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Navier-Stokes equations: a Millennium Prize Problem from the point of view of continuation of solutions

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Abstract. In this work, we investigate the local and global well-posedness of the Navier-Stokes problem in an open, bounded, and smooth subset Ω of $\mathbb{R}^N$, where $N = 2, 3$. We employ Banach scales to express the Navier-Stokes problem in a very weak form with initial data in $L^p(\Omega)^N$, where $p \geq N$. We prove the local well-posedness of solutions and provide conditions guaranteeing their existence for all $t \geq 0$. Our approach involves techniques from semilinear parabolic equations, taking into account nonlinearities with critical growth.

Keywords: parabolic semilinear problems, local and global well-posedness, nonlinearities with critical growth.

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1 Introduction

The Navier-Stokes equations serve as a mathematical model for describing the motion of a fluid within a domain Ω belonging to $\mathbb{R}^N$, where

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$N = 2$ or $N = 3$. For a given point x within this domain and a time $t \geq 0$, the objective is to determine the fluid velocity field, denoted as $u(x,t) = (u^1(x,t), \cdots, u^N(x,t)) \in \mathbb{R}^N$, and the corresponding pressure field, denoted as $\pi(x,t) \in \mathbb{R}$.

Assume that Ω is an open, bounded with smooth boundary in $\mathbb{R}^N$. Let $\Delta u = (\Delta u^1, \cdots, \Delta u^N)$ and

$$(u \cdot \nabla)u = \left(\sum_{k=1}^{N} u^k \partial_k u^1, \cdots, \sum_{k=1}^{N} u^k \partial_k u^N \right),$$

where $\partial_i = \partial/(\partial x_i)$, and $\Delta = \sum_{j=1}^{N} \partial_{jj}$ is the Laplace operator. The Navier-Stokes equations are given by

$$\begin{aligned} & u_t = \Delta u - \nabla \pi + f(t) - (u \cdot \nabla)u, \quad x \in \Omega, \\ & \operatorname{div}(u) = 0, \quad x \in \Omega, \\ & u = 0, \quad x \in \partial\Omega, \\ & u(0,x) = u_0(x), \end{aligned} \tag{1.1}$$

where $u : [0,\infty) \times \Omega \to \mathbb{R}^N$ represents the velocity field, $\pi : [0,\infty) \times \Omega \to \mathbb{R}$ the pressure and $f : [0,\infty) \to \mathbb{R}^N$ is an external force.

There are four problems regarding the Navier-Stokes equations as a Millennium Prize Problem, and the Clay Mathematics Institute (CMI) desires clarification for a minimum of one among the posed inquiries. We will discuss one of them from the perspective of the continuation of solutions. Therefore, there will remain questions that may be interesting but will not be addressed here. To find the questions precisely, see the website of Clay Mathematics Institute (CMI) and look for "Millennium Prizes" and "Navier-Stokes Equations." Essentially the problem is regarding the existence of globally defined and smooth solutions in dimension 3 for (1.1), meaning they exist for all future times. This is an open and extremely challenging problem in the fields of mathematics and physics, known as the "existence and global smoothness of 3D Navier-Stokes solutions" (in dimension 2 is already solved). The explicit formulation of the issue is as follows:

Problem A: Existence and Smoothness in $\mathbb{R}^3$: Suppose that the external force $f(t) \equiv 0$. Given $u_0 \in \mathcal{C}^\infty(\Omega)^N$ with div $u_0 = 0$, there are π and u solutions in $\mathcal{C}^\infty(\Omega \times [0, +\infty))$ with u satifying

$$\int_\Omega |u(x,t)|^2 dx < C \quad \text{for all } t \geq 0. \tag{1.2}$$

The goal of this paper is to discuss Problem A from an ODE perspective. The main idea is to see the Navier-Stokes equations as a parabolic problem with ϵ-regular nonlinearity and apply the abstract results of [3] modifying the scale. More precisely, we will rewrite (1.1) as an abstract Cauchy problem

$$\begin{aligned} &\dot{x} = Ax + f(x),\ t > t_0 \\ &x(t_0) = x_0, \end{aligned} \tag{1.3}$$

where the linear operator $-A : D(A) \subset E_0 \to E_0$ is a sectorial operator in the Banach space E_0. We will denote by E_α, $\alpha \geq 0$ the elements of a Banach scale $\{E_\alpha : \alpha \geq 0\}$ associated to the operator A (see [2]) and by e^{At} the analytic semigroup generated by A.

