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Four-dimensional zero-Hopf bifurcation for a Lorenz-Haken system

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Dedicated to Professor Renato Tribuzy on the occasion of his 75th birthday

Abstract. In this paper, we investigate a four-dimensional Lorenz-Haken system described by the following equations:

$$\dot{x} = a(y - x), \dot{y} = -cy - dz + (e - w)x, \dot{z} = dy - cz, \dot{w} = -bw + xy,$$

where a, b, c, d, and e are real parameters.

We aim to characterize the parameter values at which a zero-Hopf equilibrium point occurs at the singular points and to demonstrate the existence of periodic orbits bifurcating from these points. Additionally, we provide the stability conditions for these periodic solutions. The principal tool employed in this analysis is the averaging theory.

Keywords: Hyperchaotic systems, Zero-Hopf bifurcation, Periodic solution, Averaging theory.

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1 Introduction

In this paper, we study the zero-Hopf equilibrium points and zero-Hopf bifurcations of periodic orbit that take place at the equilibria in the Lorenz-Haken system. This system was introduced in the article of Haken [1] as a model describing the dynamics of a homogeneously broadened gain medium in a unidirectional ring cavity. In the notation given in [4], the Lorenz-Haken equations are given by

$$\begin{aligned} \dot{x} &= -\sigma(x-y) + iqx|x|^2, \\ \dot{y} &= -(1-i\delta)y + (r-z)x, \\ \dot{z} &= -bz + \mathrm{Re}(xy), \end{aligned} \tag{1.1}$$

where x, y and z are complex variables, and σ, b, q, r, δ are the real parameters. In 2019, Hayder Natiq et al.[2] derived a new 4D chaotic laser system with three equilibrium points from (1.1), since both x and z can be chosen to be real and y a complex variable, they particularly conducted numerical studies on the existence of attractors associated with stable points and the presence of chaos in the system.

In recent years, interest in the study of 4-dimensional systems that exhibit chaos, also called hyperchaotic systems (i.e., systems that can have two or more directions where the Lyapunov exponents are positive), has increased, especially due to its importance in describing nonlinear phenomena in science and engineering (see [6] and the bibliography contained therein).

In this paper, we study a four-dimensional system of differential equations which is a generalization of the system introduced in [2]

$$\begin{aligned} \dot{x} &= a(y-x), \\ \dot{y} &= -cy - dz + (e-w)x, \\ \dot{z} &= dy - cz, \\ \dot{w} &= -bw + xy, \end{aligned} \tag{1.2}$$

where x, y, z, w are state variables and a, b, c, d and e are real parameters.

Here, a zero-Hopf equilibrium is an equilibrium point of a four-dimensional autonomous differential system, which has a double zero eigenvalue and a pair of purely imaginary eigenvalues.

The outline of this paper is as follows: In Section 2, we present the statement of the main results related to characterizing the zero-Hopf equilibrium points of the system and the presence of zero-Hopf bifurcations. In Section 3, we introduce bifurcation theory to study the existence of periodic solutions. In Section 4, we apply the averaging theory to prove the principal results.

2 Statement of the main results

In the first instance, we are going to compute the equilibrium points of the Lorenz-Haken system (1.2).

Proposition 2.1. *Let $\Delta = \frac{b}{c}(c^2 + d^2 - ec)$ and $c \neq 0$. The following statements are true:*

1. *If $b = 0$, system (1.2) has a straight line of equilibria $\mathtt{p} = (0, 0, 0, w)$.*
2. *If $\Delta \leq 0$ and $b \neq 0$, system (1.2) has a unique equilibrium point $\mathtt{p}_0 = (0, 0, 0, 0)$.*
3. *If $\Delta > 0$ and $b \neq 0$, system (1.2) has three equilibria $\mathtt{p}_0 = (0, 0, 0, 0)$,*

$$\mathtt{p}_\pm \;\; = \big(\pm\sqrt{\Delta}, \pm\sqrt{\Delta}, \pm\tfrac{d}{c}\sqrt{\Delta}, -\tfrac{1}{b}\Delta \big).$$

Proposition 2.1 follows easily by direct computations.

