Matemática
Contemporânea
Vol. 55, 39–47
http://doi.org/10.21711/231766362023/rmc555

Relation between classes of graphs with interval count k

Lívia S. Medeiros [1], **Fabiano S. Oliveira** [2] and **Jayme L. Szwarcfiter** [3]

[1,2,3]IME – Universidade do Estado do Rio de Janeiro (UERJ) – Brazil
[4]COPPE, NCE, IM – Universidade Federal do Rio de Janeiro (UFRJ) – Brazil

Dedicated to Professor Jayme Szwarcfiter
on the occasion of his 80th birthday

Abstract. The subclass $\text{Len}(a_1, a_2, \ldots, a_k)$ of interval graphs consists of those that admit an interval model having precisely the interval lengths $a_1, a_2, \ldots, a_k$. For all $0 \leq a_1 < a_2 < \ldots < a_k$, and $0 \leq b_1 < b_2 < \ldots < b_k$, we prove that $\text{Len}(a_1, a_2, \ldots, a_k) \subseteq \text{Len}(b_1, b_2, \ldots, b_k)$ if and only if there exists a constant r such that $b_j = ra_j$ for all $1 \leq j \leq k$.

Keywords: Interval count, Interval graph, Interval order.

2020 Mathematics Subject Classification: 12A34, 67B89.

1 Introduction

Let $I = [\ell(I), r(I)]$ be a closed interval of the real line, where $\ell(I)$ and $r(I)$ denote respectively the *left* and *right extreme points* of I. A graph G is an *interval graph* if there exists a bijection θ of $V(G)$ to a family $\mathcal{M}$ of intervals on the real line, called a *model*, in which for all $u, v \in V(G)$ with

The authors are partially supported by FAPERJ, CNPq and CAPES.

$u \neq v$, $(u,v) \in E(G)$ if and only if $\theta(u) \cap \theta(v) \neq \emptyset$. An *order* $P = (X, \prec)$ is a binary relation $\prec$ on the set X which is irreflexive and transitive. The *interval order* of a model $\mathcal{M}$ is the order $P = (X, \prec)$ such that $X = \mathcal{M}$ and, for all $I, J \in X$, $I \prec J$ if and only if $r(I) < \ell(J)$. If P is an interval order with interval model $\mathcal{M}$ and G is the interval graph corresponding to $\mathcal{M}$, we say that P *agrees* with G. Note that if P agrees with G, any model of P is also a model of G but a model of G may not be a model of P. For the sake of convenience, when there is a model $\mathcal{M}$ of a graph G (resp. order $P = (X, \prec)$), in the context, an interval $I_x \in \mathcal{M}$ and the corresponding vertex $x \in V(G)$ (resp. element $x \in X$) are used interchangeably. The *length* of an interval I is given by $r(I) - \ell(I)$ and denoted by $|I|$. The number of distinct interval lengths in a model $\mathcal{M}$ is denoted by $IC(\mathcal{M})$, that is, $IC(\mathcal{M}) = |\{|I| \mid I \in \mathcal{M}\}|$.

Ronald Graham raised the question of how many distinct interval lengths are sufficient to represent a model of a given interval graph [5]. In other words, he suggested the problem of determining a model of a given interval graph having the smallest number $IC(G)$ of distinct interval lengths, known as the *interval count* problem. Formally, $IC(G) = \min\{IC(\mathcal{M}) \mid \mathcal{M}$ is a model of $G\}$ (resp. $IC(P) = \min\{IC(\mathcal{M}) \mid \mathcal{M}$ is a model of $P\}$). The problem of deciding efficiently whether $IC(G) = 1$ or $IC(P) = 1$ is solved, since it is equivalent to the problem of recognizing unit interval graphs and orders [1, 4]. We say that an interval x is *nested* to another interval y if there exists distinct intervals a, b such that the intersections between these intervals are like those shown in Figure 1.1(i). Figure 1.1(ii) illustrates an interval graph G and Figure 1.1(iii) an example of a model of G with two distinct interval lengths that realizes $IC(G)$.