For the nonlinearities we recall the definition of ϵ-regularity. For $\epsilon \geq 0$, we will say that f is an **ϵ-regular map** relative to the pair (E_1, E_0) if there exists $\rho > 1$, $\gamma(\epsilon)$ with $\rho\epsilon \leq \gamma(\epsilon) < 1$, and a constant c such that $f : E_{1+\epsilon} \to E_{\gamma(\epsilon)}$ and

$$\|f(x) - f(y)\|_{\gamma(\epsilon)} \leq c\|x - y\|_{1+\epsilon}(\|x\|_{1+\epsilon}^{\rho-1} + \|y\|_{1+\epsilon}^{\rho-1} + 1), \tag{1.4}$$

for all $x, y \in E_{1+\epsilon}$. In [3] the authors obtain a special class of solutions that appear when assuming that the nonlinearity is ϵ-regular: we say that $x : [0, \tau] \to E_1$ is a **weak ϵ-regular solution**, or simply an **ϵ-solution**, relative to the pair (E_1, E_0) if $x \in C([0,\tau], E_1) \cap C((0,\tau], E_{1+\epsilon})$, and $x(t)$ satisfies, for all $t \in [0, \tau]$,

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-s)} f(x(s))\, ds. \tag{1.5}$$

We borrow from [3] the following abstract result:

Theorem 1.1. *Suppose f is independent of time and is an ϵ-regular map, for some $\epsilon > 0$, relative to the pair (E_1, E_0). Then, given $y_0 \in E_1$, there exist $r = r(y_0) > 0$ and $\tau_0 = \tau_0(y_0) > 0$ such that for any $x_0 \in B_{E_1}(y_0, r)$, there exists a continuous function $x : [0, \tau_0] \to E_1$, with $x(0) = x_0$, which is the unique ϵ-solution of*

$$\begin{aligned} &\dot{x} = Ax + f(x), \ t > 0 \\ &x(0) = x_0. \end{aligned} \tag{1.6}$$

This solution satisfies

$$x \in C((0, \tau_0], E_{1+\gamma(\epsilon)}) \text{ and } t^\theta \|x(t, x_0)\|_{E_{1+\theta}} \to 0, \ \text{as } t \to 0, \ 0 < \theta < \gamma(\epsilon).$$

Additionally,

1. *If $x_0, z_0 \in B_{E_1}(y_0, r)$, then*

$$t^\theta \|x(t, x_0) - x(t, z_0)\|_{E_{1+\theta}} \leq C(\theta_0)\|x_0 - z_0\|_1, \ \forall t \in [0, \tau_0],$$

   for $0 \leq \theta \leq \theta_0 < \gamma(\epsilon)$.

2. *$x(\cdot, x_0)$ is a strong solution of (1.6) and*

$$x \in C^1((0, \tau_0], E_{\gamma(\epsilon)}) \cap C((0, \tau_0], E_{1+\gamma(\epsilon)}).$$

3. *If $\gamma(\epsilon) > \rho\epsilon$, then the existence time is uniform on bounded sets of E_1.*

The constants above depend on the following: $\tau_0 = \tau_0(y_0, A, \epsilon, \rho, \gamma(\epsilon), c, M)$, $r = r(y_0, \epsilon, \rho, \gamma(\epsilon), c, M)$, $C = C(\theta_0, \epsilon, \rho, \gamma(\epsilon), M)$.

In order to write (1.1) as an abstract Cauchy problem we recall some results from [5]. The free-divergence space is characterized by $L^p_\sigma(\Omega)^N = \overline{\mathcal{C}^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_{L^p(\Omega)^N}}$, where $\mathcal{C}^\infty_{0,\sigma}(\Omega) = \{u \in C^\infty_0(\Omega)^N : div(u) = 0\}$ and

$$L^p(\Omega)^N = L^p_\sigma(\Omega)^N \oplus \{\nabla\varphi : \varphi \in W^{1,p}(\Omega)\},$$

which is usually called by **Helmholtz decompotition**, see [5, pages 697-698]. Then the **Leray's projection** in $L^p(\Omega)^N$ is the natural projection defined by the Helmholtz decomposition on the divergence-free space, i.e.,

$$P_p : L^p(\Omega)^N \to L^p_\sigma(\Omega)^N.$$

Thus applying P_p in (1.1) we obtain an abstract Cauchy Problem in $L^p_\sigma(\Omega)^N$ given by

$$\begin{aligned} &u_t = A_p u + N(u) + f_\sigma, \quad t > 0 \\ &u(0) = u_0, \end{aligned} \tag{1.7}$$

where $A_p = P_p\Delta$ is the Stokes Operator, P_p is the Leray's projection, Δ is the Laplace operator with Dirichlet boundary condition, $N(u) = -P_p(u \cdot \nabla)u$, $f_\sigma = P_p f$. From the construction of Leray's projection, problems (1.7) and (1.1) are equivalent, see [5] for details.

