Matemática
Contemporânea
Vol. 55, 77–85
http://doi.org/10.21711/231766362023/rmc559

On the Firefighter problem of full icosahedral symmetry fullerene graphs

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Dedicated to Professor Jayme Szwarcfiter
on the occasion of his 80th birthday

Abstract. The Firefighter problem was introduced by Hartnell in 1995 and corresponds to a scenario in which a fire breaks out at one or more vertices of a graph and spreads to all adjacent vertices that have not been protected in previous steps. In this paper, we present an algorithm of the Firefighter problem to an infinite family of full icosahedral symmetry fullerene graphs, providing the surviving rate at least 50%, contributing to the question proposed by Costa in 2015 on the surviving rate of graphs with maximum degree 3.

Keywords: Firefighter problem, fullerene graphs, cubic graphs.

2020 Mathematics Subject Classification: 05C57, 91A43, 68R10.

1 Introduction

The *Firefighter problem* in a graph G begins when one or more vertices of G burns and then one or more vertices of G are chosen to be defended

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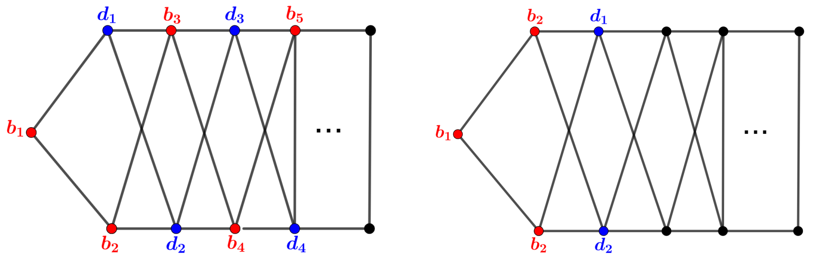
at Step 1, making it unburnable. For each new step, the fire spreads to all adjacent vertices that have not been protected in the previous steps and again, one or more vertices are chosen to be defended by firefighters (each firefighter defends one vertex), until the fire stops spreading. In this work, we will deal with the case where only one initial burned vertex and one firefighter per step is allowed.

The $MVS(G, F, d)$ is the maximum number of vertices saved in G, when fire occurs on vertices in F, with at most d defended vertices per step, looking at all possible strategies. When $F = \{r\}$, we write $MVS(G, r, d)$. From now on, in the figures, *burned* vertices, denoted b_k, $k \geq 1$, indicate the step such that the vertices were burned, and *defended* vertices, denoted d_k, $k \geq 1$, indicate the step such that the vertices were defended. Moreover, red vertices represent the burned vertices, blue vertices represent the defended vertices, and black vertices are the ones that are saved without a firefighter (vertices defended indirectly).

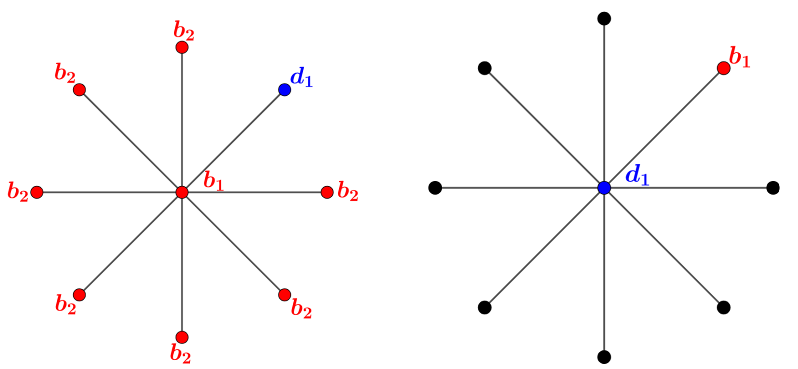
The main goals of the Firefighter problem are to maximize the number of saved vertices or to stop the spread of the fire in the smallest number of steps. The common sense of defending an adjacent vertex to the fire would not guarantee saving the maximum number of vertices. This strategy is known as a greedy algorithm. An example can be seen in Figure 1.1a, where if we defend an adjacent vertex to the fire in each step, the process would never end. On the other hand, in Figure 1.1b, defending a vertex far from the fire is the better strategy.