We observe that the two equilibria, $\mathtt{p}_\pm$, tend to the equilibrium point $\mathtt{p}$ as b approaches 0. In summary, the equilibrium point of the system (1.2) can be $\mathtt{p}_+$, $\mathtt{p}_-$, $\mathtt{p}$, or the origin. Note that for $a = b = 0$ system (1.2) has other equilibrium point $(x, 0, 0, e)$, which we will not analyze in this work. Additionally, the system (1.2) exhibits invariance under the coordinate transformation $(x, y, z, w) \to (-x, -y, -z, w)$. Consequently, the system (1.2) has rotational symmetry around the w-axis.

Due to this, in the following, we will focus on examining the possibility of $\mathtt{p}_+$ being a zero-Hopf equilibrium for certain parameter values. The same analysis will apply to the other equilibrium, $\mathtt{p}_-$.

In the next result we characterize when the equilibrium $\mathtt{p}$, $\mathtt{p}_\pm$ and the origin are zero-Hopf equilibrium of the system (1.2).

Proposition 2.2. *For the Lorenz-Haken system (1.2), the following statements hold:*

(i) $\mathtt{p}_0$ *is a zero-Hopf equilibrium if only if* $a = -2c, b = 0, d = -\frac{\sqrt{c^2+\omega^2}}{3}$ *and* $e = \frac{4c^2+\omega^2}{3c}$, *with* $\omega \in \mathbb{R}^+$,

(ii) $\mathtt{p}$ *is a zero-Hopf equilibrium if only if* $a = -2c, b = 0$ *and* $3d^2 - c^2 > 0$,

(iii) $\mathtt{p}_+$ *and* $\mathtt{p}_-$ *are zero-Hopf equilibrium if only if* $a = -2c, b = 0, d = -\frac{\sqrt{c^2+\omega^2}}{\sqrt{3}}$, *with* $\omega \in \mathbb{R}^+$.

In the rest of this section, we will study the zero-Hopf bifurcation and periodic solutions of the hyperchaotic system (1.2) at all the equilibrium points.

Theorem 2.3. *For the Lorenz-Haken system (1.2). The following statements hold.*

(i) Let

$$(a, b, d, e) = \left(-2c + \varepsilon a_1, \varepsilon b_1, -\frac{\sqrt{c^2+\omega^2}}{3} + \varepsilon d_1, \frac{4c^2+\omega^2}{3c} + \varepsilon e_1\right) \tag{2.1}$$

where $\omega > 0$ *and* $\varepsilon > 0$ *are sufficiently small parameters. If* $a_1 \neq 0$, $b_1 \neq 0$, $c \neq 0$, $\eta = 3ce_1 + 2\sqrt{3}d_1\sqrt{c^2+\omega^2} \neq 0$ *and* $\eta_1 = 3a_1\omega^2 - 2c\eta \neq 0$, *then for* $\varepsilon > 0$ *sufficiently small, the hyperchaotic system (1.2) has a zero-Hopf bifurcation at the equilibrium point located at* $\mathtt{p}_0$, *and at most three periodic orbits can bifurcate from this equilibrium when* $\varepsilon = 0$. *Moreover, the periodic solutions are stable if* $a_1 > 0, b_1 > 0$, $16\eta + 3b_1\omega^2 < 0$ *and* $4\eta_1 + 3b_1\omega^2 < 0$.

(ii) Let

$$(a, b) = (-2c + \varepsilon a_1, \varepsilon b_1), \tag{2.2}$$

where $\omega > 0$ *and* $\varepsilon > 0$ *are sufficiently small parameter. If* $a_1 \neq 0$, $b_1 \neq 0$, $c \neq 0$, $d \neq 0$, $(c^2 - d^2)(c^2 + d^2 - ce) \neq 0$, $2(c^2 - d^2) - ce \neq 0$, $3d^2 - c^2 > 0$, $c^4 - 8c^2d^2 + 7d^4 + 2cd^2e < 0$ *and* $(c^4 - 4c^2d^2 + 3d^4)(c^2 + d^2 - ce) < 0$, *then for* $\epsilon > 0$ *sufficiently small, the hyperchaotic system (1.2) has a zero-Hopf bifurcation at the equilibrium point located at* $\mathtt{p}$, *and at most four periodic orbits can bifurcate from this equilibrium when* $\varepsilon = 0$. *Moreover, the periodic solution is stable if* $a_1 > 0, b_1 > 0$, $(c^4 - 8c^2d^2 + 7d^4 + 2cd^2e) < 0$, $2c^2 - 2d^2 - ce < 0$ *and* $c^4 - d^4 - c^3e + cd^2e > 0$.