The main goal of this paper is to explain how to write (1.7) in its very weak formulation, which is essentially changing the Banach scale, transitioning from the previously employed one denoted as (E_1, E_0) to the scale (E_0, E_{-1}). Subsequently, we aim to establish that the nonlinearity within this newly introduced scale manifests as an ϵ-regular mapping to apply Theorem 1.1, this is done in Section 2. Consequently, we present a condition whose fulfillment implies the existence and uniqueness of global, as well as smooth, solutions for (1.1), which is explained in Section 3. Therefore, the resolution of this condition would, in turn, resolve the longstanding Millennium Prize Problem associated with the Navier-Stokes equations.

2 Very weak formulation for the Navier-Stokes equation

In this section, we will reformulate problem (1.7) in a weak context. In the language of Banach scales, it's essentially "translating" the problem to a negative scale. To do this, we need to analyze the equation in a functional analytic context in an attempt to express each term of the equation as a linear functional.

Let us establish the functional analytical context for (1.7). Our goal is to write (1.7) in the scale (E_0^p, E_{-1}^p), and apply Theorem 1.1 in this scale.

In the following, we will deal with these terms in order to consider the very weak formulation for (1.7), and then prove local well-posedness for it. We also recall that the Stokes Operator A_p generates an analytic semigroup in $L^p_\sigma(\Omega)^N$. More precisely, the Stokes operator $-A_p$ is sectorial in $L^p_\sigma(\Omega)^N$, for all $2 \leq p < \infty$, see [6, Chapter 3].

For this, we need to analyze the equation in a functional analytic context in an attempt to express each term of the equation as a linear functional. We will first analyze the nonlinear term $P_p(u \cdot \nabla)u$, which will be the key to obtaining a result of local regularity.

To do that, it will be useful to introduce the following notation:

$$\begin{aligned} \mathrm{d}iv(u\, u) &:= \frac{\partial}{\partial x_1}(u_1 u) + \cdots + \frac{\partial}{\partial x_n}(u_n u) \\ &= (\mathrm{d}iv(u_1 u), \cdots, \mathrm{d}iv(u_n u)). \end{aligned}$$

Then, for u with $\mathrm{d}iv\, u = 0$, we have

$$\begin{aligned} (u \cdot \nabla)u &= \frac{\partial}{\partial x_1}(u_1 u) + \cdots + \frac{\partial}{\partial x_n}(u_n u) - \left(\frac{\partial}{\partial x_1} u_1 + \cdots + \frac{\partial}{\partial x_n} u_n \right) u \\ &= \mathrm{d}iv(u\, u). \end{aligned}$$

Recall the tensor product in $\mathbb{R}^N$ and some of its properties. For $u, v \in \mathbb{R}^N$ the **tensor product** between u and v is given by $u\, v := u \otimes v := (u_i v_j)_{i,j=1}^N$, and for $u, v, w \in \mathbb{R}^N$ we have

- $(u + v) \otimes w = u \otimes w + v \otimes w$;
- $|u \otimes v|_{M_N(\mathbb{R})} \leq |u|\, |v|$;
- $u \otimes v = (v \otimes u)^t$, where A^t is the transpose matrix of $A \in M_N(\mathbb{R})$.

To prove that the nonlinearity is ϵ-regular we will need to next result.

Lemma 2.1. *Let $u, \phi : \Omega \to \mathbb{R}^N$ be smooth compactly supported in Ω with $div\phi = divu = 0$. Then*

$$\int_\Omega (u \cdot \nabla)u \cdot \phi = -\int_\Omega u \otimes u \cdot \nabla\phi,$$

where in the last term we use the following notation: $A \cdot B = \sum_{i,j=1}^n a_{ij}b_{ij}$, for $A, B \in M_N(\mathbb{R})$.