(a) (b)

Figure 1.1: Applying the greedy strategy on the left, the process would never end. Remark that on the right the process ends in 2 steps and we have $n-3$ saved vertices.

The parameter $sn(G, v)$ represents the highest number of vertices in graph G that can be saved, when the fire starts at vertex v, assuming the best strategy is used. Note that in Figure 1.2a, $sn(S_8, b_1) = 1$, while in Figure 1.2b, $sn(S_8, b_1) = 8$.



(a) (b)

Figure 1.2: The behavior of the $sn(S_8, b_1)$, where S_8 is the star graph.

Introduced by Cai and Wang [2], the *surviving rate*, denoted as $\rho(G)$, is the expected proportion of vertices that can be saved, assuming the initial vertex is selected randomly and uniformly.

$$\rho(G) = \frac{1}{|V(G)|^2} \sum_{v \in V(G)} sn(G, v)\cdot$$

In the two important surveys due to Wagner [8] and Finbow and MacGillivray [5] are presented all solved and open problems. In 2007, Finbow et al. [4] showed that finding an optimal strategy is NP-hard for trees of maximum degree three, and presented a tractable case on graphs of maximum degree three when the fire breaks out at a vertex of degree two. This implies that the Firefighter problem is hard for graphs of maximum degree three such that the fire breaks out in a vertex of degree three. In this context, Costa [3] posed the following question.

Question 1.1 (Costa [3], 2015)**.** For which subclasses of graphs with maximum degree 3 is it possible to calculate the surviving rate?

In this paper, we investigate the Firefighter problem in the class of fullerene graphs. *Fullerene graphs* are cubic, planar, 3-connected graphs containing only pentagonal or hexagonal faces. These graphs are mathematical models of diamond molecules composed only of carbon atoms, with 12 pentagonal faces and an unspecified number of hexagonal faces (0 or ≥ 2). These molecules were experimentally conceived by Kroto et al. in 1985 [7] and have remarkable physicochemical properties, widely studied in different branches of science.

Given a connected and planar graph G, we define the distance between the faces F_a and F_b of G, denoted by $dist_G(F_a, F_b)$, from the distance between their corresponding vertices a and b in G^*, dual graph of G, denoted by $dist_{G^*}(a, b)$, as follows $dist_G(F_a, F_b) = dist_{G^*}(a, b)$.

The *icosahedral fullerene graphs*, introduced by Andova and Škrekovski in 2013 [1], are the graphs such that the centers of the pentagonal faces form an icosahedron. The *full icosahedral symmetry fullerene graphs* $G_{i,0}$, where $i \geq 1$, is a perfectly symmetric icosahedral fullerene graph such that each pentagonal face equidist exactly i units to other 5 pentagonal faces. Furthermore, the path that gives the distance of length i between each pentagonal face of $G_{i,0}$ is unique.

The $20i^2$ vertices of $G_{i,0}$ are distributed in $3i$ concentric cycles (*auxiliary cycles*). It should be noted that these graphs have $10i^2-10$ hexagonal faces and always have 12 pentagonal faces. This indicates that there is a quadratic increase in the number of hexagonal faces in $G_{i,0}$, affecting the number of steps for fire spreading, which requires more steps to traverse an entire hexagonal face compared to pentagonal faces. The line that crosses the bisector of the central pentagonal face sppliting the $|V(G_{i,0)})| - 4i$ vertices into the left and right sides, is called *symmetry line* r and represented by a dashed line. As an example, we present the graph $G_{2,0}$ in Figure 1.3. In the figures, the 12 pentagonal faces are colored blue and the outer pentagonal face is represented by the blue auxiliary cycle $3i$.