(iii) Let

$$(a, b, d) = (-2c + \varepsilon a_1, \varepsilon b_1, -\frac{1}{\sqrt{3}}\sqrt{c^2 + \omega^2} + \varepsilon d_1), \tag{2.3}$$

where $\omega > 0$ *and* $\varepsilon > 0$ *are sufficiently small parameter. If* $c \neq 0$, $a_1 \neq 0$, *and* $\kappa = b_1(4c^2 - 3ce + 3\omega^2) < 0$, *then for* $\varepsilon > 0$ *sufficiently small, the hyperchaotic system (1.2) has a zero-Hopf bifurcation at the equilibrium point located at* $\mathtt{p}_\pm$, *and at most two periodic orbits can bifurcate from this equilibrium when* $\varepsilon = 0$. *Moreover, the periodic solutions are unstable.*

3 The Averaging Theory of First Order

The averaging theory provides sufficient conditions for the existence of periodic solutions of non-autonomous differential systems written in the following standard form:

$$\dot{\mathbf{x}} = \varepsilon F(t, \mathbf{x}) + \varepsilon^2 G(t, \mathbf{x}, \varepsilon), \tag{3.1}$$

where $\mathbf{x} \in D$ is an open subset of $\mathbb{R}^n$, $t \geq 0$. We assume that $F(t, \mathbf{x})$ and $G(t, \mathbf{x}, \varepsilon)$ are T-periodic in t. We define averaged function

$$f(\mathbf{x}) = \frac{1}{T}\int_0^T F(t, \mathbf{x})dt. \tag{3.2}$$

With these function the classical averaging method for finding periodic solutions can be summarized by the following theorem, which relates zeros of the first non-vanishing averaged function to the existence of periodic solutions of the non-autonomous differential system (3.1).

Theorem 3.1. *Make the following assumptions:*

(a) F*, its Jacobian* $\frac{\partial F}{\partial \mathbf{x}}$ *and its Hessian* $\frac{\partial^2 F}{\partial \mathbf{x}^2}$*;* G*, its Jacobian* $\frac{\partial G}{\partial \mathbf{x}}$ *are defined, continuous and bounded by a constant independent of* ε *in* $[0, \infty) \times D$ *and* $\varepsilon \in (0, \varepsilon_0]$.

(b) T *is a constant independent of* ε.

Then the following conclusions can be obtained:

(i) If p *is the zero of the averaged function* $f(\mathbf{x})$*, and*

$$\det\left(\frac{\partial f}{\partial \mathbf{x}}\right)\bigg|_{\mathbf{x}=p} \neq 0, \tag{3.3}$$

then there exists a T*-periodic solution* $\mathbf{x}(t, \varepsilon)$ *of system (3.1) such that* $\mathbf{x}(0, \varepsilon) \to p$ *as* $\varepsilon \to 0$.

(ii) If the eigenvalue of the Jacobian matrix $\left(\frac{\partial f}{\partial \mathbf{x}}\right)$ *has a negative real part, the periodic solution* $\mathbf{x}(t, \varepsilon)$ *is asymptotically stable.*

For more information about the averaging theory, see [3] and [5].

4 Proof of results

In this section, we will provide the proofs of Proposition (2.2) and Theorem (2.3).

Proof of proposition 2.2. The characteristic equation at the equilibrium point p_0 is obtained

$$\begin{aligned} P(\lambda) \;=\; & \lambda^4 + (a+b+2c)\lambda^3 + (2bc + c^2 + d^2 + a(b+2c-e))\lambda^2 \\ & + (b(c^2+d^2) + a(2bc + c^2 + d^2 - (b-c)e))\lambda + ab(c^2+d^2-ce). \end{aligned} \tag{4.1}$$

When $a=-2c, b=0, d=-\frac{\sqrt{c^2+\omega^2}}{3}$ and $e=\frac{4c^2+\omega^2}{3c}$, Eq.(4.1) has roots $\lambda_1=\lambda_2=0$ and $\lambda_{3,4}=\pm\omega\, i$, with $\omega\in\mathbb{R}^+$. That is, the equilibrium point p_0 is a zero-Hopf equilibrium of the hyperchaotic system (1.2).

