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Coloring game and maximum vertex degree restriction

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Dedicated to Professor Jayme Szwarcfiter
on the occasion of his 80th birthday

Abstract. The *coloring game* is a two player non-cooperative game conceived by Steven Brams, firstly published in 1981. In this game, Alice and Bob, alternately take turns properly coloring the vertices of a finite graph G with t colors. The goal of Alice is to properly color the vertices of G with t colors, and Bob does his best to prevent it. If at any point there exists an uncolored vertex without available color, then Bob wins; otherwise Alice wins. The game chromatic number $\chi_g(G)$ of G is the smallest number t such that Alice has a winning strategy. In 1991, Bodlaender showed the smallest tree T with $\chi_g(T)$ equals to 4, and in 1993 Faigle et al. proved that every tree T satisfies the upper bound $\chi_g(T) \leq 4$. In this paper, we discuss an interesting tree family with maximum degree 3 that has game chromatic number 3 for its first members but game chromatic number 4 otherwise.

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1 Introduction

Let $G = (V, E)$ be a finite, simple, undirected graph with vertex set $V = V(G)$ and edge set $E = E(G)$. In 1981, Martin Gardner [6] published for the first time a two player non-cooperative map-coloring game conceived by Steven Brams. Ten years later, this game was reinvented by Bodlaender [1] in the context of graphs, and called the *coloring game*.

The two players, Alice and Bob, take turns properly coloring an uncolored vertex of graph G by a color in a given color set with t colors (moves), such that adjacent vertices have different colors. Alice's goal is to color the input graph with the t colors, and Bob does his best to prevent it. Alice wins when the graph is completely (properly) colored with t colors; otherwise, Bob wins.

The *game chromatic number* $\chi_g^a(G)$ (or simply $\chi_g(G)$) of G is the smallest number t of colors such that Alice has a winning strategy for the graph coloring game on G when she starts the game. In the vast literature about the coloring game, it is only considered the case when Alice starts. The case when Bob starts is also studied in this paper for proof purposes. So, let $\chi_g^b(G)$ be the smallest number t of colors such that Alice has a winning strategy for the graph coloring game on G, when Bob starts the game.

Clearly, for any graph G, we have that $\chi(G) \leq \chi_g(G) \leq \Delta(G) + 1$, where $\chi(G)$ denotes the chromatic number and $\Delta(G)$ the maximum degree of graph G. So, we have that a complete graph K_n has $\chi_g(K_n) = n$, because $\chi(K_n) = \Delta(K_n) + 1 = n$, and, analogously, an independent set S_n has $\chi_g(S_n) = 1$.

Analyzing the game chromatic number of paths P_n, with n vertices, we can quickly check that $\chi_g(P_1) = 1$ and $\chi_g(P_2) = \chi_g(P_3) = 2$. For $n \geq 4$, we have that $\chi_g(P_n) = 3$ because, regardless of where Alice colors on her first turn, Bob can always assign a different color to a vertex at distance 2 from the vertex that Alice colored, forcing the third color. We recall that the distance $d(u, v)$ between two vertices u and v in a graph is the number of edges in a shortest path connecting them. Using the same idea, we have that cycles C_n have $\chi_g(C_n) = 3$, and stars $K_{1,p}$ with $p \geq 1$ are the only connected graphs satisfying $\chi_g(G) = 2$.

The coloring game has been extensively studied for different graph classes in order to obtain better upper and lower bounds for $\chi_g(G)$: toroidal grids [9], Cartesian products of some classes of graphs [2], planar graphs [10], outerplanar graphs [7], forests [3] and partial k-trees [11].

Bodlaender [1] showed an example of a tree with game chromatic number at least 4, and proved that every tree has game chromatic number at most 5. Faigle et al. [4] subsequently improved this bound by proving that every forest has game chromatic number at most 4.

Despite the variety of papers in this area, the distinction between forests with different game chromatic numbers was only considered in 2015 by Dunn et al. [3]. They characterized forests with game chromatic number 2, and suggested the characterization of forests with game chromatic number 3 and 4 as open problems, due to the difficulty concerning this subject. We contribute to their study by considering a special family of trees called caterpillars in order to define an infinite family of trees with game chromatic number 4.

A *caterpillar* $H = cat(k_1, k_2, \ldots, k_s)$ is a tree obtained from a central path v_1, v_2, v_3, $\ldots$, v_s (called *spine*) by joining k_i leaf vertices to v_i, for each $i = 1, \ldots, s$; and with number of vertices $n = s + \sum_{i=1}^{s} k_i$. We consider caterpillars with $k_1 = k_s = 0$. For $i = 2, \ldots, s-1$, if $k_i \geq 1$, then we say that the vertex v_i has k_i adjacent *leg leaves*.

Our motivation to focus on caterpillars relies on the fact that Bodlaender [1] proved the existence of a tree with $\chi_g(T) \geq 4$ by considering the caterpillar $H_d = cat(0, 2, 2, 2, 2, 0)$ depicted in Figure 1.1. Actually, Dunn et al. [3] proved that the caterpillar H_d is the smallest tree such that $\chi_g(T) = 4$, and is the unique tree with fourteen vertices and game chromatic number 4. We observe that this tree has maximum degree 4.

