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The maximal transitive subtournaments of a digraph: the τ operator

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Dedicated to Professor Jayme Szwarcfiter
on the occasion of his 80th birthday

Abstract. We introduce the maximal transitive subtournament operator τ of a digraph D. We study some basic properties of the operator and exhibit infinite families of convergent and divergent digraphs under τ. It is proved that for every $p \in \mathbb{N}$ there exists an infinite family of finite τ-periodic digraphs of period p.

Keywords: maximal transitive subtournament digraph, τ operator

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1 Introduction and preliminaries

Throughout this paper we consider *simple* digraphs (or oriented graphs), i.e. without loops or symmetric edges. We will use standard terminology for digraphs [2] and graph dynamics [7]. For convenience we recall some useful definitions and notation.

Let $D = (V, A)$ be a digraph. An arc from $u \in V(D)$ to $v \in V(D)$ is denoted by $u \to v$. The *in-degree* and the *out-degree* of a vertex $v \in V(D)$ are denoted by $d^-(v)$ and $d^+(v)$, respectively. A vertex $v \in V(D)$ is a *source* if $d^-(v) = 0$ and a *sink* if $d^+(v) = 0$.

A *tournament* on n vertices is an orientation of the complete graph K_n. A tournament $T = (V, A)$ is called *transitive* if the condition $u \to v$ and $v \to w$ are arcs of T, implies that $u \to w$ is an arc of T. It is well-known that a transitive tournament is acyclic and its vertices may be totally ordered by the arc relation (see [2]). Consequently, T has exactly one source and one sink denoted by f and s, respectively.

Let D be a digraph. We consider the set of the inclusion-wise maximal transitive subtournaments (that we call *tt-cliques*) of D, denoted by $\mathbb{T}(D)$. The *maximal transitive subtournament digraph* (or the *tt-clique digraph*) of D, denoted by $\tau(D)$, is the digraph such that $V(\tau(D)) = \mathbb{T}(D)$ and if $T_1, T_2 \in \mathbb{T}(D)$, then $T_1 \to T_2$ is an arc of $\tau(D)$ if $f_1,\ s_2 \notin V(T_1) \cap V(T_2)$ and $s_1,\ f_2 \in V(T_1) \cap V(T_2)$ (where f_i is the source and s_i is the sink of T_i for $i = 1, 2$). We say that $\tau(D)$ is the *tt-clique operator* or the τ *operator* in brief. Notice that if D is a digraph with no arcs, then $\tau(D) = \emptyset$.

In a sense, this operator is a corresponding notion to the widely studied clique operator of graphs. The *clique graph* $K(G)$ is the graph whose vertex set is the set of its cliques (i.e. its maximal complete subgraphs) and two cliques of G are adjacent if their vertex intersection is nonempty. We call K the *clique operator*. A characterization of clique graphs was given by Roberts and Spencer in [8]. Among the extensive literature on the clique operator, see the book [7] by Prisner and the survey [9] by Szwarcfiter.

Given a digraph D, the *line digraph* $\overrightarrow{L}(D)$ of D has the arcs of D as its vertices and if $u \to v$ and $w \to x$ are arcs of D, then there is an arc $(u, v) \to (w, x)$ in $\overrightarrow{L}(D)$ if and only if $v = w$. Let $m(D)$ denote the maximum order of a tt-clique of D. Observe that if $m(D) \leq 2$, then $\tau(D) = \overrightarrow{L}(D)$. There are several characterizations of line digraphs (see the book [1] by Bagga and Beineke and the already mentioned book [7]).

By convention, $\tau^0(D) = D$ and $\tau^n(D) = \tau(\tau^{n-1}(D))$ for every $n \geq 1$. Following the theory of graph operators, let us recall some definitions. A digraph is *τ-divergent* if $\lim_{n\to\infty} |\tau^n(D)| = \infty$, otherwise, it is *$\tau$-convergent*. Equivalently, a digraph is *τ-convergent* if $\tau^m(D) \cong \tau^{m+p}(D)$ for some integers $m \geq 0$ and $p \geq 1$. The smallest such numbers m and p are called the *transition index* and the *period* of D (under τ), respectively. In this case, we say that D *converges* to the set $\{\tau^{m+k}(D) : 0 \leq k \leq p-1\}$. If $p = 1$, we simply say that D *converges* to $\tau^m(D)$. If $m = 0$, then D is called *τ-periodic* of period p and when $m = 0$ and $p = 1$, D is said to be *self-τ-convergent* or *τ-invariant*.

