Matemática
Contemporânea
Vol. 55, 67–76
http://doi.org/10.21711/231766362023/rmc558

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Bounds on the Edge-Sum Distinguishing Game

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Dedicated to Professor Jayme Szwarcfiter on the occasion of his 80th birthday

Abstract. In 2017, Tuza introduced a graph labeling game called *Edge-Sum Distinguishing Game* (ESD Game). Two players, Alice and Bob, alternately assign an unused label $f(v) \in \{1, \ldots, s\}$ to an unlabeled vertex v of a graph G, and the induced edge label $\phi(uv)$ of an edge $uv \in E(G)$ is given by $\phi(uv) = f(u) + f(v)$. Alice's goal is to end up with an injective vertex labeling of all vertices of G that induces distinct edge labels, and Bob's goal is to prevent this. In this work, we show bounds on the number of consecutive positive integer labels necessary for Alice to win the ESD game on a simple graph G.

Keywords: graph labeling, labeling game.

2020 Mathematics Subject Classification: 05C57, 05C78, 91A46.

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1 Introduction

In this paper, all graphs $G = (V(G), E(G))$ are finite, undirected, simple such that $n = |V(G)|$ and $m = |E(G)|$.

A *vertex* (*edge*) labeling of a graph G is an assignment of labels (elements of a set) to the vertices (edges) of G, that induces an edge labeling (vertex labeling) to the edges of G, satisfying some constraints.

Many graph labelings based on sums of integer labels have been proposed [8]. For example, in 1970, Kotzig and Rosa [14] introduced the notion of *magic valuation*, which is a vertex labeling $f\colon V(G) \to \{1, \dots, n\}$ of a graph G such that $S = \{f(u) + f(v)\colon uv \in E(G)\}$ consists of m consecutive integers. This labeling was later rediscovered by Enomoto et al. [5] and renamed as *super edge-magic labeling*.

Stewart [17] calls an edge labeling $f\colon E(G) \to \mathbb{Z}$ of a simple graph G *supermagic* if (1) distinct edges have distinct values assigned, and their labels comprise consecutive integers; (2) the sum of values assigned to all edges incident to a given vertex x is the same for all vertices v of G. In 1976, Sedláček [16] showed that Mobius ladders M_n have a supermagic labeling when $n \geq 3$ and n is odd.

In 1980, Graham and Sloane [10] defined the *harmonious labeling* as an injective function $f\colon V(G) \to \mathbb{Z}_m$ in which each edge $uv \in E(G)$ is labeled with $\phi(uv) = (f(u) + f(v)) \bmod m$, so that the resulting edge labels are distinct. They also conjectured that every tree has a harmonious labeling. Very limited results on this conjecture are known [6, 20].

Using a computer, Aldred and McKay [1] showed that all trees with at most 26 vertices have a harmonious labeling. Helm graphs, web graphs and odd cycles have been shown to have harmonious labelings by Gnanajothi [9].

Chang, Hsu and Rogers [4] investigated variations of harmonious labeling. They defined an injective labeling f of a graph G with q vertices to be *strongly c-harmonious* if the vertex labels are from $\{0, 1, \dots, q-1\}$ and the edge labels induced by $f(u)+f(v)$ for each edge uv are $c, \dots, c+q-1$.

In 1990, Harary [11] introduced the notion of *sum graph*. A graph G is called sum graph if there is an bijective labeling f from $V(G)$ to a set of positive integers S such that $xy \in E(G)$ if and only if $f(x) + f(y) \in S$. Every sum graph must contain isolated vertices because the vertex with the highest label in a sum graph can not be adjacent to any other vertex.

In 1990, Hartsfield and Ringel [13] introduced *antimagic labelings* motivated by magic labelings. Given a graph G with m edges, a *antimagic labeling* of G is an injective edge labeling $f\colon E(G) \to \{1, \dots, m\}$ such that the sums of the labels of the edges incident to each vertex are distinct. They conjectured that every graph except K_2 has an antimagic labeling.

For the reader interested in more examples of labelings constructed from sums of integer labels or in results on (anti)magic labelings, harmonious labelings and super edge-magic labelings, we suggest Gallian's dynamic survey [8].

In 2017, Tuza [19] introduced the *Edge-Sum Distinguishing labeling* (ESD labeling), defined as follows: given a graph G and a set of consecutive integer labels $\mathcal{L} = \{1, 2, \dots, s\}$, an ESD labeling of G is an injective labeling $f\colon V(G) \to \mathcal{L}$ such that, when we assign the edge label $\phi(uv) = f(u) + f(v)$ for each edge $uv \in E(G)$, the (induced) edge labeling ϕ is injective. We note that the set of all possible edge labels induced by the vertex labeling f is represented by $\mathcal{L}_E = \{3, 4, \dots, 2s - 1\}$. Figure 1.1 exhibits a graph with an ESD labeling.

