Matemática
Contemporânea
Vol. 55, 30–38
http://doi.org/10.21711/231766362023/rmc554

On the complexity of the k-independence number and the h-diameter of a graph

Aida Abiad [1,2,3] and **Hidde Koerts** [4]

[1]Department of Mathematics and Computer Science, Eindhoven University of Technology, The Netherlands
[2]Department of Mathematics: Analysis, Logic and Discrete Mathematics, Ghent University, Belgium
[3]Department of Mathematics and Data Science, Vrije Universiteit Brussel, Belgium
[4]Department of Combinatorics and Optimization, University of Waterloo, Canada

Dedicated to Professor Jayme Szwarcfiter
on the occasion of his 80th birthday

Abstract. The k-independence number of a graph, which extends the classical independence number, is the maximum size of a set of vertices at pairwise distance greater than k. The associated decision problem is known to be NP-complete for general graphs, and it is also known to remain NP-complete for regular bipartite graphs when $k \in \{2,3,4\}$ and for planar bipartite graphs of maximum degree 3 for all $k \geq 2$. We continue this line of research by showing that the problem remains NP-complete when considering several other graph classes. Moreover, we establish a new connection between the k-independence number and the h-diameter, which is a natural generalization of the graph diameter. Finally, we use this new link to show the (parametrized) complexity of the decision problem associated to computing the h-diameter.

Keywords: complexity, k-independence number, h-diameter.

2020 Mathematics Subject Classification: 05C69, 68Q17, 68Q27.

1 Introduction

Let graph G and integer $k \geq 1$ be given. A set of vertices $S \subseteq V(G)$ is said to be *k-independent* if all vertices in S have a pairwise distance greater than k. A natural extension of Maximum Independent Set problem concerns the *Maximum k-Independent Set Problem* (for short, k-ISP), the determination of a maximum k-independent set. Hence, it is equivalent to Maximum Independent Set for $k = 1$. Moreover, the *k-independence number* of G (sometimes also called the *distance-k independence number*), denoted as $\alpha_k(G)$, is thus the cardinality of a maximum k-independent set.

The k-independence number of a graph has received a considerable amount of attention over the years. From the complexity point of view, Kong and Zhao [8] showed that for every $k \geq 2$, determining $\alpha_k(G)$ is NP-complete for general graphs, and that this problem remains NP-complete for regular bipartite graphs when $k \in \{2, 3, 4\}$ [7]. The complexity of the decision problem k-ISP has further been investigated by Eto, Guo and Miyano [5], who showed that the problem remains NP-complete for planar bipartite graphs of maximum degree 3 for all $k \geq 2$. Additionally, they proved that in chordal graphs, the problem is NP-hard for all even $k \geq 2$, but polynomial time solvable for odd $k \geq 1$. Finally, they consider the complexity of the problem when parameterised by the size of the k-independent set, and show that the problem is then W[1]-hard on chordal graphs for any even $k \geq 2$.

In this paper, we continue this line of research by showing that the k-ISP remains NP-complete on several other graph classes. We will also show that α_k is closely related to another graph parameter, the socalled h-diameter. The *h-diameter* of a graph G, denoted by $D_h(G)$, was introduced by Chung, Delorme and Solé [2] as an extension of the classical diameter, and it is defined as the largest pairwise minimum distance of a set of h vertices in G, i.e., the best possible distance of a code of size h in G. Note that $D_2(G) = D(G)$. While there exist polynomial time algorithms to calculate the diameter of a graph [1], to our knowledge, the complexity of the h-diameter (and the more general version known as $(s_1, s_2, \ldots, s_r)$-diameter) has not previously been established. Thus, in the second part we will use this connection between both parameters for showing the (parametrized) complexity of the decision problem associated to computing the h-diameter ($(s_1, s_2, \ldots, s_r)$-diameter), which we refer to by h-DP.

2 k-Independent set problem

To show NP-completeness of k-ISP, Kong and Zhao [8] provide a reduction from the regular Independent Set Problem for any fixed k. In this paper, we analyse two slightly adapted versions of the Kong and Zhao [8] reduction to prove further results. We will refer to these as *Reduction 1* and *Reduction 2*.

