k \in \mathbb{N}$.

Note that $t_{l_k} \in [a, 0]$. Thus,

$$\{x_{l_k}\} \subset \bigcup_{t_{l_k} \in [a,0]} E_{t_{l_k}} \subset \bigcup_{t \in [a,0]} E_t.$$

From hypotheses, $\bigcup_{t \in [a,0]} E_t \in K(X)$, then $\{x_{l_k}\}$ possess a convergent subsequence.

($\Leftarrow$) Suppose that the family $\{E_t\}_{t \in \mathbb{R}}$ is backwards compact. Then, $\bigcup_{t \leq 0} E_t$ is pre-compact, or equivalently, $E := \overline{\bigcup_{t \leq 0} E_t}$ is compact. Note that $E \neq \emptyset$ once by hipotheses, $\{E_t\}_{t \in \mathbb{R}}$ is a family of nonempty sets.

For simplicity, we will show that $\lim_{t \longrightarrow -\infty} dist(E_t, E) = 0$ by using sequences. Consider the sequence $t_n \downarrow -\infty$. Since each E_{t_n} is compact, for any $n \in \mathbb{N}$, we can choose $x_n \in E_{t_n}$ such that

$$d(x_n, E) = dist(E_{t_n}, E), \ \forall \ n \in \mathbb{N}. \tag{3.20}$$

Note that $x_n \in E_{t_n} \subset E$. As E is compact, the sequence $\{x_n\}$ possess a convergent subsequence, i.e., $x_{n_k} \longrightarrow x \in E$ as $k \longrightarrow +\infty$. Then,

$$\lim_{k \longrightarrow +\infty} d(x_{n_k}, x) = 0. \tag{3.21}$$

On the other hand, we have

$$dist(x_{n_k}, E) \leq d(x_{n_k}, x), \ \forall \ n \in \mathbb{N}.$$

Passing the limit in both sides of the above inequality and using (3.21) we obtain

$$\lim_{k \longrightarrow +\infty} dist(x_{n_k}, E) \leq \lim_{k \longrightarrow +\infty} d(x_{n_k}, x) = 0.$$

Then,

$$\lim_{k \longrightarrow +\infty} dist(x_{n_k}, E) = 0.$$

From (3.20) we have $\lim_{k \to +\infty} dist(E_{t_{n_k}}, E) = 0$, which means in other word that, $\lim_{t \to -\infty} dist(E_t, E) = 0$, as we wanted to prove.

$\square$

3.1.4 Construction of the limit-set of the pullback attractor

Here, we shall compare the global attractor $\mathcal{A}_\infty$ with the so-called limit-set $\mathcal{A}(\infty)$ of the pullback attractor.

Definition 3.16. [18] The limit-set $\mathcal{A}(\infty)$ of the pullback attractor $\mathcal{A} := \{\mathcal{A}(t) : t \in \mathbb{R}\}$ is defined in the following way:

$$\mathcal{A}(\infty) := \bigcap_{t \in \mathbb{R}} \overline{\bigcup_{r \geq t} \mathcal{A}(r)}.$$

Similarly, one can define the set

$$\mathcal{A}(-\infty) := \bigcap_{t \in \mathbb{R}} \overline{\bigcup_{r \leq t} \mathcal{A}(r)}.$$

Proposition 3.17. *[23] The following characterization holds:*

$$\mathcal{A}(\infty) = \bigcup_{r_n \uparrow \infty} \{x \in X : \exists \ x_n \in \mathcal{A}(r_n) \ \text{such that} \ x_n \longrightarrow x\}. \tag{3.22}$$

Proof. Let us show first that

$$\bigcap_{t\in\mathbb{R}} \overline{\bigcup_{r\geq t}\mathcal{A}(r)} \subset \bigcup_{r_n\uparrow\infty} \{x \in X : \exists\, x_n \in \mathcal{A}(r_n) \text{ such that } x_n \longrightarrow x\}.$$