There are many results on divergent, convergent and self-clique graphs. However, the behaviour of the iterated clique operator is far from being characterized. On the other hand, the behavior of the iterated line digraph has been completely characterized by Beineke (see for example [1]) and Hemminger [5].

Applications of (transitive) tournaments include the study of voting theory and social choice theory, game theory and computer science (see [6] and its references).

In the following sections, we study some basic properties of the τ operator and exhibit infinite families of convergent and divergent digraphs under τ. In particular we prove that for every $p \in \mathbb{N}$ there exists an infinite family of finite τ-periodic digraphs of period p.

2 Properties of the τ operator

Let D be a digraph. Recall that if $\mathbb{T}(D) = \{T_i : i = 1, ..., k\}$, then f_i and s_i denote the source and the sink of T_i, respectively. For brevity, if $u_1 \to v$, ..., $u_k \to v$ $(k \geq 2)$ are arcs of D, we write $u_1, ..., u_k \to v$. Analogously, we use $u \to v_1, ..., v_k$ instead of $u \to v_1$, ..., $u \to v_k$.

Proposition 2.1. *The τ operator is not surjective.*

Proof. Define the digraph H by setting $V(H) = \{T_1, T_2, T_3, T_4\}$ and $A(D) = \{T_1 \to T_2, T_3, T_4; T_2 \to T_3; T_3 \to T_4; T_4 \to T_2\}$. We claim that $H \notin \mathrm{Im}(\tau)$. For a contradiction, suppose that there exists $\tau^{-1}(H)$ (accordingly, T_1, T_2, T_3 and T_4 are the tt-cliques of $\tau^{-1}(H)$). By the definition of τ, the sources $f_2, f_3, f_4 \in T_1$. Since $T_2 \to T_3$, $T_3 \to T_4$ y $T_4 \to T_2$, we have that $f_2 \to f_3 \to f_4 \to f_2$ is a directed cycle of $\tau^{-1}(H)$. This is a contradiction. □

Consider the following generalization of the digraph H used in the proof of Proposition 2.1. Let $\overrightarrow{C}_n$ be a directed cycle with vertex set $\{T_0, T_1, ..., T_{n-1}\}$ $(n \geq 3)$ and arcs $T_i \to T_{i+1}$ (the sum is taken modulo n). Define the digraph H_n by $V(H_n) = \{T_0, T_1, ..., T_{n-1}\} \cup \{T_v\}$. We construct H_n by taking a copy of $\overrightarrow{C}_n$ and T_v such that $T_v \to T_i$ for every $i \in \{0, 1, ..., n-1\}$.

If H_n is a subdigraph of a digraph D, then $f_i \in T_v$ in $\tau^{-1}(H)$ for every $i \in \{0, 1, ..., n-1\}$, where f_i is the source of the tt-clique T_i in $\tau^{-1}(H)$. Analogously as in the previous proof, the set of sources f_i induces a directed cycle in T_v, which is impossible. This proves the following:

Theorem 2.2. *Let D be a digraph and H_n a subdigraph of D. Then $D \notin Im(\tau)$.*

We conjecture that $D \in \mathrm{Im}(\tau)$ if and only if D does not contain H_n as a subdigraph.

Recall that $m(D)$ is the maximum order of a tt-clique of D. Using the definition of the τ operator, it is straightforward to prove the following:

Theorem 2.3. *$m(D) \geq m(\tau(D))$ for every digraph D.* □

Denote by $\overleftarrow{D}$ the *converse* of a digraph D, i.e. $\overleftarrow{D}$ is obtained from D by reversing every arc of D. From the definitions of τ and of the converse of a digraph, we have that $\tau(\overleftarrow{D}) = \overleftarrow{\tau(D)}$.

Recall that a digraph D is *strongly connected* if for every pair of vertices u and v there is a directed path from u to v and from v to u. Notice that a strongly connected digraph has neither a source nor a sink.

In general, if D is strongly connected, then $\tau(D)$ is *not* necessarily strongly connected as the following example shows. Let D be a digraph defined by $V(D) = \{0, 1, 2, 3, 4, 5\}$ and $A(D) = \{0 \to 1 \to 2 \to 3 \to 5 \to 0, 0 \to 2 \to 4 \to 0, 4 \to 5 \to 1 \to 3, 4 \to 3\}$. Then $\mathbb{T}(D) = \{T_0 = [012], T_1 = [123], T_2 = [243], T_3 = [435], T_4 = [450], T_5 = [501]\}$, where $[xyz]$ denotes the transitive tournament on 3 vertices such that $x \to y \to z$ and $x \to z$. Hence $V(\tau(D)) = \mathbb{T}(D)$ and $A(\tau(D)) = \{T_0 \to T_2 \to T_3 \to T_5 \to T_1, T_4 \to T_5 \to T_0 \to T_1, T_4 \to T_0\}$. Observe that T_4 is a source and T_1 is a sink. We conclude that $\tau(D)$ is not strongly connected.