Reduction 1: Let a graph G be given. We construct a graph G' as follows. All edges in G are first replaced by a path of length k, where the length of a path is the number of edges it contains. Then, if k is even, we add edges between all the midpoints of these paths. Thus, these midpoints form a clique in G'. Otherwise, if k is odd, the midpoint of the paths is not well-defined. Instead, we consider the two vertices closest to the theoretical midpoint of the path. We add edges from all these points to a single additional vertex. This reduction differs only from the reduction provided by Kong and Zhao in that for the case where k is odd, we connect the two vertices closest to the midpoint of each edge replacement path with an additional vertex, whereas Kong and Zhao add a connection to only one of the vertices on each edge replacement path. The correctness analysis is highly similar and omitted here.

Reduction 2: This only applies if k is even and at least 4. Let again a graph G be given. As in Reduction 1, in the construction of G', all edges in G are replaced by a path of length k. However, instead of adding edges between all midpoints of these added paths, we connect each pair of added paths by a small gadget. For P_1 and P_2 two edge-replacing paths, we add additional vertices $v_{P_1P_2}$ and $v_{P_2P_1}$. These two vertices are then connected to the $(\frac{k}{2}-1)^{\text{th}}$ and $(\frac{k}{2}+1)^{\text{th}}$ vertices along P_1 and P_2 respectively. Finally, an edge is added between the two vertices $v_{P_1P_2}$ and $v_{P_2P_1}$. Note that we thus add $\binom{|E(G)|}{2}$ such gadgets. We again omit the correctness analysis.

Based on the structure preserving properties of the two reductions, as well as inherent structural features introduced by the reductions, we can now show that k-ISP is still NP-hard for several graph classes. We will use $d_G(u,v)$ to denote the distance between a pair of vertices u,v in a graph G.

Lemma 2.1. *k-ISP is NP-complete*

(i) for K_4-free graphs for all $k \geq 1$ odd,

(ii) and for K_3-free graphs for all $k \geq 4$ even.

Proof. (i) Note that as k-ISP is equivalent to ISP for $k=1$, and as ISP is known to be NP-complete even for K_3-free graphs [10], it suffices to consider $k \geq 3$. We will prove the result by showing that the graphs resulting from Reduction 1 are K_4-free for $k \geq 3$ odd. Let G be a graph and let G' be the graph resulting from applying Reduction 1 to G. To show that G' is K_4-free for all $k \geq 3$ odd, we consider which vertices in $V(G')$ could form a clique of size 4.

First, we consider the vertices in $V(G)$. By construction, such a vertex $v \in V(G)$ is adjacent to exactly one vertex per edge in $E(G)$ connecting to v. As these vertices are on different edge-replacing paths, and as these paths are not directly connected for k odd, the largest clique in G' containing v is of size at most 2. Next, we consider the added vertex connected to the midpoints of all edge-replacing paths, which we denote by $v_c \in V(G') \backslash V(G)$. By the construction provided in the reduction, v_c is adjacent to exactly two vertices on each edge-replacing path. As these paths are not directly connected, the largest clique in G containing v_c is of size at most 3. Finally, we consider the added vertices on the edge-replacing paths. Let $u \in V(G') \setminus V(G)$ be such a vertex. Because no vertex in $V(G) \cup \{v_c\}$ is contained in a clique of size 4, a clique must exist of only vertices on the edge-replacing paths. As no edge connects two such paths for k odd, no such clique exists. Therefore, G' is K_4-free, as desired.

(ii) We will consider the structure of the graph G' resulting from Reduction 2. As for (i), we consider the different categories of vertices. Again, it holds that for all vertices v_1, v_2 in $V(G)$, if $(v_1, v_2) \in E(G)$, it must hold that $d_{G'}(v_1, v_2) = k$. If $(v_1, v_2) \notin E(G)$, it must moreover hold that $d_{G'}(v_1, v_2) > k$. By the described construction, for v_1, v_2 in $\hat{V} = V(G') \setminus V(G)$ the set of added vertices and $k \geq 6$, the distance between v_1 and v_2 in G' is at most k. As these are the same properties that hold and are used in the correctness proof of the original reduction, we conclude that this additional reduction is valid as well.