Take an arbitrary $w \in \bigcap_{t\in\mathbb{R}} \overline{\bigcup_{r\geq t}\mathcal{A}(r)}$. Consider a sequence $t_n \uparrow +\infty$, i.e.,

$$t_1 < t_2 < t_3 < \cdots < t_n < \cdots.$$

Then $w \in \overline{\bigcup_{r\geq t_n} \mathcal{A}(r)}$ for all $n \in \mathbb{N}$. So, for each $n \in \mathbb{N}$, there is $\{w_j^n\} \subset \bigcup_{r\geq t_n} \mathcal{A}(r)$ such that $w_j^n \longrightarrow w$ as $j \longrightarrow \infty$. Note that, $w_j^n \in \mathcal{A}(s_j^n)$ for some $s_j^n \geq t_n$. Considering, $r_j := s_j^j$ where $r_j \geq t_j$ it follows that $x_j := w_j^j \in \mathcal{A}(r_j)$ and $x_j \longrightarrow w$ as $j \longrightarrow \infty$, as we wanted to show.

Now, let us show that

$$\bigcup_{r_n\uparrow\infty} \{x \in X : \exists\, x_n \in \mathcal{A}(r_n) \text{ such that } x_n \longrightarrow x\} \subset \bigcap_{t\in\mathbb{R}} \overline{\bigcup_{r\geq t}\mathcal{A}(r)}.$$

Let $x \in \bigcup_{r_n\uparrow\infty} \{x \in X : \exists\, x_n \in \mathcal{A}(r_n) \text{ such that } x_n \longrightarrow x\}$. Then, there is a sequence $\{x_n\}$ with $x_n \in \mathcal{A}(r_n)$ for all $n \in \mathbb{N}$, such that $x_n \longrightarrow x$ and $r_n \uparrow +\infty$.

Let $t \in \mathbb{R}$ be arbitrarily fixed. We want to show that $x \in \overline{\bigcup_{r\geq t}\mathcal{A}(r)}$. Since $r_n \uparrow +\infty$, there is $n_1 \in \mathbb{N}$ such that $n \geq n_1 \Longrightarrow r_n \geq t$.

So, with $n \geq n_1$, $x_n \in \mathcal{A}(r_n)$, $r_n \geq t$ and $x_n \to x$ as $n \longrightarrow \infty$.

Renumbering the sequence $y_1 = x_{n_1}$, $y_2 = x_{n_1+1}$, $y_3 = x_{n_1+2}$,... and $\tilde{r}_1 = r_{n_1}$, $\tilde{r}_2 = r_{n_1+1}$, $\tilde{r}_3 = r_{n_1+2}$,... we have $y_n \in \mathcal{A}(\tilde{r}_n)$, $\tilde{r}_n \geq t$ and $y_n \longrightarrow x$. Thus, $x \in \overline{\bigcup_{r\geq t}\mathcal{A}(r)}$. Once t was arbitrary, we can conclude that $x \in \bigcap_{t\in\mathbb{R}} \overline{\bigcup_{r\geq t}\mathcal{A}(r)}$, as we wanted to show.

Therefore, we have

$$\bigcap_{t\in\mathbb{R}} \overline{\bigcup_{r\geq t}\mathcal{A}(r)} = \bigcup_{r_n\uparrow\infty} \{x \in X : \exists\, x_n \in \mathcal{A}(r_n) \text{ such that } x_n \longrightarrow x\}.$$

$\square$

Proposition 3.18. *[18] Consider S an evolution process possessing a forward compact pullback attractor $\mathcal{A}$ and a semigroup T having a global attractor $\mathcal{A}_\infty$. Suppose that S is asymptotically autonomous to T, then we have $\mathcal{A}(\infty) \subset \mathcal{A}_\infty$.*

To have the equality $\mathcal{A}(\infty) = \mathcal{A}_\infty$, we need stronger conditions.