Proof. Fix $\Gamma = \partial\Omega$, then

$$\begin{aligned}
\int_\Omega (u \cdot \nabla)u \cdot \phi dx = \int_\Omega & \mathrm{d}iv(u\,u) \cdot \phi\, dx \\
&= \int_\Omega (\mathrm{d}iv(u_1 u), \cdots, \mathrm{d}iv(u_N u)) \cdot (\phi_1, \cdots, \phi_N)\, dx \\
&= -\int_\Omega \left[(u_1 u) \cdot \nabla\phi_1 + \cdots + (u_N u) \cdot \nabla\phi_N\right] dx \\
&\quad + \int_\Gamma \left[\phi_1 (u_1 u)_\Gamma + \cdots + \phi_N (u_N u)_\Gamma\right] d\sigma \\
&= -\int_\Omega [u_1 u \ \cdots \ u_N u] \cdot [\nabla\phi_1 \ \cdots \nabla\phi_N] dx + \int_\Gamma 0\, d\sigma \\
&= -\int_\Omega \begin{bmatrix} u_1u_1 & \cdots & u_1u_N \\ \vdots & \ddots & \vdots \\ u_Nu_1 & \cdots & u_Nu_N \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial\phi_1}{\partial x_1} & \cdots & \frac{\partial\phi_N}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial\phi_1}{\partial x_N} & \cdots & \frac{\partial\phi_N}{\partial x_N} \end{bmatrix} dx \\
&= -\int_\Omega u \otimes u \cdot \nabla\phi dx,
\end{aligned}$$

where, for each $i = 1, \cdots, N$, $(u_i u)_\Gamma$ is the normal component of $u_i u$ in Γ. ∎

Let $m \in \mathbb{N}$. For all $1 < p < \infty$ we define $E_0^p := L_\sigma^p(\Omega)^N$ endowed with norm $\|\cdot\|_{L^p(\Omega)^N}$. As we have seen before, A_p is closed and densely defined, which makes it possible to consider the scale of interpolation-extrapolation of order m generated by (E_0^p, A_p), i.e.,

$$\{(E_\alpha^p,\, A_{p,\alpha})\, ;\, \alpha \geq -m\}.$$

We denote $(A_p)^\#$ as the dual operator of A_p. In [5, page 698] we have the following characterization $(A_p)^\# = A_{p^*}$ and $D(A_p^\#) = E_1^{p^*}$. Since

$(A_p)^{\#}$ is closed in $E_0^{p^*}$ and $\rho(A_p) = \rho((A_p)^{\#})$, it is possible to consider the interpolation-extrapolation scale generated by $((E_0^p)^{\#}, (A_p)^{\#})$, which can be written as

$$\{(E_\alpha^{p^*}, A_{p^*,\alpha})\, ;\, \alpha \geq -m\}.$$

Since $1 < p < \infty$, $L_\sigma^p(\Omega)^N$ is reflexive and we have

$$E_{-\alpha}^p = (E_\alpha^{p^*})^{\#},$$

see [2]. We will use this relation to introduce the functional analytical form to the problem (1.7).

Theorem 2.2. *If* $1 > \alpha > 0$ *and* $q = \frac{Np}{N-2\alpha p}$, *then* $E_\alpha^p \hookrightarrow L^q(\Omega)^N$.

Now we are ready to rewrite (1.7) in E_{-1}^p. Taking the scalar product of (1.7) with a test function $\phi \in C_{0,\sigma}^\infty(\Omega)$ and integrating over Ω, we have

$$\frac{d}{dt}\int_\Omega u\cdot\phi = \int_\Omega A_p u\cdot\phi + \int_\Omega N(u)\phi + \int_\Omega f_\sigma\cdot\phi, \quad t>0. \tag{2.1}$$

Now, since $\phi \in L_\sigma^{p^*}(\Omega)^N$ we have $P_{p^*}\phi = \phi$. Moreover,

$$\int_\Omega P_p\Delta u\cdot\phi = \int_\Omega \Delta u\cdot\phi = \int_\Omega u\cdot\Delta\phi.$$

From Lemma 2.1

$$\int_\Omega N(u)\cdot\phi = -\int_\Omega (u\cdot\nabla)u\cdot\phi = \int_\Omega u\otimes u\cdot\nabla\phi.$$