We will consider the structure of the graph resulting from this reduction for k even. Let G be a graph and let G' be the graph resulting from applying the reduction to G. To show that G' is K_3-free for all $k \geq 4$ even, we consider which vertices in $V(G')$ could form a clique of size 3.

First, we consider the vertices in $V(G)$. By construction, such a vertex $v \in V(G)$ is directly connected to exactly a single vertex per edge in E connecting to v. As these vertices are on different edge-replacing paths, and as these paths are not directly connected in the considered reduction, the largest clique in G' containing v is of size at most 2. Next, we consider the vertices on the edge-replacing paths between vertices in V. Any such vertex $v_p \in V' \setminus V$ is either connected to two other vertices on the path, or one other vertex on the path and one vertex in V. The $(\frac{k}{2}-1)^{\text{th}}$ and $(\frac{k}{2}+1)^{\text{th}}$ vertices on the path are additionally connected to one of the added vertices connecting the edge-replacing paths. As none of these neighbors share an edge by the reduction's construction, the largest clique in G' containing v_p is of size at most 2. Finally, we consider the added vertices connecting the edge-replacing paths. Such a vertex $v_c \in V' \setminus V$ is connected to three other vertices; another vertex connecting edge-replacing paths and two non-adjacent vertices on an edge-replacing path. As thus also for v_c it holds that its neighbors do not share an edge, the largest clique in G' containing v_p is of size at most 2. There does not exist vertices in V' which can be contained in a clique of size 3. Therefore, G' is K_3-free. Then, as thus every graph resulting from the reduction for $k \geq 4$ is K_3-free, the lemma must hold. □

A graph G is *maximal clique irreducible* if every maximal clique in G contains an edge that is not contained in any other maximal clique.

Lemma 2.2. *k-ISP is NP-complete for maximal clique irreducible graphs for all $k \geq 2$.*

Proof. We will consider the structure of the path resulting from Reduction 1. Let G be a graph and let G' be the graph resulting from applying the reduction to G. To show that G' is maximal clique irreducible for all $k \geq 2$, we consider which maximal cliques contain individual edges in G'. We will consider the cases for k odd and even separately.

First, we consider the case where k is even. Then the graph G' contains two types of edges, those between two vertices in $V(G)$ along edge-replacing paths, and those connecting the midpoints of the edge-replacing paths. The edges in the edge-replacing paths and those connecting the path graph to the vertices in $V(G)$, form maximal cliques of size 2 by construction. The midpoints of all the edge-replacing paths form a maximal clique of size $|E(G)|$ containing exactly the edges of the second type. Thus, each edge is contained in only a single maximal clique in G' for k even.

Next, we consider the case where k is odd. As shown in the proof of Lemma 2.1(i), G' does not contain a clique of size 4. Trivially, an edge in a maximal clique of size 2 cannot be contained in another maximal clique. Thus, we consider only cliques of size 3. By the construction provided in the reduction, such cliques only occur between the two midpoints of an edge-replacing path and v_c, for $v_c \in V(G')\setminus V(G)$ the added vertex connected to the midpoints of all edge-replacing paths. Due to no two of the edge-replacing paths being connected by an edge, these maximal cliques cannot share an edge. Thus, each edge is contained in only a single maximal clique in G' for $k \geq 2$ odd.

Then, because each edge in $E(G')$ is contained in only a single maximal clique in G' for all $k \geq 2$, every graph resulting from the reduction for $k \geq 2$ is maximal clique irreducible. Therefore, the lemma must hold. □

Note that there exist classes of graphs such that α can be computed in polynomial time, but for which the computation of α_k for $k \geq 2$ is NP-hard. This holds, for example, for the class of regular bipartite graphs. The independence number of bipartite graphs is well known to be computable in polynomial time. For instance, using that the minimum vertex cover is the complement of the maximum independent set, and that by Kőnig's Theorem the number of vertices in the minimum vertex cover is equal to the number of edges in a maximum matching [9], we can use polynomial time algorithms for maximum matchings to compute α in polynomial time [6]. In contrast, for $k \geq 2$, Kong and Zhao show that computing α_k remains NP-hard for regular bipartite graphs [8]. Moreover, Eto, Guo and Miyano [5] show that approximating α_k to within a factor of $n^{1/2-\epsilon}$ for bipartite graphs on n vertices is NP-hard.