An $\{a_1, \ldots, a_k\}$-*model* is an interval model $\mathcal{M}$ in which $|I_v| \in \{a_1, \ldots, a_k\}$ for all $I_v \in \mathcal{M}$. The graph class that admits an $\{a_1, \ldots, a_k\}$-model is denoted by $\textsc{Len}(a_1, \ldots, a_k)$, $a_1 < \ldots < a_k$. The class $\textsc{Len}(0, 1)$ was characterized in [7]. Almost two decades later, authors in [6] described a linear time recognition algorithm for this class. In [3], a polynomial-time algorithm using linear programming was developed to determine whether,

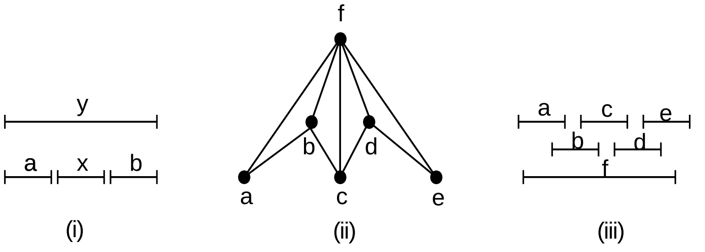

Figure 1.1: In (i) a visual representation of x nested to y, in (ii) an interval graph G and in (iii) a model of G that realizes the interval count of G.

given a connected graph G and a bipartition of $V(G)$ into the sets A and B, $G \in \text{Len}(a,b)$ for some pair of constants $a < b$ and $|I_v| = a$ for all $v \in A$ and $|I_v| = b$ for all $v \in B$. In [2], the inclusion relationship between the different classes $\text{Len}(a,b)$ obtained by varying the parameters a and b was investigated, and shown that $\text{Len}(a',b') \not\subseteq \text{Len}(a,b)$ if and only if $\frac{a'}{b'} \neq \frac{a}{b}$ for all $0 \leq a' < b'$ and $0 \leq a < b$.

This work investigates the more general inclusion relation between the classes $\text{Len}(a_1,\ldots,a_k)$ obtained by varying the parameters $a_1,\ldots,a_k$.

We start by examining the particular case $k = 2$ (Theorems 1.1 and 1.2).

Theorem 1.1 (Francis, Medeiros, Oliveira and Szwarcfiter [2])**.** $\text{Len}(0,k) \not\subseteq \text{Len}(a,b)$ *and* $\text{Len}(a,b) \not\subseteq \text{Len}(0,k)$, *for all* $k > 0$ *and* $0 < a < b$.

The proof consists of constructing a special $\{0,k\}$-model and showing that the corresponding interval graph does not admit an $\{a,b\}$-model, and also the converse.

We build a $\{0,k\}$-model shown schematically in Figure 1.2(i). In this figure, there are $\lceil \frac{b}{a} \rceil + 2$ intervals of length 0 plus an interval of length k, producing an $\{0,k\}$-model. It is shown that the interval graph corresponding to this model does not belong to $\text{Len}(a,b)$. On the other hand, an $\{a,b\}$-model is shown in Figure 1.2(ii). This model consists of a P_5 and a universal vertex. It is shown that the corresponding interval graph does not belong to $\text{Len}(0,k)$.

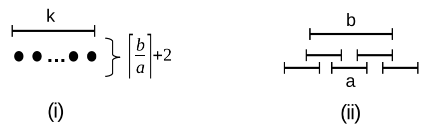

Figure 1.2: (i) G admits a $\{0, k\}$-model but not an $\{a, b\}$-mode, and (ii) G admits an $\{a, b\}$-model but not a $\{0, k\}$-model.

Theorem 1.2 (Francis, Medeiros, Oliveira and Szwarcfiter [2])**.** $\textsc{Len}(a', b') \nsubseteq \textsc{Len}(a, b)$ *for all* $0 < a' < b'$ *and* $0 < a < b$ *for which* $\frac{b'}{a'} \neq \frac{b}{a}$.

The proof consists in, assuming that $\frac{b'}{a'} < \frac{b}{a}$, showing the following: (i) constructing a special $\{a', b'\}$-model whose corresponding interval graph does not admit an $\{a, b\}$-model; and (ii) constructing an $\{a, b\}$-model whose corresponding interval graph does not admit an $\{a', b'\}$-model. Note that this is equivalent to constructing two special $\{a', b'\}$-models whose corresponding interval graph does not admit an $\{a, b\}$-model for the cases $\frac{b'}{a'} < \frac{b}{a}$ and $\frac{b'}{a'} > \frac{b}{a}$, by exchanging the pair (a', b') by (a, b) and vice-versa in the construction (ii). We outline the construction of such models as they will be employed in the demonstration of our result.