Hence,

$$\int_\Omega N(u)\cdot\phi = \int_\Omega u\otimes u\cdot\nabla\phi.$$

In this way, we obtain the **very weak formulation** for the Navier-Stokes equation

$$\frac{d}{dt}\int_\Omega u\cdot\phi = \int_\Omega u\cdot\Delta\phi + \int_\Omega u\otimes u\cdot\nabla\phi + \int_\Omega f_\sigma\cdot\phi, \quad t>0. \tag{2.2}$$

Therefore, for each $u \in L_\sigma^p(\Omega)^N$ define $A_{p,-1}u$, $N_{-1}(u)$, and $f_{\sigma,-1}$ as elements of $E_{-1}^p = (E_1^{p^*})^{\#}$ so that

$$E_1^{p^*} \ni \phi \mapsto A_{p,-1}u(\phi) = \int_\Omega u\Delta\phi,$$

$$E_1^{p^*} \ni \phi \mapsto N_{-1}(u)(\phi) = \int_\Omega u \otimes u \ \nabla\phi,$$

and

$$E_1^{p^*} \ni \phi \mapsto f_{\sigma,-1}(\phi) = \int_\Omega f_\sigma \cdot \phi,$$

i.e., we are considering each term of (1.7) as a linear functional defined in $E_1^{p^*}$ taking values in $\mathbb{R}$.

Hence, the equation (1.7) in the scale E_{-1}^p is given by

$$\begin{aligned} &u_t = A_{p,-1}u + N_{-1}(u) + f_{\sigma,-1}, \quad t > 0 \\ &u(0) = u_0. \end{aligned} \tag{2.3}$$

Therefore, the operator $A_{p,-1}$, with $D(A_{p,-1}) = L^p_\sigma(\Omega)^N$

$$A_{p,-1} : L^p_\sigma(\Omega)^N \subset E^p_{-1} \to E^p_{-1}$$

is sectorial, see [2].

Once we arrive at this point, by Theorem 1.1, the local well-posedness of (2.3) in $L^p_\sigma(\Omega)^N$ follows if we prove that $N_{-1}(\cdot)$ is a ϵ-regular map relatively to the pair (E_0^p, E_{-1}^p). Since $f_{\sigma,-1}$ is only time dependent we could assume that $f_{\sigma,-1}$ is locally Hölder continuous to apply the non-autonomous version of Theorem 1.1, see [3] for this case. Thus, from now on, we assume that $f_{\sigma,-1} = 0$.

Theorem 2.3. *$N_{-1}(\cdot)$ is ϵ-regular relatively to the pair (E_0^p, E_{-1}^p) for all $p \geq N$, with $\gamma(\epsilon) = \frac{p-N}{2p} + 2\epsilon$, being:*

- *double critical for $p = 2$, i.e., ϵ-regular for all $\epsilon \in (0, \frac{1}{4})$;*
- *critical for $p = N$ with $N \geq 3$, i.e., ϵ-regular for all $\epsilon \in [0, \frac{1}{4})$;*
- *subcritical for $p > N$, i.e., ϵ-regular for all $\epsilon \in [0, \frac{N}{4p})$.*

Proof. Given $u \in E_\epsilon^p$, we will prove that there exists $r > 1$ such that $u \in L^{2r}(\Omega)^N$. Indeed, by Theorem 2.2 we have $r = \frac{Np}{2N-4\epsilon p}$ is the desired value of r. From the condition that $r > 0$ we must have $\epsilon < \frac{N}{2p}$. Since

$r > 1$, we also have $\epsilon > \frac{N(2-p)}{4p}$. Let r^* be such that $1/r + 1/r^* = 1$, thus $r^* = \frac{Np}{Np-2N+4\epsilon p}$.

Now, to prove that N_{-1} is ϵ-regular, we first show that $N_{-1} : E^p_\epsilon \to E^p_{\eta-1}$, where η is the candidate to $\gamma(\epsilon)$. Indeed, since $E^p_{\eta-1} = (E^{p^*}_{1-\eta})^*$ we must have that η satisfies $\eta = \frac{p-N}{2p} + 2\epsilon$ so that $|\nabla\phi| \in L^{r^*}(\Omega)$. In fact, $\phi \in E^{p^*}_{1-\eta} \subset H^{2-2\eta,p^*}(\Omega)^N$ hence, $|\nabla\phi| \in H^{1-2\eta,p^*}(\Omega)$ and from Theorem 2.2 we must have η satisfying

$$r^* = \frac{Np}{N-(1-2\eta)p^*},$$

and since $r^* = \frac{Np}{Np-2N+4\epsilon p}$, we see that $\eta = \frac{p-N}{2p} + 2\epsilon$.