Proposition 3.19. *[18] Under the same assumptions of Proposition 3.18, we have $\mathcal{A}(\infty) = \mathcal{A}_\infty$ if we further assume the following conditions:*

(i) *$\mathcal{A}(\infty)$ is a forward attracting set for $S(\cdot, 0)$, i.e., for each $B \in \mathfrak{B}$,*

$$\lim_{t \longrightarrow +\infty} dist\big(S(t,0)B, \mathcal{A}(\infty)\big) = 0;$$

(ii) *$S(\cdot, 0)$ converges to T uniformly on bounded sets of X, i.e., for each $B \in \mathfrak{B}$,*

$$\lim_{t \longrightarrow +\infty} \sup_{x \in B} d\big(S(t,0)x, T(t)x\big) = 0.$$

Remark 3.20. Proposition 3.19 holds true if (i) and (ii) hold true only for $B = \mathcal{A}_\infty$.

Theorem 3.21. *[19] Consider S an evolution process possessing a pullback attractor $\mathcal{A}$.*
(i) $\lim_{t \longrightarrow \infty} dist\big(\mathcal{A}(t), \mathcal{A}(\infty)\big) = 0$ *if and only $\mathcal{A}$ is forward compact.*

(ii) $\lim_{t \longrightarrow -\infty} dist\big(\mathcal{A}(t), \mathcal{A}(-\infty)\big) = 0$ *if and only $\mathcal{A}$ is backward compact.*

The next lemma present a special case where the global attractor is only a unitary set. In this case we obtain lower semicontinuity.

Lemma 3.22. *[18] Under the same assumptions of Proposition 3.18, if the global attractor is a single point, i.e., $\mathcal{A}_\infty = \{x_0\}$, then $\mathcal{A}(\infty) = \mathcal{A}_\infty$. In this case, we also have the lower semicontinuity:*

$$dist\big(\mathcal{A}_\infty, \mathcal{A}(t)\big) \longrightarrow 0 \text{ as } t \longrightarrow +\infty.$$

3.1.5 Construction of the lower limit-set of the pullback attractor

In this subsection, we will see the definition of the so-called lower limit-set of the pullback attractor and present a proposition where the lower upper semicontinuity holds if and only if the global attractor is equal to the lower limit-set of the pullback attractor.

Definition 3.23. [18] We define the lower limit-set $\mathcal{A}_L(\infty)$, of the pullback attractor $\mathcal{A} := \{\mathcal{A}(t)\}_{t\in\mathbb{R}}$, by:

$$\mathcal{A}_L(\infty) = \bigcap_{r_n \uparrow \infty} \{x \in X : \exists\ x_n \in \mathcal{A}(r_n) \text{ such that } x_n \longrightarrow x\}. \tag{3.23}$$

Proposition 3.24. *[18] Under the same assumptions of Proposition 3.18, the lower semicontinuity holds, i.e.,*

$$\lim_{t \longrightarrow +\infty} dist\big(\mathcal{A}_\infty, \mathcal{A}(t)\big) = 0 \tag{3.24}$$

if, and only if, $\mathcal{A}_\infty = \mathcal{A}_L(\infty)$. *In either case,* $\mathcal{A}_\infty = \mathcal{A}(\infty)$.

Theorem 3.25. *[19] Consider S an evolution process possessing a forward compact pullback attractor $\mathcal{A}$ with a nonempty lower limit-set $\mathcal{A}_L(\infty)$. Then, $\mathcal{A}_L(\infty)$ is compact and such that*

$$\lim_{t \longrightarrow \infty} dist\big(\mathcal{A}_L(\infty), \mathcal{A}(t)\big) = 0.$$

Remark 3.26. The reader can find some more similar results involving upper and lower limit-sets of pullback attractors in [19].

4 Applications

Let us start with three examples of applications where the time dependence is on the external forcing term. The reader can also find asymptotically autonomous problems in Section 8.6.2 of [2], where non-autonomous semi-linear problems with the explicit dependence on time occurring only in the external forcing term are presented.