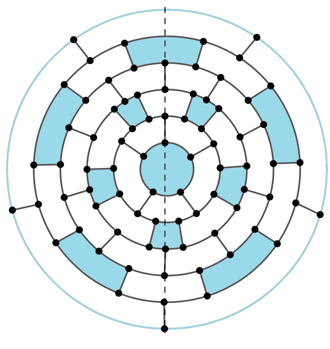


Figure 1.3: $G_{2,0}$ consisting of $20 \cdot 2^2 = 80$ vertices, $3 \cdot 2 = 6$ auxiliary cycles and $10 \cdot 2^2 - \ 10 = 30$ hexagonal faces.

2 Surviving Rate

In 2023, we showed that the maximum number of defended vertices in $G_{1,0}$, the smallest fullerene graph, is equal to 9 and, consequently, the surviving rate is 45% [6]. This result motivated Theorem 2.1, which provides a lower bound for the surviving rate in full icosahedral symmetry fullerene graphs $G_{i,0}$, for $i \geq 2$.

Theorem 2.1. *Every full icosahedral symmetry fullerene graph has*

$\rho(G_{i,0}) \geq$ 50%, $i \geq 2$.

Idea of the proof. This result is based on the algorithm that separates the mostly burned vertices to the left of the symmetry line, from the mostly defended vertices to the right of the symmetry line. Furthermore, the algorithm defends at least a half of the $4i$ vertices of the $G_{i,0}$ that intersect the symmetry line (*vertices of the symmetry line*). We define v_k as a vertex of the symmetry line r in the auxiliary cycle c_k, i.e., $v_k \in c_k \cap r$, $k \in \{1, \ldots, 3i\}$. The vertex u^a belonging to the auxiliary cycle c_k is called *vertex adjacent to the left of the symmetry line* r if this is on the left of r and $dist(u^a, v_k)$ is the maximum distance among all vertices of c_k. Let u be a vertex of the auxiliary cycle c_k. We define the distance from u to the symmetry line r as follows:

$$dist(u,r) = min \begin{cases} dist(u, v_k), \text{such that } v_k \in c_k \cap r \\ dist(u, u^a) + \dfrac{1}{2}, \text{such that } u^a \in c_k \end{cases}$$

At each step i we choose a vertex to be defended d_i always keeping the symmetry line r as a reference. If any burned vertex belongs to the symmetry line, then we defend the vertex of the same auxiliary cycle adjacent to the burned vertex on the right of r; otherwise, when the burned vertices do not belong to r, we consider the burned vertex with the shortest distance from the symmetry line, and we defend the vertex on the right of it. We observe that we do not have two burned vertices in the same step and with the same distance from the symmetry line, since the fire does not spread uniformly throughout the graph $G_{i,0}$. The algorithm ensures that the fire remains, in almost all steps, on the left of the symmetry line. Therefore, the surviving rate $\rho(G_{i,0}) \geq 50\%, i \geq 2$. □

We present the algorithm applied to $G_{2,0}$ and $G_{3,0}$ in Figure 2.1 and to $G_{4,0}$ in Figure 2.2.

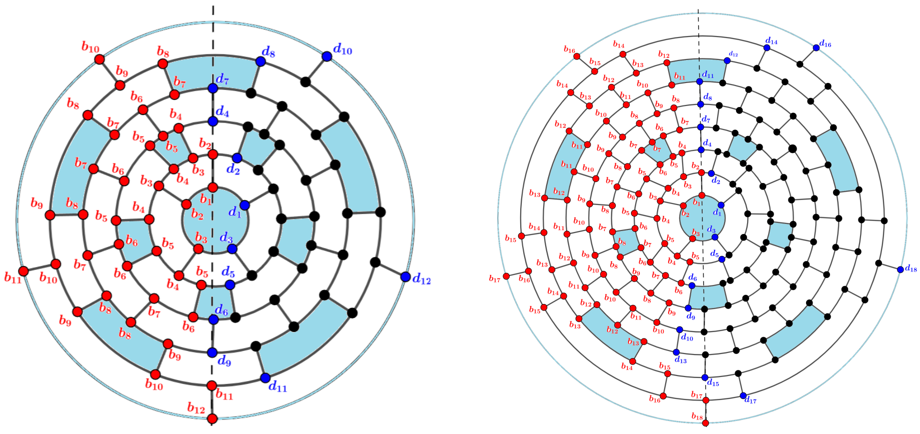


Figure 2.1: Algorithm that ensures surviving rates $\rho(G_{2,0}) \geq 50\%$ and $\rho(G_{3,0}) \geq 53\%$, respectively.