Then, for $u, v \in E^p_\epsilon$

$$\begin{aligned}\left|\int_\Omega u\otimes v\nabla\phi\right| &\le \int_\Omega |u||v||\nabla\phi| \\ &\le \left(\int_\Omega (|u||v|)^r\right)^{\frac{1}{r}} \left(\int_\Omega |\nabla\phi|^{r^*}\right)^{\frac{1}{r^*}} \\ &\le \|u\|_{L^{2r}}\|v\|_{L^{2r}}\||\phi|\|_{1,r^*} \\ &\le c\|u\|_{E^p_\epsilon}\|v\|_{E^p_\epsilon}\||\phi|\|_{E^{p^*}_{1-\eta}}.\end{aligned}$$

Thus, if $u = v$, we have $\|N_{-1}(u)\|_{E^p_{\eta-1}} \le c\|u\|^2_{E^p_\epsilon}$. From the above estimate

$$\begin{aligned}&\left|\int_\Omega u\otimes u\nabla\phi - \int_\Omega v\otimes v\nabla\phi\right| \\ &\le \left|\int_\Omega u\otimes(u-v)\nabla\phi\right| + \left|\int_\Omega (u-v)\otimes v\nabla\phi\right|,\end{aligned}$$

which implies

$$\|N_{-1}(u) - N_{-1}(v)\|_{E^p_{\eta-1}} \le c(\|u\|_{E^p_\epsilon} + \|v\|_{E^\epsilon_p})\|u-v\|_{E^p_\epsilon}.$$

□

It is interesting to note that N_{-1} is truly double critical for $p = N$. Next, we investigate the ϵ-regular differentiability of N_{-1}; that is, we consider the differentiability of the function N_{-1} as a map from X^ϵ_p to $H^{-2+2\epsilon,p}_{0,\sigma}(\Omega, \mathbb{R}^N)$.

Proposition 2.4. *For $u \in X_p^\epsilon$ and $h \in X_p^\epsilon$*

$$E_{1-\eta}^{p'} \ni \phi \mapsto \int_\Omega (u \otimes h + h \otimes u) \cdot \nabla\phi \in \mathbb{R}, \tag{2.4}$$

defines an element of $E_{\eta-1}^p$ which we will denote by $DN_{-1}(u)h$ and

$$\|N_{-1}(u+h) - N_{-1}(u) - DN_{-1}(u)h\|_{E_{\eta-1}^p} \le \|h\|_{E_\epsilon^p}^2. \tag{2.5}$$

Proof. It is clear from what we have done previously that (2.4) defines an element of $E_{\eta-1}^p$ and that

$$E_\epsilon^p \ni h \mapsto DN_{-1}(u)h \in E_{\eta-1}^p$$

is a bounded linear operator for each $u \in E_\epsilon^p$. This bounded linear operator is denoted by $DN_{-1}(u)$. It is also clear that

$$E_\epsilon^p \ni u \mapsto DN_{-1}(u) \in \mathcal{L}(E_\epsilon^p, E_{\eta-1}^p)$$

is continuous.

It remains to prove (2.5). Note that

$$[N_{-1}(u+h) - N_{-1}(u) - DN_{-1}(u)h](\phi) = \int_\Omega h \otimes h \cdot \nabla\phi.$$

Hence, it is also clear that (2.5) holds from the computations in Proposition 2.4 and it follows that N_{-1} is continuously differentiable. Note that, from Proposition 2.4, $u^* = 0$ will be an asymptotically stable equilibrium for (2.3). □

As a consequence of Theorem 1.1 we have:

Corollary 2.5. *Given $p \ge N$ and $u_0 \in L_\sigma^p(\Omega)^N$, there exists a unique ϵ-solution for (2.3) passing by u_0 which is defined in a maximal interval of existence $[0, \tau_{u_0})$, with*

$$u \in C([0,\tau_{u_0}), E_0^p) \cap C((0,\tau_{u_0}), E_{\gamma(\epsilon)}^p) \cap C^1((0,\tau_{u_0}), E_{\gamma(\epsilon)-1}^p).$$

Moreover,

1. *If $p > N$ and $\limsup_{t \to \tau_{u_0}} \|u(t, u_0)\|_p < \infty$, then $\tau_{u_0} = \infty$, where $\|\cdot\|_p = \|\cdot\|_{L^p(\Omega)^N}$.*