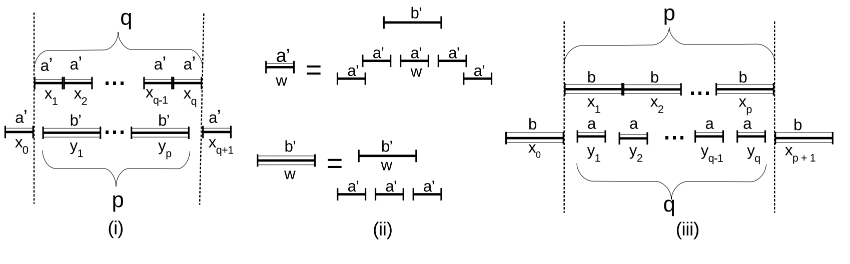

Figure 1.3: Scheme of the model $\mathcal{M}$ used in the Theorem 1.2. The label above each interval represents its length, and below its identification.

So, assume that $\frac{b'}{a'} < \frac{b}{a}$. Since there exists a rational number between any two real numbers, there exist positive integers p, q such that $\frac{b'}{a'} < \frac{q}{p} < \frac{b}{a}$. Clearly, $p < q$. We build an $\{a', b'\}$-model $\mathcal{M}$ using the integers p and q. Let $\mathcal{M}$ be the model of Figure 1.3(i). In this model, there is a path

$x_0, x_1, \ldots x_q, x_{q+1}$ with $q+2$ intervals of length a'. Furthermore, there is an independent set $\{y_1, \ldots, y_p\}$ with p intervals of length b'. In $\mathcal{M}$, there are also more intervals than are explicitly represented. Each interval drawn as double and triple bars denotes that there are more intervals associated with it. If w is a double bar interval, then we add five more intervals to the model such that w and such additional intervals consist of the submodel depicted in the upper part of Figure 1.3(ii). The actual left and right extremes of these additional intervals are omitted because they can be chosen arbitrarily, provided that the intervals form a submodel isomorphic to that of Figure 1.3(ii) and their sizes are as prescribed. If w is a triple bar interval, then we add three more intervals associated with w to form the submodel given by the bottom of Figure 1.3(ii). This completes the description of $\mathcal{M}$. It is shown that the interval graph G corresponding to $\mathcal{M}$, although in $\textsc{Len}(a', b')$, does not belong to $\textsc{Len}(a, b)$.

We prove that $\textsc{Len}(a, b) \nsubseteq \textsc{Len}(a', b')$ in a similar way. Let $\mathcal{M}$ be the $\{a, b\}$-model schematized in Figure 1.3(iii). In this model, there is a path x_0, x_1, $\ldots x_p$, x_{p+1} with $p + 2$ intervals of length b. In addition, there is an independent set $\{y_1, \ldots, y_q\}$ with q intervals of length a. In $\mathcal{M}$, there are also more intervals than are explicitly represented. Again, the double bar and triple bar intervals are to be replaced by a scheme similar to that shown in Figure 1.3(ii), with the change that all a'-s in that figure are to be replaced by a-s and all b'-s by b-s. It is shown that the interval graph G corresponding to $\mathcal{M}$, although in $\textsc{Len}(a, b)$, does not belong to $\textsc{Len}(a', b')$.

In summary, such a proof provides a construction of an $\{a', b'\}$-model that cannot be transformed into an $\{a, b\}$-model that corresponds to the same interval graph. The exact construction to be employed, out of the two possible ones, depends on whether $\frac{b'}{a'} < \frac{b}{a}$ or the other way around. We will denote such a construction by $\mathcal{M}(a', b', a, b)$, regardless the pair p, q chosen in the construction, which we will employ in the next section.

Additionally, note that if $\frac{b'}{a'} = \frac{b}{a}$, then $\textsc{Len}(a', b') = \textsc{Len}(a, b)$.

2 Inclusion relationship between classes

We investigate the inclusion between the classes $\textsc{Len}(a_1, \ldots, a_k)$, for all $0 \leq a_1 < \ldots < a_k$. If S represents the sequence $a_1, \ldots, a_k$, then we also denote $\textsc{Len}(a_1, \ldots, a_k)$ by $\textsc{Len}(S)$.