3 Convergence and divergence under τ

We first consider acyclic digraphs. Recall that every acyclic digraph has a source and a sink. According to the definition of τ operator, it is straightforward to show that $\tau(D)$ is acyclic whenever D is an acyclic digraph. Observe that K_1 is acyclic.

The *distance* from vertex u to vertex v in D is the length of the shortest directed path from u to v. A longest directed path of D is a directed path of maximum length. We denote by $l(D)$ the length of a longest directed path of D.

Proposition 3.1. *Let D be a connected acyclic digraph such that $|V(D)| \geq 2$. Then $l(D) > l(\tau(D))$.*

Proof. Consider $l = l(D)$ and $l' = l(\tau(D))$. Let $T_0 \to T_1 \to \ldots \to T_{l'}$ be a directed path of length l' in $\tau(D)$. The vertices $f_0, f_1, \ldots f_{l'}$ and $s_{l'}$ induce a directed path in D of length $l' + 1$. Therefore, $l' + 1 \leq l$. □

Let D be an acyclic digraph of order n. Notice that $\tau^k(D)$ is acyclic and $l(\tau^k(D)) > l(\tau^{k+1}(D))$ for every $k \in \{1, ..., n\}$. Hence there exists $m \leq k$ such that $\tau^m(D) = K_1$. We have the following consequence.

Corollary 3.2. *Every acyclic digraph is τ-convergent.*

Let $\mathbb{Z}_n$ be the group of the residues modulo n and $\emptyset \neq J \subseteq \mathbb{Z}_n \setminus \{0\}$. The *circulant digraph* $\overrightarrow{C}_n(J)$ is defined by $V(\overrightarrow{C}_n(J)) = \mathbb{Z}_n$ and $i \to j$ is an arc of $\overrightarrow{C}_n(J)$ if $j - i \in J$ with $i, j \in \mathbb{Z}_n$. Since we only deal with simple digraphs, we set $|J \cap \{i, -i\}| \leq 1$ for every $i \in \mathbb{Z}_n$ to avoid symmetric arcs.

We say that a digraph D is *vertex-transitive* if for every pair of vertices $u, v \in V(D)$, there exists an automorphism of D that maps u to v. In particular, circulant digraphs $\overrightarrow{C}_n(J)$ are vertex-transitive and $\phi(u) = u + k \pmod n$ for every $k \in \mathbb{Z}_n$ is an automorphism of $\overrightarrow{C}_n(J)$. We define the interval $[k, l] \subseteq \mathbb{Z}_n$ $(k \neq l)$ by $[k, l] = \{k, k+1, ..., l\} \pmod n$ (i.e. every sum is taken modulo n, for example, $[11, 3] = \{11, 12, 0, 1, 2, 3\}$ in $\mathbb{Z}_{13}$).

Theorem 3.3. *$\overrightarrow{C}_n(1, 2, ..., k)$, where $n \geq 5$ and $2 \leq k \leq \lfloor (n-1)/2 \rfloor$, is τ-invariant.*

Proof. Let $D = \overrightarrow{C}_n(1, 2, ..., k)$. Notice that $T_0 = [0, k]$ is a tt-clique of D. Moreover, $T_i = [i, k+i]$ induces a tt-clique for every $i \in \mathbb{Z}_n$ since D is vertex-transitive. Then $\mathbb{T}(D) = \{T_i : i \in \mathbb{Z}_n\}$. Consider $\tau(D)$ whose vertex set is $\mathbb{T}(D)$ and $T_i \to T_{i+j}$ is an arc of $\tau(D)$ for every $j = 1, ..., k$. Define the digraph homomorphism $\phi : D \to \tau(D)$ by $\phi(i) = T_i$ for every $i \in \mathbb{Z}_n$. It is straightforward to check that ϕ is a digraph isomorphism. $\square$

Let $D = \overrightarrow{C}_n(1, 2, ..., k)$. Define D_a and D_b to be the digraphs for which $V(D_a) = \mathbb{Z}_n \cup \{a\}$, $V(D_b) = \mathbb{Z}_n \cup \{b\}$ and $A(D_a) = A(D) \cup \{a \to 0, 1\}$, $A(D_b) = A(D) \cup \{0, 1 \to b\}$, respectively. We recall that $[u, v, w]$ denotes the transitive tournament T such that $V(T) = \{u, v, w\}$ and $A(T) = \{u \to v \to w, u \to w\}$.