Faigle et al. [4] proved that $\chi_g(T) \leq 4$, for trees T, and stated that this result can be extended to forests F, that is, $\chi_g(F) \leq 4$. Dunn et al. [3] asked whether the maximum degree is relevant to characterize graphs with

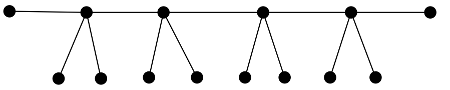


Figure 1.1: The caterpillar H_d satisfies $\chi_g(H_d) = 4$.

game chromatic number equal to 3. Recall that the smallest tree with game chromatic number 4 has maximum degree 4. In a previous extended abstract [5], we have defined necessary and sufficient conditions for a tree with maximum degree 4 to have game chromatic number 4.

In the present paper, we contribute to this maximum degree question by analyzing, in Section 2, a family of caterpillars with maximum degree 3 and without vertex of degree 2. We analyze caterpillars in which we can use a strategy for Alice to win with 3 colors in Theorem 2.1, but this strategy fails for a sufficiently large s.

Our work establishes that the required maximum degree to ensure game chromatic number 4 is in fact 3. Finally, we present in Section 3 a sketch of Theorem 3.1 about members of the family which have game chromatic number 4, and further questions.

2 Trees with game chromatic number 3

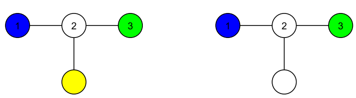
We recall that, by Faigle et al. [4], every caterpillar H, which is not a star, has $3 \leq \chi_g(H) \leq 4$. According to our notation, the star with $n-1$ leaves, for $n \geq 4$, is denoted $K_{1,n-1} = cat(0, n-3, 0)$.

It is an open challenge to characterize the caterpillars with game chromatic number respectively equal to 3 and to 4.

We refer to Figure 2.1, where vertex v_i is simply labeled i. In the coloring game on a caterpillar, a player is forced to use four colors if, during the game, there exists an induced subgraph isomorphic to a *claw*, the caterpillar $cat(0, 1, 0)$, with its leaves colored with different colors, as depicted in Figure 2.1(a). Thus, Bob's strategy is to obtain a previously partially colored claw subgraph as in Figure 2.1(b) to start coloring on it.

Let G be a graph, Z be a previously colored vertex set of $V(G)$, and (G, Z) be the partially colored graph. We say that Alice (resp. Bob) plays on (G, Z), if Alice (resp. Bob) colors the uncolored vertices of $V(G) \setminus Z$. Let $\chi_g^a(G, Z)$ (resp. $\chi_g^b(G, Z)$) be the smallest number t of colors such that Alice has a winning strategy for the graph coloring game on G, when Alice (resp. Bob) starts playing on a graph G with a previously colored set of vertices $Z \subseteq V(G)$.

Let $C = cat(0, 1, 0)$ be a claw and $Z = \{v_1, v_3 \mid c(v_1) \neq c(v_3)\}$ be a previously colored set of vertices of C. We claim that $\chi_g^a(C, Z) = 3$ and $\chi_g^b(C, Z) = 4$. We refer to Figure 2.1(b). First, we observe that, since vertex v_2 can not be properly colored with the two colors previously given to v_1 and to v_3, at least three colors are necessary. If Bob starts playing on (C, Z), then he colors the unique leg leaf (adjacent to v_2) with a third color, forcing a fourth color in v_2. If Alice starts playing on (C, Z), then she colors v_2 with a third color, which ensures that she is able to win the game with 3 colors.



(a) (b)

Figure 2.1: (a) The partially colored claw forces the game chromatic number to be 4; (b) claw-situation.

We define the *claw-situation* as an ordered pair (C, Z), where $C = cat(0, 1, 0)$ is a claw and $Z = \{v_1, v_3 \mid c(v_1) \neq c(v_3)\}$ is a previously colored set of vertices (see Figure 2.1(b)). The claw-situation generalizes the key tool implicitly used by Bodlaender to build the smallest tree with game chromatic number 4. We strengthen the tool in order to define an infinite family of trees with game chromatic number 4 and maximum degree 3.

We shall focus on the tree family H_s composed by caterpillars with $k_1 = k_s = 0$ and $k_i = 1$, for $2 \leq i \leq (s - 1)$. We study how Alice can win the game with 3 colors in caterpillars with maximum degree 3 and without vertices of degree 2. We apply this winning strategy to H_7 and H_9.

Theorem 2.1. *The game chromatic number of H_7 and H_9 is equal to 3.*

Sketch of the proof: First, we consider H_7. Alice colors v_4 with color 1 in her first move, as in Figure 2.2. In Table 2, we analyse Bob and Alice's possible next moves. By the symmetry of the graph, Bob playing at v_5, v_6, v_7, λ_5 or λ_6 is analogous. For every possible Bob's first move, Alice avoids claw-situations, and plays so that 3 colors are enough to color the graph H_7.