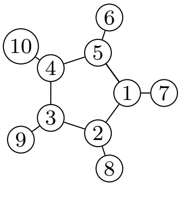


Figure 1.1: A graph with an edge-sum distinguishing labeling.

The ESD labeling was later investigated by Bok and Jedličková [2], who determined the minimum positive integer s for which many classical families of graphs admit an ESD labeling $f\colon V(G) \to \{1, \dots, s\}$.

Graph labelings are usually investigated from the perspective of determining whether a given graph has a required labeling or not [8]. An alternative perspective is to analyze graph labeling problems from the point of view of combinatorial games [3, 12, 18]. In fact, Tuza introduced the ESD labeling in connection to the study of a combinatorial game related to graph labelings with sums. In his seminal paper, Tuza [19] surveyed the area of graph labeling games and presented two graph labeling games with sums [3, 12] based on magic labelings. Tuza [19] also proposed new variants of graph labeling games such as the *Graceful game*, studied by Frickes et al. [7], the *Edge-Difference Distinguishing game*, later investigated by Oliveira et al. [15], and the *Edge-Sum Distinguishing game*.

The *Edge-Sum Distinguishing game (ESD game)* is a type of *maker-breaker* game, where the players have opposite goals. In this game, Alice and Bob alternately assign a previously unused label $f(v) \in \mathcal{L} = \{1, \ldots, s\}$ to an unlabeled vertex v of a given graph G. If both ends of an edge $vw \in E(G)$ are already labeled, then the (induced) label $\phi(vw)$ of the edge vw is defined as $\phi(vw) = f(v) + f(w)$. A move is *legal* if after it all edge labels are distinct. Only legal moves are allowed in this game. Alice (the *maker*) wins if the graph G is fully ESD labeled, and Bob (the *breaker*) wins if he can prevent this (that is, Bob wins if, at some point, no more legal moves are allowed and the graph is not fully ESD labeled). We refer to Figure 1.2 for an example of ESD game played on a Fan graph F_3 with $\mathcal{L} = \{1, \ldots, 9\}$.

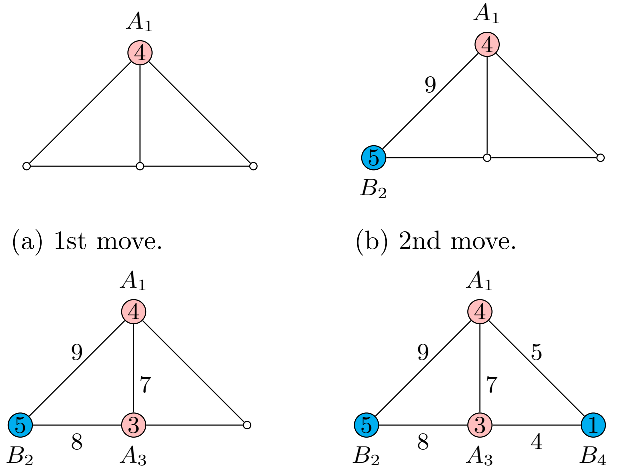
In 2017, Tuza [19] posed the following questions about the ESD game.

Question 1.1. Given a graph G and a set of consecutive non-negative integer labels $\mathcal{L} = \{1, \ldots, s\}$, for which values of s can Alice win the ESD game?

Question 1.2. If Alice can win the ESD game on a graph G with the set of labels $\mathcal{L} = \{1, \ldots, s\}$, can she also win with $\mathcal{L} = \{1, \ldots, s+1\}$?

We define the *edge-sum distinguishing game number* $\sigma_g(G)$ of a graph G as the least positive integer s such that Alice has a winning strategy for the ESD game on G using the set of labels $\{1, \ldots, s\}$, independently of which player starts the game.

In this work, we investigate winning strategies for Alice and Bob on the *ESD game* on a simple connected graph G and we partially answer Tuza's questions presenting bounds for the number of consecutive non-negative integer labels necessary for Alice to win the ESD game on a graph G.



(a) 1st move. (b) 2nd move.

(c) 3rd move. (d) 4th move.

Figure 1.2: ESD game played on a Fan graph F_3 with $\mathcal{L} = \{1, \ldots, 9\}$. Alice's moves A_1 and A_3, and Bob's moves B_2 and B_4.