Example 4.1. Consider the following nonautonomous reaction diffusion equation on $\mathbb{R}$:

$$\frac{\partial u}{\partial t} - \Delta u + \lambda u + f(u) = g(x,t), \tag{4.1}$$

with the initial condition $u(\tau) = u_0$, where $\lambda > 0$ and the nonlinearity $f \in C^1(\mathbb{R},\mathbb{R})$ satisfying

$$f(u)u \geq 0, \;\; f(0) = 0, \;\; f'(u) \geq -c, \; c \geq 0, \;\; |f'(u)| \leq c(1+|u|^p)$$

with $p \geq 0$. The nonautonomous forcing term $g \in L^2_{loc}(\mathbb{R}, L^2(\mathbb{R}))$ satisfies the following tempered condition

$$\int_{-\infty}^{0} e^{\lambda s}\|g(\cdot,s\|^2 ds < \infty.$$

By using Corollary 3.7 it was proved in [5] that if there is $g_0(\cdot) \in L^2(\mathbb{R})$ such that

$$\lim_{\tau\to\infty}\int_{\tau}^{\infty} \|g(\cdot,s) - g_0\|^2 ds = 0,$$

then $\lim_{t\to\infty} dist(\mathcal{A}(t), \mathcal{A}_\infty) = 0$, where the semi-distance is on $L^2(\mathbb{R})$, $\mathcal{A} := \{\mathcal{A}(t)\}_{t\in\mathbb{R}}$ is the pullback attractor associated to problem (4.1) and $\mathcal{A}_\infty$ is the global attractor of the autonomous limit equation

$$\frac{\partial u}{\partial t} - \Delta u + \lambda u + f(u) = g_0(x).$$

.

Example 4.2. Consider the following nonautonomous weakly dissipative $p-$Laplace equation:

$$\frac{\partial u}{\partial t} + Au + \lambda u + f(x,u) = g(x,t), \; t \geq \tau \in \mathbb{R}, \; x \in \mathbb{R}^n, \tag{4.2}$$

with the initial condition $u(\tau) = u_0$, where $\lambda > 0$ and the main operator $A : W^{1,p}(\mathbb{R}^n) \to W^{-1,p'}(\mathbb{R}^n)$ is a $p-$Laplace operator defined by

$$Au(v) := \sum_{i=1}^{n}\int_{\mathbb{R}^n} |u_{x_i}|^{p-2}u_{x_i}v_{x_i}dx,$$

for $u, v \in W^{1,p}(\mathbb{R}^n)$ and $p > 2$.

The source term satisfies the following condition:

Assumption F. f is a continuously differentiable function satisfying: for all $(x,s) \in \mathbb{R}^n \times \mathbb{R}$,

$$f(x,s)s \geq \alpha_1|s|^q + \psi_1(x), \quad |f(x,s)| \leq \alpha_2|s|^{q-1} + \psi_2(x), \quad \frac{\partial f}{\partial s}(x,s) \geq -\alpha_3,$$

where $q > 2,\ \alpha_1, \alpha_2, \alpha_3 > 0$ are constants, $\psi_2 \in L^2(\mathbb{R}^n)$ and $\psi_1 \in L^1(\mathbb{R}^n) \cap L^{I+1}(\mathbb{R}^n)$, with $I = I(p,q) := \left\{\dfrac{p-2}{q-2}\right\}$, where $\{r\}$ denotes the minimal integer no lesser than r.

The external forcing term satisfies the following two conditions:

Assumption G1. g is uniformly tempered in $L^2(\mathbb{R}^n) \cap L^{I+1}(\mathbb{R}^n)$: $\lim_{t\to\infty} G_2(t) < \infty$ and $\lim_{t\to\infty} G_{I+1}(t) < \infty$, where all the functions $t \mapsto G_i(t)$ are increasing and defined by

$$G_i(t) := \sup_{s\leq t} \int_{-\infty}^{0} e^{\lambda r}\|g(\cdot, r+s)\|^i_{L^i} dr, \ \text{ for } \ i = 2, \cdots, I+1.$$

Assumption G2. There is a function $g_\infty \in L^2(\mathbb{R}^n) \cap L^{I+1}(\mathbb{R}^n)$ such that

$$\lim_{\tau\to\infty} \int_0^\infty e^{-2\alpha_3 s}\|g(\cdot, \tau+s) - g_\infty(\cdot)\|^2 ds = 0,$$

where α_3 is the positive constante given in Assumption **F**.