Theorem 2.1. *Let $S_1 = a_1, \ldots, a_k$ and $S_2 = b_1, \ldots, b_k$ be two sequences such that $a_1, b_1 \geq 0$ and, for all $1 \leq i < k$, $a_i < a_{i+1}$ and $b_i < b_{i+1}$. Then, $\textsc{Len}(S_1) \nsubseteq \textsc{Len}(S_2)$ if and only if there is no constant r such that $b_j = ra_j$ for every $1 \leq j \leq k$.*

Proof. The proof of the necessary condition is trivial. For the converse, consider the cases:

1. $(a_1 \neq 0$ and $b_1 \neq 0)$ or $(a_1 = b_1 = 0)$.

   Let $i = 1$ if $a_1 \neq 0$ and $b_1 \neq 0$, or $i = 2$ otherwise. Note that, by this choice, $a_i \neq 0$ and $b_i \neq 0$. Let $a_j \in S_1$ and $b_j \in S_2$ with $i < j \leq k$, such that $\frac{a_j}{a_i} \neq \frac{b_j}{b_i}$. Note that such a value of j certainly exists, because if $\frac{a_j}{a_i} = \frac{b_j}{b_i}$ for all $i < j \leq k$, then there would exist $r = \frac{b_i}{a_i}$ such that $b_j = (\frac{b_i}{a_i})a_j = ra_j$ for every $1 \leq j \leq k$, contradicting the hypothesis of the theorem. Let $\mathcal{M}(S_1, S_2)$ be the model obtained from $\mathcal{M}(a_i, a_j, b_i, b_j)$ by redefining the double and triple bar intervals in the model $\mathcal{M}$ of Figure 1.3 as described below (see Figure 2.1):

   (a) For each double bar interval, labeled $a' = a_i$ in $\mathcal{M}$, create a chain of k nested intervals, with lengths $a_1, \ldots, a_k$, as specified in Figure 2.1(i) if $i = 1$ and as specified in Figure 2.1(ii) if $i = 2$.

   (b) For each triple bar interval, labeled $b' = a_j$ in $\mathcal{M}$, create a chain of k nested intervals, of length $a_1, \ldots, a_k$, as specified in Figure 2.1(iii). Note that the submodels shown in Figures 2.1 are equal except for the position of the original interval in the nested chain (the length shown in the double bar intervals represent the i-th smallest length of the chain, while in the triple bar intervals, the j-th smallest length of the chain).

As before, the actual left and right extremes of these additional intervals are omitted because they can be chosen arbitrarily, provided that the intervals form a submodel isomorphic to that of Figure 2.1.

Let G be the interval graph corresponding to $\mathcal{M}(S_1, S_2)$. Consider the double or triple bar interval transformations of Figure 2.1. In these models, notice the intervals that form the nesting chain with k intervals (the smallest length a_1, nested in another of length a_2, and so on until nesting to the one of length a_k). Let $C \subset \mathcal{M}(S_1, S_2)$ be the set of such intervals in these chains. By construction, in any model $\mathcal{M}$ of G, we have that intervals of C having length a_i in $\mathcal{M}(S_1, S_2)$ have the i-th smallest length of $\mathcal{M}$. If there exists a S_2-model $\mathcal{M}'$ of G, then all C intervals of length a_i and a_j in the S_1-model $\mathcal{M}(S_1, S_2)$ of G have respectively length b_i and b_j in $\mathcal{M}'$. This is equivalent to saying that there is a submodel of $\mathcal{M}(S_1, S_2)$, isomorphic to $\mathcal{M}(a_i, a_j, b_i, b_j)$, which can be transformed from an $\{a_i, a_j\}$-model into a $\{b_i, b_j\}$-model, contradicting the construction of Theorem 1.2.

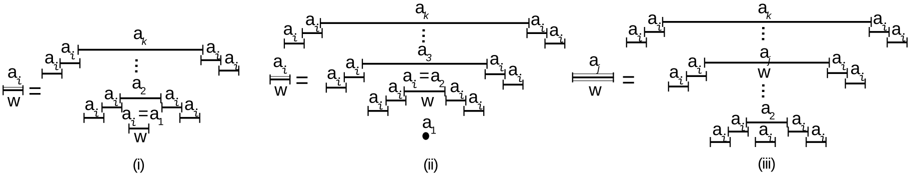

Figure 2.1: The model $\mathcal{M}(a_1, \ldots, a_k, b_1, \ldots, b_k)$, obtained from the model $\mathcal{M}(a_i, a_j, b_i, b_j)$ by replacing the exchanges in Figure 1.3(ii) with those in Figure 2.1.