(ii) The characteristic equation at the equilibrium point $\mathtt{p}$ is obtained

$$P(\lambda)=\lambda^4+(a+2c)\lambda^3+(c^2+d^2+a(c-\frac{d^2}{c}))\lambda^2. \tag{4.2}$$

When $a=-2c, b=0$, Eq.(4.2) has roots $\lambda_1=\lambda_2=0$, $\lambda_{3,4}=\pm\sqrt{3d^2-c^2}i$.

(iii) The characteristic equation at the equilibrium point p_+ is obtained

$$\begin{aligned} P(\lambda) &= \lambda^4+(a+b+2c)\lambda^3+(c^2+d^2+a(b+c-\tfrac{d^2}{c})+b(c-\tfrac{d^2}{c}+e))\lambda^2 \\ &\quad +b(ce+a(-c-\tfrac{3d^2}{c}+2e))\lambda-2ab(c^2+d^2-ce). \end{aligned} \tag{4.3}$$

When $a=-2c, b=0, d=-\frac{\sqrt{c^2+\omega^2}}{\sqrt{3}}$, Eq.(4.3) has roots $\lambda_1=\lambda_2=0$, $\lambda_{3,4}=\pm\sqrt{3d^2-c^2}i$.

Proposition 2.2 is proved. □

Proof of statement (i) of Theorem 2.3. First, we assume the condition (2.1). Then, we can write the Lorenz-Haken system (1.2) in the standard form (3.1) in order to use the averaging theory for detecting its periodic solutions. We start by writing the linear part of the Lorenz-Haken system (1.2) when $\epsilon=0$ in its Jordan normal form.

$$\begin{pmatrix} 0 & -\omega & 0 & 0 \\ \omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

so consider the linear change of variables

$$\begin{aligned} x &= \frac{2c(\sqrt{3}c\omega\overline{Y}+\sqrt{3}\omega^2\overline{X}-3c\sqrt{c^2+\omega^2}\overline{Z})}{3\omega^2\sqrt{c^2+\omega^2}}, \\ y &= \frac{\sqrt{3}c\omega^2\overline{X}+\sqrt{3}\omega^3\overline{Y}+2c^2(\sqrt{3}\omega\overline{Y}-3\sqrt{c^2+\omega^2}\overline{Z})}{3\omega^2\sqrt{c^2+\omega^2}}, \\ z &= \frac{1}{3}(\overline{X}+\frac{c}{\omega^2}(-2\omega\overline{Y}+2\sqrt{3}\sqrt{c^2+\omega^2}\overline{Z})), \\ w &= \overline{W}. \end{aligned}$$

In addition, by letting $(\overline{X}, \overline{Y}, \overline{Z}) = \varepsilon(X, Y, Z)$, and subsequently introducing cylindrical coordinates $X = r\cos\theta$ and $Y = r\sin\theta$ (where $r > 0$) with θ as the new independent variable, the system (1.2) takes the following form in these new independent variables:

$$
\begin{aligned}
\frac{dr}{d\theta} &= \frac{\varepsilon}{3c\omega^4\sqrt{c^2+\omega^2}}(cr\omega^3(\sqrt{3}cd_1 - a_1\sqrt{c^2+\omega^2})\cos^2\theta + \omega\sqrt{c^2+\omega^2}\cos\theta \\
&\quad (-6c^3 d_1 Z + r\omega(6c^2(e_1 - W) + a_1\omega^2)\sin\theta) + c\sin\theta(6cz(-(2c^2 \\
&\quad + \omega^2)d_1\sqrt{c^2+\omega^2} - \sqrt{3}c(e_1 - W)(c^2+\omega^2)) + r\omega(2c\omega(\sqrt{3}cd_1 \\
&\quad + a_1\sqrt{c^2+\omega^2})\cos\theta + ((4c^3 + 3c\omega^2)\sqrt{3}d_1 + (6c^2(e_1 - W) - 2a_1\omega^2) \\
&\quad \sqrt{c^2+\omega^2})\sin\theta)) + O(\varepsilon^2) \\
&= \varepsilon F_1(\theta, r, Z, W) + O(\varepsilon^2), \\
\frac{dZ}{d\theta} &= \frac{\varepsilon}{18c^2\omega^3\sqrt{c^2+\omega^2}}(-12c^3 Z(2\sqrt{3}(c^2+\omega^2)d_1 + 3c(e_1 - W)\sqrt{c^2+\omega^2}) \\
&\quad + \sqrt{3}cr\omega^2(4c^2(a_1 + 3e_1 - 3W) + a_1\omega^2)\cos\theta + r\omega(6(4c^3 + c\omega^2)d_1 \\
&\quad \sqrt{c^2+\omega^2} - \sqrt{3}(12c^4(-e_1 + W) + 4a_1c^2\omega^2 + a_1\omega^4))\sin\theta) + O(\varepsilon^2) \\
&= \varepsilon F_2(\theta, r, Z, W) + O(\varepsilon^2), \\
\frac{dW}{d\theta} &= \frac{\varepsilon}{3\omega^5(c^2+\omega^2)}((c^2+\omega^2)(12c^4Z^2 + 2c^2r^2\omega^2 - 3b_1W\omega^4) \\
&\quad + cr\omega(-2c^3r\omega\cos 2\theta - 2\sqrt{3}cZ\sqrt{c^2+\omega^2}(3c\omega\cos\theta + (4c^2+\omega^2)\sin\theta) \\
&\quad + 3c^2r\omega^2\sin 2\theta + r\omega^4\sin 2\theta)) + O(\varepsilon^2) \\
&= \varepsilon F_3(\theta, r, Z, W) + O(\varepsilon^2).
\end{aligned}
\tag{4.4}
$$

Using the notations of the averaging theory described in Theorem 3.1, we have $t = \theta$, $T = 2\pi$, $\mathbf{x} = (r, z, w)$, $F(\theta, r, Z, W) = (F_1(\theta, \mathbf{x}), F_2(\theta, \mathbf{x}), F_3(\theta, \mathbf{x}))$ and $F(\theta, r, Z, W) = (F_1(\theta, \mathbf{x}), F_2(\theta, \mathbf{x}), F_3(\theta, \mathbf{x}))$ and

$$
\begin{aligned}
f(r, Z, W) &= \frac{1}{2\pi}\int_0^{2\pi} F(\theta, r, Z, W)d\theta \\
&= (f_1(r, Z, W), f_2(r, Z, W), f_3(r, Z, W)).
\end{aligned}
$$

Where the components of $f(r, Z, W)$ are given as follows:

$$\begin{aligned}
f_1(r, Z, W) &= \frac{r(6c^2(e_1 - W) - 3a_1\omega^2 + 4\sqrt{3}cd_1\sqrt{c^2+\omega^2})}{6\omega^3},\\
f_2(r, Z, W) &= -\frac{2cZ}{3\omega^3}(3c(e_1 - W) + 2\sqrt{3}d_1\sqrt{c^2+\omega^2}),\\
f_3(r, Z, W) &= \frac{12c^4Z^2 + 2c^2r^2\omega^2 - 3b_1W\omega^4}{3\omega^5},
\end{aligned}$$

and solving the nonlinear system given by $f(r, Z, W) = 0$ we can conclude that the system has the next four solutions

$$\begin{aligned}
s_0 &= (0, 0, 0),\\
s_{1,2} &= (0, \mp\frac{1}{2\sqrt{3}c^{5/2}}\sqrt{b_1\omega^4(3ce_1 + 2\sqrt{3}d_1\sqrt{c^2+\omega^2})}, e_1 + \frac{2d_1\sqrt{c^2+\omega^2}}{\sqrt{3}c}),\\
s_3 &= (\frac{1}{2c^2}\sqrt{b_1\omega^2(6c^2e_1 - 3a_1\omega^2 + 4\sqrt{3}cd_1\sqrt{c^2+\omega^2})}, 0, e_1 + \frac{1}{6c^2}(-3a_1\omega^2\\
&\quad + 4\sqrt{3}cd_1\sqrt{c^2+\omega^2})).
\end{aligned}$$

The first solution s_0 corresponds to the equilibrium at the origin. For the other three solutions, we get

(i) For the solution s_1 and s_2 when $c \neq 0$, $s_{1,2}$ are real solutions. The Jacobian of solution $s_{1,2}$ is

$$\det\big(\frac{\partial f}{\partial \mathbf{x}}(s_1)\big) = \det\big(\frac{\partial f}{\partial \mathbf{x}}(s_2)\big) = \frac{2a_1b_1c\big(3ce_1 + 2\sqrt{3}d_1\sqrt{c^2+\omega^2}\big)}{3\omega^5}.$$