2 Results on the ESD Game

We begin this section by presenting bounds for the edge-sum distinguishing game number. Bok and Jedličková [2] proved that if G is a graph with maximum degree Δ, then $\sigma_g(G) \leq (\Delta^2 + 1)n + \Delta\binom{n-1}{2}$. In the next theorem, we improve their result, by presenting a better upper bound for the parameter $\sigma_g(G)$.

Theorem 2.1. *If G is a graph on n vertices and m edges, then*

$$\sigma_g(G) \leq n + \max\{d(u)(m - d(u))\colon u \in V(G)\}.$$

Sketch of the proof: Let G be a graph on n vertices and m edges, and let $\mathcal{L} = \{1, \ldots, s\}$ be a set of consecutive integer labels such that $s \geq n + \max\{d(u)(m - d(u))\colon u \in V(G)\}$. Alice (or Bob) starts playing the ESD game on G, and our objective is to show a winning strategy for Alice.

At the beginning of the game, every vertex $w \in V(G)$ has a list of available labels $L(w) = \mathcal{L}$. At each round of the game, a player (Alice or Bob) chooses an unlabeled vertex and assigns to it an available label α such that $1 \leq \alpha \leq s$. Right after a player's move, the sets of available labels of the remaining unlabeled vertices are updated to maintain the property that these sets only contain available labels for the respective vertices.

At the j-th move, a player (Alice or Bob) chooses an unlabeled vertex $v_j \in V(G)$ and assigns an available label $f(v_j)$ to v_j. Right after the j-th move, the set of available labels $L(u)$ of each remaining unlabeled vertex $u \in V(G)$ is updated. Only unused vertex labels and vertex labels that cannot generate repeated edge labels in future iterations can remain in each set. The sets of available labels are updated according to the following two steps:

(1) for every unlabeled vertex $u \in V(G)$, remove $f(v_j)$ from $L(u)$. Note that, since an ESD labeling is injective, the label $f(v_j)$ cannot be assigned to more than one vertex;

(2) for every unlabeled vertex $u \in V(G)$ and for every labeled vertex $u' \in N(u)$, delete from $L(u)$ every label ℓ such that $\ell + f(u') = \phi(e)$, for every edge $e \in E(G)$ that has both endpoints labeled.

Based on the two steps previously described, we determine the maximum number of labels that are deleted, throughout the game, from each set of available labels. We conclude that at most $(n-1) + \max\{d(u)(m - d(u))\colon u \in V(G)\}$ labels are deleted from each set of available labels. Since $|\mathcal{L}|$ is greater than this value, the result follows. □

We recall that Tuza proposed two questions about the game. For

Tuza's Question 1.1, on Theorem 2.1, we present a tight upper bound for the number of labels necessary for Alice to win the game on general graphs. We developed computational experiments to check the result of the game for some small graphs. We applied the backtracking technique (an intelligent exhaustive search), growing the tree of partial solutions by branching on the set of all possible moves and pruning the branching process as soon as we decide which player wins the game.

We observe that, in this algorithm, at each turn of the game, a player needs to choose one vertex in the set of remaining vertices and a label in the set of remaining labels. Thus, at the k-th turn, the player has $(n-k+1)\cdot(s-k+1)$ possible choices. Since $s \geq n$, we have $\Omega(n! \times n!)$ possible configurations for the whole game. For this reason, we just considered graphs with at most 10 vertices to compute with our backtracking algorithm.

In our tests, we did not find a counterexample in which Alice wins the game with s labels but does not win with $s+1$ labels, and this means that Tuza's Question 1.2 remains an open problem. Finally, given a graph G, we asked what is the minimum s such that Alice wins the ESD game independently of which player started the game, i.e., the $\sigma_g(G)$. Our algorithm computed the exact value of $\sigma_g(G)$ of G for the first members in graph classes such as stars $K_{1,n-1}$, paths P_n, cycles C_n, and wheels W_n. These results are summarized in Table 2.1.

Acknowledgements

This study was financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001 and CAPES-PrInt project number 88881.310248/2018-01, CNPq and FAPERJ.

Graph G	$\sigma_g(G)$
$K_{1,n-1}$, $n \geq 2$	n
P_n, $n \leq 3$	n
P_n, $4 \leq n \leq 8$	$n+1$
P_n, $9 \leq n \leq 10$	$n+2$
C_n, $n = 3$	n
C_n, $4 \leq n \leq 5$	$n+1$
C_n, $6 \leq n \leq 9$	$n+2$
C_n, $n = 10$	$n+3$
W_{n-1}, $4 \leq n \leq 5$	$2n-2$
W_{n-1}, $6 \leq n \leq 10$	$2n-1$

Table 2.1: Graphs and their respective ESD game number.

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