2. ($a_1 \neq 0$ and $b_1 = 0$) or ($a_1 = 0$ and $b_1 \neq 0$).

   Let $\mathcal{M}(S_1, S_2)$ be the model obtained from Figure 1.2 according with the following cases:

   - If $a_1 = 0$ and $b_1 \neq 0$.

Construct the model sketched in Figure 1.2(i) and nest the interval a_2 (which is mapped to interval k of Figure 1.2(i)) to a chain of k nested intervals, with lengths $a_3, \ldots, a_k$, as specified in Figure 2.2(i).

- If $a_1 \neq 0$ and $b_1 = 0$.

  Construct the model sketched in Figure 1.2(ii) and nest the interval a_2 (which is mapped to interval b of Figure 1.2(ii)) to a chain of k nested intervals, with lengths $a_3, \ldots, a_k$, as specified in Figure 2.2(ii).

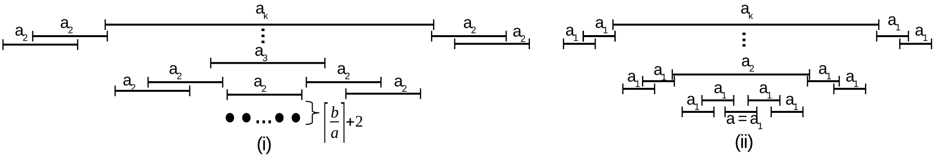

Figure 2.2: The model $\mathcal{M}(a_1, \ldots, a_k, b_1, \ldots, b_k)$ obtained from the model sketched in Figure 1.2(i) or (ii) by forming a nesting chain of k intervals.

Let G be the interval graph corresponding to $\mathcal{M}(S_1, S_2)$. If there exists a S_2-model $\mathcal{M}'$ of G, then all C intervals of length a_1 and a_2 in the S_1-model $\mathcal{M}(S_1, S_2)$ of G have respectively length b_1 and b_2 in $\mathcal{M}'$. This is equivalent to saying that there exists a submodel of $\mathcal{M}(S_1, S_2)$, isomorphic to that of Figure 1.2(i) or (ii), which can be transformed from an $\{a_1, a_2\}$-model into a $\{b_1, b_2\}$-model, which contradicts the construction of Theorem 1.1.

□

3 Conclusion

Motivated by the inclusion relation between classes $\textsc{Len}(a, b)$ studied in [2], we investigate the inclusion of classes $\textsc{Len}(a_1, \ldots, a_k)$ with $0 \leq a_1 < \ldots < a_k$. We show that $\textsc{Len}(a_1, \ldots, a_k) \subseteq \textsc{Len}(b_1, \ldots, b_k)$ if and only if there exists a constant r such that $b_j = ra_j$ for all $1 \leq j \leq k$. Recognizing whether given graph belongs to $\textsc{Len}(a_1, \ldots, a_k)$ for all $k \geq 2$ and all $0 < a_1 < \ldots < a_k$ remains an open problem. We note that the problem of recognizing whether the interval count of an interval graph is equal to k is open for all fixed $k \geq 2$. Since the complexity of those recognition problems are unknown for decades, new results using approaches such as parameterized, randomized, or approximation algorithms would be of interest.

References

[1] P.C. Fishburn. *Interval Orders and Interval Graphs.* John Wiley and Sons, 1985.

[2] M. C. Francis, L. S. Medeiros, F. S. Oliveira, and J. L. Szwarcfiter. On subclasses of interval count two and on Fishburn's conjecture. *Discrete Applied Mathematics*, 323:236–251, 2022.

[3] F. Joos, C. Lowenstein, F. S. Oliveira, D. Rautenbach, and J. L. Szwarcfiter. Graphs of interval count two with a given partition. *Information Processing Letters*, 114:542–546, 2014.

[4] P. Klavík, Y. Otachi, and J. Sejnoha. On the classes of interval graphs of limited nesting and count of lengths. *Algorithmica*, 81:1490–1511, 2019.

[5] R. Leibowitz, S. F. Assmann, and G. W. Peck. The interval count of a graph. *SIAM J. Alg. Discr. Meth.*, 3(4):485–494, 1982.

[6] D. Rautenbach and J. L Szwarcfiter. Unit and single point interval graphs. *Discrete Applied Mathematics*, 160(10):1601–1609, 2012.

[7] D. Skrien. Chronological orderings of interval graphs. *Discrete Applied Mathematics*, 8:69–83, 1984.