(ii) For the solution s_3 when $c \neq 0$, s_3 is a real solution. The Jacobian of the solution s_3 is

$$\det\big(\frac{\partial f}{\partial x}(s_3)\big) = \frac{a_1b_1\big(-6c^2e_1 + 3a_1\omega^2 - 4\sqrt{3}cd_1\sqrt{c^2+\omega^2}\big)}{3\omega^5},$$

and from hypothesis we have $\det(\frac{\partial f}{\partial \mathbf{x}}(s_j)) \neq 0$, $j = 1, 2, 3$. Thus, the result follows by applying theorem 3.1 and going back through the change of variables (4.4).

To determine the type of stability of the two periodic solutions, we look at the eigenvalues of the Jacobian matrices $\frac{\partial f}{\partial \mathbf{x}}(s_{1,2})$. The eigenvalues are given as follows: $\lambda_1 = -\frac{a_1}{2\omega}$,

$$\lambda_{2,3} = -\frac{3b_1 \pm \sqrt{3}\omega\sqrt{\frac{b_1(48c^2e_1+3b_1\omega^2+32\sqrt{3}cd_1\sqrt{c^2+\omega^2})}{\omega^4}}}{6\omega^3}.$$

On the other hand, the eigenvalues of the Jacobian matrix $\frac{\partial f}{\partial \mathbf{x}}(s_3)$ are given as follows: $\widetilde{\lambda_1} = -\frac{a_1}{\omega}$ and

$$\widetilde{\lambda_{2,3}} = -\frac{3b_1\omega^3 \pm \sqrt{3}\sqrt{b_1\omega^4(3(4a_1+b_1)\omega^2 - 8c(3ce_1 + 2\sqrt{3}d_1\sqrt{c^2+\omega^2}))}}{6\omega^3}.$$

The stability of periodic solutions follows by imposing a negative real part to all eigenvalues of the Jacobian matrix $\partial f(s_j)/\partial x$, $j = 1, 2, 3$ and from the hypothesis. The periodic solutions are stable if $a_1 > 0, b_1 > 0$, $16\eta + 3b_1\omega^2 < 0$ and $4\eta_1 + 3b_1\omega^2 < 0$. $\square$

Proof of statement (ii) of Theorem 2.3. First, we translate p to the origin of coordinates doing $(x, y, z, w) = (\overline{x}, \overline{y}, \overline{z}, \overline{w}) + p$, then we introduce the scaling $(\overline{x}, \overline{y}, \overline{z}, \overline{w}) = \varepsilon(X, Y, Z, W)$. We start by writing the linear part of the Lorenz-Haken system (1.2) when $\epsilon = 0$ in its Jordan normal form, i.e,

$$\begin{pmatrix} 0 & -\sqrt{3d^2-c^2} & 0 & 0 \\ \sqrt{3d^2-c^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

so consider the linear change of variables

$$\begin{aligned} x &= \frac{-6d^2X + 2c^2(3W+X) - 2c\sqrt{-c^2+3d^2}Y}{3(c^2+3d^2)}, \\ y &= -\frac{1}{3(c^3-3cd^2)}(3cd^2X - c^3(6W+X) + \sqrt{-c^2+3d^2}(c^2+3d^2)Y), \\ z &= \frac{d(-3d^2X + c^2(-6W+X) + 2c\sqrt{-c^2+3d^2}Y)}{-3c^3+9cd^2}, \\ w &= Z. \end{aligned}$$

Following the idea of the previous demonstration we can use cylindrical coordinates. In order to put this system in the form (3.1), we take θ as the new independent variable and then we have

$$\begin{aligned}\frac{dr}{d\theta} &= \varepsilon F_1(\theta, r, Z, W) + O(\varepsilon^2),\\ \frac{dZ}{d\theta} &= \varepsilon F_2(\theta, r, Z, W) + O(\varepsilon^2),\\ \frac{dW}{d\theta} &= \varepsilon F_3(\theta, r, Z, W) + O(\varepsilon^2).\end{aligned} \tag{4.5}$$