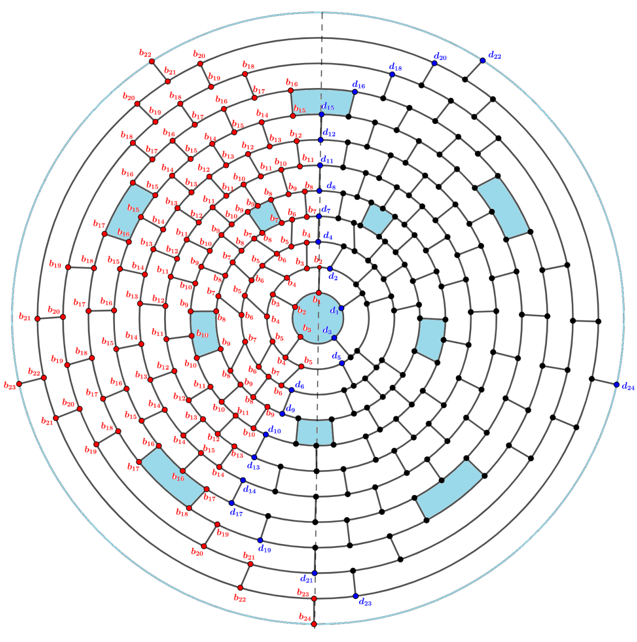


Figure 2.2: Algorithm that ensures surviving rate $\rho(G_{4,0}) \geq 55\%$.

3 Final Remarks

Notice that the behavior of the burned vertices in each step seems to be as on Table 3.1. Based on this table, Equation 3.1 indicates a lower bound to the number of burned vertices, while Equation 3.2 indicates a lower bound to the surviving rate of $G_{i,0}, i \geq 2$.

Step	1	2	...	$2i+1$	...	...	...	$4i$	...	$6i-1$	$6i$	Total
#Burneds	1	2	...	$2i+1$	$2i+1$	...	$2i+1$	$2i+1$	...	2	1	$8i^2+4i$

Table 3.1: Number of burned vertices at each step in $G_{i,0}, i \geq 2$. Red entries mean the same number $2i+1$ of burned vertices during $2(i-1)$ steps.

$$2\sum_{n=1}^{2i+1} i + (2i+1)(i-1)2 \geq 8i^2 + 4i \tag{3.1}$$

$$\rho(G_{i,0}) \geq \frac{20i^2 - (8i^2+4i)}{20i^2} = \frac{12i^2 - 4i}{20i^2} = \frac{3}{5} - \frac{1}{5i}, i \geq 2\cdot \tag{3.2}$$

Asymptotically, the surviving rate of $G_{i,0}, i \geq 2$, is at most 60% and lead us to propose the following conjecture.

Conjecture 3.1. *Every full icosahedral symmetry fullerene graph has* $50\% \leq \rho(G_{i,0}) \leq 60\%$, $i \geq 2$.

Acknowledgement

This work is supported by the Brazilian agencies CNPq (Conselho Nacional de Desenvolvimento Científico e Tecnológico) (Grant numbers: 313797/2020-0), FAPERJ (Fundação de Amparo à Pesquisa do Estado do Rio de Janeiro) (Grant numbers: ARC E-26/010.002674/2019, JCNE E-26/201.360/2021) and CAPES (Coordenação de Aperfeiçoamento de Pessoal de Nível Superior) (Grant number: 88887.668259/2022-00).

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