Theorem 3.4. *D_a and D_b are τ-invariant.*

Proof. Notice that $V(\tau(D_a)) = \mathbb{T}(D_a) = \mathbb{T}(D) \cup [a, 0, 1]$ and $V(\tau(D_b)) = \mathbb{T}(D_b) = \mathbb{T}(D) \cup [0, 1, b]$, where $\mathbb{T}(D) = \{T_i : i \in \mathbb{Z}_n\}$. Therefore, $A(\tau(D_a)) = A(\tau(D)) \cup \{[a, 0, 1] \to T_0, T_1\}$ and $A(\tau(D_b)) = A(\tau(D)) \cup \{T_{n-k}, T_{n-k+1} \to [0, 1, b]\}$ with $T_i = [i, k+i] \in \mathbb{T}(D)$ for $i \in \{0, 1, n-k, n-k+1\}$. If we define $\phi : D_a \to \tau(D_a)$ by $\phi(i) = T_i$ with $i \in \mathbb{Z}_n$

and $\phi(a) = [a, 0, 1]$, then it is routine to prove that ϕ is a digraph isomorphism. Similarly, let $\psi : D_b \to \tau(D_b)$ be a digraph homomorphism such that $\psi(i) = T_{n-k+i}$ with $i \in \mathbb{Z}_n$ and $\psi(b) = [0, 1, b]$. It is straightforward to show that ψ is an isomorphism. □

We emphasize that geometrically the isomorphism ψ of the previous proof is a *rotation* of D_b identified by sending $0 \in \mathbb{Z}_n$ to $n - k \in \mathbb{Z}_n$.

In [3], Escalante proved that for every $p \in \mathbb{N}$ there exist infinitely many finite connected K-periodic graphs of period p (see also Theorem 14.17 of [7]). We show an analogous result for digraphs under τ.

We define a digraph $H = D_a \cup D_b$, i.e. $V(H) = \mathbb{Z}_n \cup \{a, b\}$ and $A(H) = A(D) \cup \{a \to 0, 1\} \cup \{0, 1 \to b\}$.

Theorem 3.5. *Let $n \geq 5$ and $2 \leq k \leq \lfloor (n-1)/2 \rfloor$. Then H is τ-periodic of period $p = \frac{n}{\gcd(n,k)}$.*

Proof. From the definition of H (the sums are taken modulo n throughout the proof), $V(\tau(H)) = \mathbb{T}(H) = \mathbb{T}(D) \cup \{[a, 0, 1], [0, 1, b]\}$ and

$$\begin{aligned} A(\tau(H)) = A(\tau(D)) \cup \{[a, 0, 1] \to T_0, T_1\} \cup \\ \{T_{n-k}, T_{n-k+1} \to [0, 1, b]\} \cup \{[a, 0, 1] \to [0, 1, b]\}. \end{aligned}$$

We relabel the vertices of $\tau(H)$. Let $f : V(\tau(H)) \to \mathbb{Z}_n \cup \{a, b_1\}$ be the bijection such that $f(T_i) = i$ for $i \in \mathbb{Z}_n$, $f([a, 0, 1]) = a$ and $f([0, 1, b]) = b_1$. Therefore $V(\tau(H)) = \mathbb{Z}_n \cup \{a, b_1\}$ and

$$A(\tau(H)) = A(D) \cup \{a \to 0, 1\} \cup \{(n-k), (n-k+1) \to b_1\} \cup \{a \to b_1\}.$$

Notice that the arc $a \to b_1$ is a tt-clique of $\tau(H)$. Then, it is a vertex of $\tau^2(H)$ denoted by $[a, b_1]$. Accordingly,

$$\begin{aligned} V(\tau^2(H)) &= \mathbb{T}(\tau(H)) = \mathbb{T}(D) \cup \{[a, 0, 1], [n-k, n-k+1, b_1], [a, b_1]\}, \\ A(\tau^2(H)) &= A(\tau(D)) \cup \{[a, 0, 1] \to T_0, T_1\} \cup \\ &\qquad \{T_{n-2k}, T_{n-2k+1} \to [n-k, n-k+1, b_1]\}, \end{aligned}$$