Using the notations of the averaging theory described in Theorem 3.1, we have $t = \theta$, $T = 2\pi$, $\mathbf{x} = (r, Z, W)$,

$$F(\theta, r, Z, W) = (F_1(\theta, \mathbf{x}), F_2(\theta, \mathbf{x}), F_3(\theta, \mathbf{x}))$$

and

$$\begin{aligned} f(r, Z, W) &= \frac{1}{2\pi}\int_0^{2\pi} F(\theta, r, Z, W)d\theta \\ &= (f_1(r, Z, W), f_2(r, Z, W), f_3(r, Z, W)),\end{aligned}$$

where the components of $f(r, Z, W)$ are given as follows:

$$\begin{aligned} f_1(r, Z, W) &= \frac{r(a_1(c^2-3d^2)-2c^2Z)}{2(-c^2+3d^2)^{3/2}},\\ f_2(r, Z, W) &= \frac{1}{6\sqrt{-c^2+3d^2}(c^3-3cd^2)^2}(3a_1b_1(c^4-4c^2d^2+3d^2)(c^2\\ &\quad +d^2+3d^4)2c^2+(-2d^2(c^2-3d^2)r^2+12c^4w^2-3b_1(c^2\\ &\quad -3d^2)(2c^2-2d^2-ce)Z)),\\ f_3(r, Z, W) &= \frac{2c^2WZ}{(-c^2+3d^2)^{3/2}},\end{aligned}$$

and solving the nonlinear system given by $f(r, Z, W) = 0$, we can conclude that the system has the next four solutions

$$\begin{aligned} s_1 &= (0, \frac{a_1(c^2-d^2)(c^2+d^2-ce)}{2c^2(2c^2-2d^2-ce)}, 0),\\ s_2 &= \Big(\frac{\sqrt{3}\sqrt{a_1}}{2cd}\sqrt{-b_1(c^4-8c^2d^2+7d^4+2cd^2e)}, \frac{1}{2}a_1(1-\frac{3d^2}{c^2}), 0\Big),\\ s_{3,4} &= \Big(0, 0, \pm\frac{\sqrt{a_1}}{2\sqrt{2}c^3}\sqrt{-b_1(c^4-4c^2d^2+3d^4)(c^2+d^2-ce)}\Big).\end{aligned}$$

The solution s_j, $j = 1, \ldots, 4$ exist if and only if $c \neq 0$, $d \neq 0$, and $2(c^2 - d^2) - ce \neq 0$. On the other hand, the solution $s_1 \neq (0, 0, 0)$ if only if $(c^2 - d^2)(c^2 + d^2 - ce) \neq 0$, and the solutions s_2 and $s_{3,4}$ are real if only if $c^4 - 8c^2d^2 + 7d^4 + 2cd^2e < 0$ and $(c^4 - 4c^2d^2 + 3d^4)(c^2 + d^2 - ce) < 0$. For the four solutions, we get

$$
\begin{aligned}
\det\Big(\frac{\partial f}{\partial \mathbf{x}}(s_1)\Big) &= \frac{a_1^2 b_1 (c^2 - d^2)(c^2 + d^2 - ce)(c^4 - 8c^2d^2 + 7d^4 + 2cd^2e)}{2(-c^2 + 3d^2)^{9/2}(2c^2 - 2d^2 - ce)}, \\
\det\Big(\frac{\partial f}{\partial \mathbf{x}}(s_2)\Big) &= \frac{a_1^2 b_1 (c^4 - 8c^2d^2 + 7d^2 + 2cd^2e)}{(-c^2 + 3d^2)^{7/2}}, \\
\det\Big(\frac{\partial f}{\partial \mathbf{x}}(s_3)\Big) &= \det\Big(\frac{\partial f}{\partial \mathbf{x}}(s_4)\Big) = \frac{a_1^2 b_1 (c^2 - d^2)(c^2 + d^2 - 2ce)}{(-c^2 + 3d^2)^{7/2}},
\end{aligned}
$$

and from hypothesis we have $\det(\frac{\partial f}{\partial \mathbf{x}}(s_j))$, $j = 1, \ldots, 4$. Thus, the result follows by applying theorem 3.1 and going back through the change of variables (4.5).

The stability of periodic solutions follows by imposing a negative real part to all eigenvalues of the Jacobian matrix $\partial f(s_j)/\partial x$, $j = 1, \ldots, 4$. □

Proof of statement (iii) of Theorem 2.3. First we assume the condition (2.3), we can write the Lorenz-Haken system (1.2) in the standard form (3.1) in order to use the averaging theory for detecting its periodic solutions.