2. *Let $u_0 \in E_0^N$ with $N = 3$ and $N^+ = N/(1 - 4\epsilon)$ for $\epsilon \in (0, 1/4)$. Under these conditions, if $\limsup_{t \to \tau_{u_0}} \|u(t)\|_{N^+} < \infty$, then $\tau_{u_0} = \infty$.*

Proof. Thanks to Lemma 2.3, it is possible to apply Theorem 1.1 for the pair (E_0^p, E_{-1}^p). Note that for $p > N$ we have $\gamma(\epsilon) > \epsilon$ which is the subcritical case and the first item is proved.

For the case $p = N$, which is the critical case, we deal as follows. For all $t_0 > 0$, we have $u(t_0) \in E_{2\epsilon}^N$, since $E_{2\epsilon}^N \hookrightarrow L^q(\Omega)^N$ for $q = NN/(N - 4\epsilon N)$, if $q = N^+$ we have $u(t_0) \in E_0^q$.

Now, we know that the problem

$$\begin{aligned} &\dot{v} = A_q v + N_{-1} v, \ t > 0, \\ &v(0) = u(t_0), \end{aligned} \tag{2.6}$$

has a unique solution $v \in C((0, \tau_{v_0}), E_{2\epsilon}^{N^+})$. Since u is the maximal solution, we see that u is a continuation for v. Moreover, since $N^+ > N$ we are in the conditions of the first item, and therefore if we assume the condition on the norm $\|\cdot\|_{N^+}$, we obtain the same conclusion of item (1) and the proof is complete.

□

Remark 2.6. There is a similar result in [4], where it is proven that if $u_0 \in E_0^3$ and the solution satisfies an estimate in the norm of L^3, then $\tau_{u_0} = \infty$.

3 Existence and global smoothness of 3D Navier-Stokes Equations

Now, we are in a position to discuss Problem A, via Corollary 2.5.

Considering the problem (2.3) with initial data u_0 in L^3 and $p = N = 3$, a solution with infinite maximal existence time can be obtained by establishing an estimate of this solution in an L^{3^+} norm, as we demonstrated in the previous section.

If the maximal interval of existence is $[0, +\infty)$, it is possible to obtain a classical solution, via a "bootstrapping" procedure. Indeed, let $u_0 \in L^3(\Omega)^3$, and suppose that the maximal existence time of the solution is $\tau_{u_0} = \infty$.

By Corollary 2.5, we have $u \in C((0,\infty), E^3_{2\epsilon})$, for $\epsilon \in (0, 1/4)$. Choose $\epsilon \in (1/8, 1/4)$. Note that the embedding

$$E^3_{2\epsilon} \hookrightarrow L^p(\Omega)^3$$

holds for $p = 3/(1-4\epsilon)$.

Now, for any $t_0 > 0$, we have $u(t_0) \in L^p(\Omega)^3$, and since $p > 3$, considering the problem

$$\begin{aligned} &v_t = A_{p,-1}v + N_{-1}(v), \quad t > 0 \\ &v(0) = u(t_0), \end{aligned}$$

we obtain the solution $v \in C((0,\infty), E^p_{\gamma(\epsilon)})$, where $\gamma(\epsilon) = (p-3)/2p + 2\epsilon$.

As $\gamma(\epsilon) = 4\epsilon$ and u is a maximal solution of (2.3), and v is a continuation, we obtain $u(t) \in E^p_{4\epsilon}$ for all $t \geq t_0$, and since t_0 is arbitrary, we obtain this property for all $t > 0$. Now, as $8\epsilon - 3/p = \nu > 0$ by well-known embedding theorems, see [1, 6], we obtain $E^p_{\gamma(\epsilon)} \hookrightarrow \mathcal{C}^\nu(\Omega)^N$. By the same procedure, we can obtain the solution u having an arbitrary number of derivatives in the strong sense, i.e., $u \in C((0,\infty), C^\infty(\Omega)^N)$.

Remember that by the construction of the Leray's projection, [5], it is possible to obtain the pressure being smooth as well.

Hence, we infer the subsequent result.

Proposition 3.1. *Considering the problem* (2.3) *with initial data in* $L^3(\Omega)$*, to solve Problem A, it is sufficient to show that the solution is globally defined, i.e., the existence time is infinite.*

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