By using the Theorem 3.3 and Proposition 3.18 it was proved in [18] that under the assumptions **F**, **G1** and **G2** the nonautonomous $p-$Laplacian equation has a pullback attractor that converges upper semicontinuously in $L^2(\mathbb{R}^n)$ to the global attractor of the autonomous limit equation with the external forcing term g_∞.

Example 4.3. Consider the following nonautonomous Ginzburg-Landau equation:

$$\frac{\partial u}{\partial t} = (\lambda + i\alpha(t))\Delta u - (\kappa + i\beta(t))|u|^2 u + \gamma u + f(t,x), \ t \geq s \in \mathbb{R}, \ x \in \Omega, \tag{4.3}$$

with homogeneous Dirichlet boundary condition and the initial condition $u(\tau) = u_0$, where $\lambda, \kappa, \gamma > 0$ are constants, where Ω is smooth bounded domain in $\mathbb{R}^n$, $n = 1, 2$ and the unknown u is a complex-valued function.

The variable coefficients satisfy the following condition:

Hypothesis A. $\alpha(\cdot) \in C(\mathbb{R}, \mathbb{R})$ and $\beta(\cdot) \in C_b(\mathbb{R}, \mathbb{R})$.

The external force f satisfy the following condition:

Hypothesis B. $f \in L_{loc}(\mathbb{R}, L^2(\Omega))$ is **forward tempered** in the following sense:

$$\sup_{r \geq t} \int_{-\infty}^{r} e^{\gamma(q-r)} \|f(q, \cdot)\|^2 dq < \infty, \quad \forall\, t \in \mathbb{R}.$$

By using Theorem 3.21 (i) it was proved in [19] that under the hypotheses **A** and **B** the nonautonomous Ginzburg-Landau equation has a forward compact pullback attractor $\mathcal{A}$ in $L^2(\Omega) \times L^2(\Omega)$ that converges upper semicontinuously to the limit-set $\mathcal{A}(\infty)$ as $t \to +\infty$, i.e., $\lim\limits_{t \longrightarrow \infty} dist\big(\mathcal{A}(t), \mathcal{A}(\infty)\big) = 0$. Also, by using Theorem 3.25, it was proved the lower semicontinuous convergence to $\mathcal{A}_L(\infty)$ as $t \to +\infty$, i.e, $\lim\limits_{t \longrightarrow \infty} dist\big(\mathcal{A}_L(\infty), \mathcal{A}(t)\big) = 0$.

If instead of hypothesis **B**, the external force f would satisfies

Hypothesis B'. $f \in L_{loc}(\mathbb{R}, L^2(\Omega))$ is **backward tempered** in the following sense:

$$\sup_{r \leq t} \int_{-\infty}^{r} e^{\gamma(q-r)} \|f(q, \cdot)\|^2 dq < \infty, \quad \forall\, t \in \mathbb{R},$$

then by using item (ii) of Theorem 3.21 it was proved in [19] that under the hypotheses **A** and **B'** that the nonautonomous Ginzburg-Landau equation has a backward compact pullback attractor $\mathcal{A}$ such that

$$\lim_{t \longrightarrow -\infty} dist\big(\mathcal{A}(t), \mathcal{A}(-\infty)\big) = 0.$$

In this example all the semi-distances were on the space $L^2(\Omega) \times L^2(\Omega)$.

In the last example the time-dependence will be on the main operator:

Example 4.4. [13] Consider the following quasi-linear parabolic equation with variable exponents:

$$\frac{\partial u}{\partial t}(t) - \operatorname{div}\Big(D(t,x)|\nabla u(t)|^{p(x)-2}\nabla u(t)\Big) + |u(t)|^{p(x)-2}u(t) = B(u(t)), \tag{4.4}$$

with $u(\tau) = \psi_\tau$, on a bounded smooth domain Ω in $\mathbb{R}^n$ for some $n \geq 1$ with a homogeneous Neumann boundary condition, where the exponent $p(\cdot) \in C(\bar{\Omega}, \mathbb{R})$ satisfies $p^+ := \max_{x\in\bar{\Omega}} p(x) \geq p^- := \min_{x\in\bar{\Omega}} p(x) > 2$ and the initial condition $u(\tau) \in H := L^2(\Omega)$.