First, we translate p_+ to the origin of coordinates doing $(x, y, z, w) = (\overline{x}, \overline{y}, \overline{z}, \overline{w}) + p_+$, then we introduce the scaling $(\overline{x}, \overline{y}, \overline{z}, \overline{w}) = \varepsilon(X, Y, Z, W)$. We start by writing the linear part of the Lorenz-Haken system (1.2) when $\epsilon = 0$ in its Jordan normal form,i.e,

$$
\begin{pmatrix}
0 & -\omega & 0 & 0 \\
\omega & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
$$

so consider the linear change of variables

$$
\begin{aligned}
x &= \frac{2c(-\sqrt{3}cZ + c\omega Y + \omega^2 X)}{\sqrt{3}\omega^2},\\
y &= \frac{c\omega^2 X + \omega^3 Y + c^2(-2\sqrt{3}Z + 2\omega Y)}{\sqrt{3}\omega^2},\\
z &= \frac{\sqrt{c^2+\omega^2}(2\sqrt{3}cZ - 2c\omega Y + \omega^2 X)}{3\omega^2},\\
w &= W.
\end{aligned}
$$

Following the idea of the previous demonstration, we can use cylindrical coordinates. In order to put this system in the form (3.1), we take θ as the new independent variable and then we have a system similar to (4.5). Using the notations of the averaging theory described in Theorem 3.1, we have $t = \theta$, $T = 2\pi$, $\mathbf{x} = (r, Z, W)$, $F(\theta, r, Z, W) = (F_1(\theta, \mathbf{x}), F_2(\theta, \mathbf{x}), F_3(\theta, \mathbf{x}))$ and

$$
\begin{aligned}
f(r, Z, W) &= \frac{1}{2\pi}\int_0^{2\pi} F(\theta, r, Z, W)d\theta\\
&= (f_1(r, Z, W), f_2(r, Z, W), f_3(r, Z, W)),
\end{aligned}
$$

where the components of $f(r, Z, W)$ are given as follows:

$$
\begin{aligned}
f_1(r, Z, W) &= -\frac{r(2c^2W + a_1\omega^2)}{2\omega^3},\\
f_2(r, Z, W) &= \frac{2c^2WZ}{\omega^3},\\
f_3(r, Z, W) &= \frac{24c^4Z^2 + c(4c^3r^2 - 12b_1cW + 9b_1eW)\omega^2 + (4c^2r^2 - 9b_1W)\omega^4}{6\omega^5},
\end{aligned}
$$

and solving the nonlinear system given by $f(r, Z, W) = 0$ we can conclude that the system has the next two solutions

$$
\begin{aligned}
s_0 &= (0, 0, 0),\\
s_1 &= (\frac{1}{2c^2}\sqrt{\frac{3}{2}}\sqrt{a_1}\omega\sqrt{\frac{b_1(-4c^2 + 3ce - 3\omega^2)}{c^2+\omega^2}}, 0, -\frac{a_1\omega^2}{2c^2}).
\end{aligned}
$$

Moreover, Moreover, the Jacobian determinant of f at s_1 is given by $\det(\frac{\partial f}{\partial \mathbf{x}}(s_1)) = \frac{a_1^2 b_1(4c^2 - 3ce + 3\omega^2)}{2\omega^5}$ and from hypothesis we have $\det(\frac{\partial f}{\partial \mathbf{x}}(s_1)) \neq 0$. Thus, the result follows by applying theorem 3.1.

To determine the stability of the periodic solution, one needs to calculate the eigenvalues of the Jacobian matrix $\partial F(s_1)/\partial x$. The eigenvalues are as follows: $\lambda_1 = -\frac{a_1}{\omega}$, and $\lambda_{2,3} = -\frac{1}{4\omega^3}(\kappa \pm \sqrt{\kappa(\kappa + 8a_1\omega^2)})$ are real if $a_1 > 0$ and $\kappa > 0$. Regardless of the sign of $\kappa(\kappa + 8a_1\omega^2)$, at least one of the eigenvalues has a positive real part in this case. Therefore, the periodic solution is unstable. This completes the Proof of Theorem 2.3. $\square$

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