since $T_{n-2k} = [n-2k, n-k]$, $T_{n-2k+1} = [n-2k+1, n-k+1]$ and $[a, b_1]$ is an isolated vertex of $\tau^2(H)$ such that $\tau([a, b_1]) = \emptyset$. Relabeling again the vertices, we obtain that $V(\tau^2(H)) = \mathbb{T}(\tau(H)) = \mathbb{T}(D) \cup \{a, b_2, [a, b_2]\}$ (note that b_2 is the new label for $[n-k, n-k+1, b_1]$) and

$$A(\tau^2(H)) = A(D) \cup \{a \to 0, 1\} \cup \{(n-2k), (n-2k+1) \to b_2\}.$$

We continue this procedure and for $p \geq 2$, we obtain that

$$V(\tau^p(H)) = \mathbb{T}(\tau^{p-1}(H)) = \mathbb{T}(D) \cup \{a, b_{p-1}\} \text{ and}$$
$$A(\tau^p(H)) = A(D)) \cup \{a \to 0, 1\} \cup \{(n-pk), (n-pk+1) \to b_{p-1}\},$$

where b_{p-1} is the new label for the vertex $[n-(p-1)k, n-(p-1)k+1, b_{p-2}]$. We remark that an isolated vertex $[a, b_{p-1}]$ appears in $\tau^p(H)$ if and only if $n-(p-1)k \equiv 1 \pmod{n}$. This means that there exists the arc $[a, 0, 1] \to [1, 2, b_{p-1}]$ and we proceed as with the case of $\tau(H)$.

Observe that $\tau^p(H) = H$ if and only if $n - pk \equiv 0 \pmod{n}$ (recall that $0 \to b$ is an arc of H). Equivalently, $pk \equiv 0 \pmod{n}$. If $\gcd(n, k) = 1$, then $p \equiv 0 \pmod{n}$ and H is τ-periodic of period $p = n$. If $\gcd(n, k) = d \geq 2$, then $pk \equiv 0 \pmod{n} \Leftrightarrow p\frac{k}{d} \equiv 0 \pmod{\frac{n}{d}} \Leftrightarrow p \equiv 0 \pmod{\frac{n}{d}}$ and H is τ-periodic of period $p = \frac{n}{\gcd(n,k)}$. □

As a consequence of Theorem 3.5, for every $p \in \mathbb{N}$ such that $p \geq 3$ there exists an infinite family of finite τ-periodic digraphs of period p. For the remaining case $p = 2$ we state the following proposition whose proof is left to the reader.

Let $n \geq 2$ and define the digraph J_{2n+1} by $V(J_{2n+1}) = \mathbb{Z}_{2n+1} \cup \{a, b\}$ and $A(J_{2n+1}) = A(\overrightarrow{C}_{2n+1}(1, 2) \cup \{a \to 0, 1\} \cup \{n, (n+1) \to b\}$. Similarly, for $n \geq 3$ we define J_{2n} by $V(J_{2n}) = \mathbb{Z}_{2n} \cup \{a, b\}$ and $A(J_{2n}) = A(\overrightarrow{C}_{2n}(1, 2) \cup \{a \to 0, 1\} \cup \{(n+1), (n+2) \to b\}$. The proof of the next proposition is left to the reader.

Proposition 3.6. *$\tau^2(J_{2n+1}) \cong J_{2n+1}$ for every $n \geq 2$ and $\tau^2(J_{2n}) \cong J_{2n}$ for every $n \geq 3$.* □

From Theorems 3.3 and 3.5 and Proposition 3.6, we have the following:

Theorem 3.7. *For every $p \in \mathbb{N}$ there exists an infinite family of finite τ-periodic digraphs of period p.* □

Finally, we exhibit without proof an infinite family of divergent digraphs. Let $D = \overrightarrow{C}_n(1, ..., k)$ and $D' = \overrightarrow{C}'_n(1, ..., k)$ be the circulants such that $V(D) = \mathbb{Z}_n$ and $V(D') = \mathbb{Z}'_n = \{0', 1', ..., n' - 1\}$. Define a digraph F_{2n} such that $V(F_{2n}) = \mathbb{Z}_n \cup \mathbb{Z}'_n$ and $A(F_{2n}) = A(D) \cup A(D') \cup \{0 \to 0', 1 \to 0', 1 \to 1'\}$. Therefore, $|V(\tau(F_{2n})| = |V(F_{2n})| + 2$ and in general, $|V(\tau^{m+1}(F_{2n})| = |V(\tau^m(F_{2n})| + 2$ for every $m \in \mathbb{N}$. Hence, F_{2n} is τ-divergent of linear growth for every $n \geq 10$.

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