Motivated by problem (4.4), we study the asymptotic behavior of an abstract non-autonomous evolution equation in a Hilbert space H of the form

$$\frac{\partial u}{\partial t}(t) + A(t)u(t) = B(u(t)), \quad u(\tau) = \psi_\tau. \tag{4.5}$$

compared with that of an autonomous evolution equation

$$\frac{\partial v}{\partial t}(t) + A_\infty v(t) = B(v(t)), \quad v(0) = \psi_0, \tag{4.6}$$

where the operators A, A_∞ and B satisfy the following assumptions.

Assumption A For each $\tau \in \mathbb{R}$ there exists a nonincreasing function $g_\tau : [0, +\infty) \to [0, +\infty)$ such that $g_\tau(0) \to 0$ as $\tau \to +\infty$ and

$$\langle A(t+\tau)u(t+\tau) - A_\infty v(t), u(t+\tau) - v(t)\rangle \geq -g_\tau(t), \text{ for all } t \in \mathbb{R}^+,\ \tau \in \mathbb{R},$$

for any solution u of (4.5) and v of (4.6).

Assumption B The mapping $B : H \to H$ is globally Lipschitz, i.e., there exists $L \geq 0$ such that

$$\|B(x_1) - B(x_2)\|_H \leq L\|x_1 - x_2\|_H \quad \text{for all } x_1, x_2 \in H.$$

We will suppose that the process $\{U(t,\tau) : t \geq \tau\}$ generated by problem (4.5) has a pullback attractor $\mathfrak{A} = \{\mathcal{A}(t) : t \in \mathbb{R}\}$ and that the semigroup $\{T(t) : t \geq 0\}$ generated by problem (4.6) has a global autonomous attractor $\mathcal{A}_\infty$ in the Hilbert space H.

Lemma 4.5. *[13] Suppose that Assumption A is satisfied. Then $U(t+\tau,\tau)\psi_\tau \to T(t)\psi_0$ in H as $\tau \to +\infty$ for any $t \geq 0$ whenever $\psi_\tau \to \psi_0$ in H as $\tau \to +\infty$.*

Proof. Fix $t \geq 0$ arbitrarily and take $T \geq t$. Subtracting equation (4.6) from the equation (4.5) gives

$$\frac{d}{dt}(u(t+\tau)-v(t)) + A(t+\tau)u(t+\tau) - A_\infty v(t) = B(u(t+\tau)) - B(v(t))$$

for a.e. $t \in [\tau, T]$. Multiplying by $u(t+\tau)-v(t)$ and taking the inner product, then using Assumption A, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t+\tau)-v(t)\|_H^2 \leq L\|u(t+\tau)-v(t)\|_H^2 + g_\tau(t).$$

Integrating this last inequality from 0 to t and using that g_τ is nonincreasing, gives

$$\begin{aligned}\|u(t+\tau)-v(t)\|_H^2 \quad &\leq \quad \|\psi_\tau - \psi_0\|_H^2 + 2tg_\tau(0) \\ &\qquad\qquad + 2L\int_0^t \|u(s+\tau)-v(s)\|_H^2 ds.\end{aligned}$$

Hence, by the Gronwall inequality, there is a positive constant $K = K(T)$ such that

$$\|u(t+\tau)-v(t)\|_H^2 \leq \left(\|\psi_\tau-\psi_0\|_H^2 + g_\tau(0)\right) K.$$

Since $\psi_\tau \to \psi_0$ in H and $g_\tau(0) \to 0$ as $\tau \to +\infty$, the result follows. $\square$

The above result is applied here to the quasi-linear parabolic equation with spatially variable exponents (4.4) in the Hilbert space $H := L^2(\Omega)$. The existence of a pullback attractor for the problem (4.4) was proved in [14].