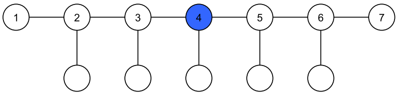


Figure 2.2: H_7 after Alice's first move.

Table 2.1: Possibilities of how Bob plays in his first move, and plays next.

Bob's first turn	Alice's second turn
$c(v_1) = 1$	$c(\lambda_2) = 1$
$c(v_1) = 2$ or $c(\lambda_2) = 2$ or $c(\lambda_4) = 2$	$c(v_2) = 1$
$c(v_2) = 1$ or $c(\lambda_3) = 1$ or $c(v_2) = 2$ or $c(\lambda_3) = 2$	$c(v_3) = 3$
$c(\lambda_2) = 1$	$c(v_1) = 1$
$c(v_3) = 2$	$c(v_1) = 2$

Now, we use an analogous argument to establish that $\chi_g^a(H_9) = 3$.

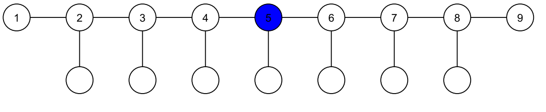


Figure 2.3: H_9 after Alice's first move.

Alice colors v_5 with color 1 in her first move, as in Figure 2.3. If Bob colors v_1 (or λ_2) with color 1, then Alice colors v_3 with color 1. If Bob colors v_1 (or λ_2) with color 2, then Alice colors λ_2 (resp. v_1) with color 2. The other plays have already been analyzed in H_7, we can use the exact same plays. □

By the previous result, it would be expected that, at least for s odd, each caterpillar $H = cat(k_1, ..., k_s)$ with $k_1 = k_s = 0$ and $k_i = 1$, for $2 \leq i \leq s-1$ should have $\chi_g^a(H) = 3$. But this it is not always true, as we can see next.

3 Towards a full dichotomy for maximum degree 3 trees

In this section, we observe that Theorem 2.1, fails for a sufficiently large s. As we see, Theorem 3.1 presents an infinite family of trees with maximum degree 3 and game chromatic number 4. It is still a work in progress to achieve a full dichotomy for maximum degree 3 trees.

Theorem 3.1. *Let H be a caterpillar $cat(k_1, ..., k_s)$, with $k_1 = k_s = 0$ and $k_i = 1$, for $2 \leq i \leq s-1$. If $s \geq 40$, then $\chi_g^a(H) = \chi_g^b(H) = 4$.*

Sketch of the proof: This proof is based on the claw-situation, and it is constructed in stages, with two of them as the main ones:
(i) If H is a caterpillar $cat(k_1, ..., k_s)$, $k_1 = k_s = 0$ and $k_i = 1$, for $2 \leq i \leq s-1$, $s \geq 10$ and let $Z = \{v_1, v_s \mid c(v_1) \neq c(v_s)\}$, then $\chi_g^b(H, Z) = 4$;
(ii) If H is a caterpillar $cat(k_1, ..., k_{20})$, $k_1 = k_{20} = 0$ and $k_i = 1$, for $2 \leq i \leq 19$ and let $Z = \{v_1, v_{20} \mid c(v_1) \neq c(v_{20})\}$, then $\chi_g^a(H, Z) = 4$.

In fact, if Alice starts the game by coloring a spine vertex v_i, then Bob colors another spine vertex v_j such that $d(v_i, v_j) = 19$ and $c(v_j) \neq c(v_i)$, and the result $\chi_g^a(H) = 4$ follows by (i) and (ii).

If Alice starts the game by coloring a leg leaf λ_i, then Bob colors a spine vertex v_{i-1} or v_{i+1} with a different color of λ_i. Now, to avoid a claw-situation, Alice is forced to color the spine vertex v_i, and the result $\chi_g^a(H) = 4$ follows as in the previous case.

On the other hand, if Bob starts the game, then Bob colors v_{20}. Depending on Alice's second move, Bob colors next v_1 or v_{40} with a different color of v_{20}. Again, the result $\chi_g^b(H) = 4$ follows by (i) and (ii). □

We recall that, in the proof of Theorem 3.1, we use the tools $\chi_g^b(H, Z)$ and $\chi_g^a(H, Z)$. We are able to ensure that $\chi_g^a(H) = 4$, for $cat(k_1, ..., k_s)$, with $s \geq 40$, $k_1 = k_s = 0$ and $k_i = 1$, for $2 \leq i \leq s-1$. But first, we show strategies for $\chi_g^a(H) = 3$ when $s = 7$ and $s = 9$. Therefore, we leave as an open question what is the minimum s that guarantees that $\chi_g^a(H) = 4$, for $cat(k_1, ..., k_s)$, with $k_1 = k_s = 0$ and $k_i = 1$, for $2 \leq i \leq s-1$.

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