We assume that B satisfies Assumption **B** and that the coefficient D satisfies the following Assumption:

Assumption D $D : [\tau, T] \times \Omega \to \mathbb{R}$ is a function in $L^\infty([\tau, T] \times \Omega)$ such that

(D1) there are positive constants, β and M such that $0 < \beta \leq D(t, x)$ for almost all $(t,x) \in [\tau, T] \times \Omega$.

(D2) $D(t,x) \geq D(s,x)$ for each $x \in \Omega$ and $t \leq s$ in $[0,T]$.

(D3) $D(t+\tau,\cdot) \to D^*(\cdot)$ in $L^\infty(\Omega)$ as $\tau \to +\infty$, for any $t \geq 0$.

Assumptions (D1)-(D2) imply that the pointwise limit $D^*(x)$ as $t \to \infty$ exists and satisfies $0 < \beta \leq D^*(x)$ for almost all $x \in \Omega$. Then the problem (4.4) with $D^*(x)$ is autonomous and has a global autonomous attractor as a particular case of the results in [14].

It will be shown that the dynamics of the original non-autonomous problem is asymptotically autonomous and its pullback attractor converges upper semicontinuously to the autonomous global attractor $\mathcal{A}_\infty$ of the problem

$$\frac{\partial v}{\partial t}(t) - \operatorname{div}\left(D^*|\nabla v(t)|^{p(x)-2}\nabla v(t)\right) + |v(t)|^{p(x)-2}v(t) = B(v(t)),\ v(0) = \psi_0. \tag{4.7}$$

In particular, consider the operators

$$\begin{aligned} A(t)u &:= -\operatorname{div}\left(D(t)|\nabla u|^{p(x)-2}\nabla u\right) + |u|^{p(x)-2}u, \\ A_\infty v &:= -\operatorname{div}\left(D^*|\nabla v|^{p(x)-2}\nabla v\right) + |v|^{p(x)-2}v, \end{aligned}$$

and apply Theorem 3.3 to the quasi-linear parabolic problem with variable exponents (4.4).

Proposition 4.6. *[14]* $\overline{\cup_{\tau\in\mathbb{R}}\mathcal{A}(\tau)}$ *is a compact subset of* H.

Theorem 4.7. *[13] If* $\{\psi_\tau : \tau \in \mathbb{R}\}$ *is a bounded set in* X *and* $\psi_\tau \to \psi_0$ *in* H *as* $\tau \to +\infty$, *then Assumption A is satisfied.*

The next result gives the desired asymptotic upper semicontinuous convergence.

Theorem 4.8. $\lim_{t\to+\infty} \operatorname{dist}(\mathcal{A}(t), \mathcal{A}_\infty) = 0$, *where the semi-distance is on* $L^2(\Omega)$.

Proof. Suppose that $\psi_\tau \in \mathcal{A}(\tau)$ and $\psi_\tau \to \psi_0$ in H. Using the invariance of the pullback attractor and the uniform estimates of the global solution provided in [14] it follows that $\{\psi_\tau : \tau \in \mathbb{R}\}$ is a bounded set in X. Theorem 4.7 then guarantees that Assumption A is satisfied. Thus, from Lemma 4.5, $U(t+\tau,\tau)\psi_\tau \to T(t)\psi_0$ in H as $\tau \to +\infty$, for any $t \geq 0$. Theorem 3.3 then yields $\lim_{t\to+\infty} \mathrm{dist}(\mathcal{A}(t), \mathcal{A}_\infty) = 0$. $\square$

5 Final remarks

In some evolution problems we do not have guaranty of uniqueness of a global solution, this is for example the case when dealing with differential inclusions. In this situation multivalued dynamical systems have to be used rather than only single-valued ones. The majority of the abstract results in Section 3 were recently extended to the multivalued scenario and we refer the reader to [6, 15, 16, 24]. It is worth to emphasize that in the work [6] new aspects were also incorporated like equi-attraction and possible periodicity of the pullback attractors.

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