Observe also that for $k \geq D(G)$, where $D(G)$ is the diameter of a graph G, it trivially holds that $\alpha_k(G) = 1$. More generally, for larger k, the density of the power graph G^k increases. However, as the Independent Set Problem is not fixed-parameter tractable for general graphs [3], the k-Independent Set Problem does not allow a polynomial algorithm parameterized by the density.

3 h-Diameter

In this section we consider the complexity of h-DP. That is, the problem of deciding whether for a graph G and a fixed h and p, there exist a set of vertices $S \subseteq V(G)$ such that $|S| = h$ and the distance between vertices in S in G is at least p. Equivalently, the problem can be stated as whether $D_h(G) \geq p$ holds. The following fact will be useful in this section and it follows directly from the parameter definitions:

Lemma 3.1. *$D_h(G) \geq p$ if and only if $\alpha_{p-1}(G) \geq h$.*

Using Lemma 3.1, the complexity of h-DP follows directly from the complexity of k-ISP seen in Section 2.

Theorem 3.2. *h-DP is NP-complete.*

Moreover, we observe that the relation from Lemma 3.1 implies that h-DP is $W[1]$-hard when parameterized by h, noting that k-ISP is $W[1]$-hard when parameterized by k as shown by Eto, Guo and Miyano [5]. Next, we show that h-DP is in fact contained in this parameterized complexity class.

Theorem 3.3. *For general h, h-DP is in the $W[1]$ complexity class parameterized by p.*

Proof. To show that the decision problem associated with the computation of the h-diameter is contained in the $W[1]$ complexity class, we will provide a parameterized reduction to Weighted Circuit Satisfiability of the set of circuits having weft at most 1 and constant depth.

Let a graph G be given. To construct the circuit, we first create an input for every vertex in $V(G)$. Next, we invert all inputs. In the next layer of the circuit, an or-gate is added for each pair of inputs $v_1, v_2 \in V(G)$ with $d_G(v_1, v_2) < h$. Each such gate is then connected with the inverted inputs of the two corresponding vertices. Finally, the or-gates are connected to a single and-gate, which functions as the output. By design, this final gate is the only gate in the circuit with an input degree greater than two, satisfying the weft requirement. The construction of the circuit is illustrated in Figure 3.1.

Next, we will show that the h-diameter of G is at least ℓ if and only if the circuit is satisfiable for some assignment of weight ℓ. We observe that the circuit outputs `True` if and only if all or-gates evaluate to `True`. Moreover, such an or-gate evaluates to `True` if and only if at most one of the associated inputs is `True`. By the construction, two inputs are associated to an or-gate if and only if the distance between the associated vertices in G is smaller than h. Thus, the circuit outputs `True` if and only if inputs are set to `True` such that the associated vertices are at least at distance h of one another in G. Therefore, the circuit is satisfiable for some assignment of weight ℓ if and only if there exist ℓ vertices in G which are pairwise at a distance of at least h, which is equivalent to $D_h(G) \geq \ell$.

It remains to show that the reduction is a valid parameterized reduction. For the translation of the graph to the corresponding circuit, the all-pairs shortest paths can be computed in $O(|V(G)|^3)$ with the Floyd-Warshall algorithm. As the subsequent construction of the circuit can be completed in $O(|V(G)|^2)$, the reduction is of polynomial complexity only dependent on the number of vertices in the graph. Then, as by the formulation of the reduction the parameter p

associated to the h-diameter decision problem instance is equal to the weight of the corresponding instance of Weighted Circuit Satisfiability, the reduction is a valid parameterized reduction.

As we have thus provided a parameterized reduction from the decision problem associated with the computation of the h-diameter to Weighted Circuit Satisfiability of the set of circuits with weft at most 1, we conclude that said decision problem parameterized by p is contained in the $W[1]$-hardness class. □

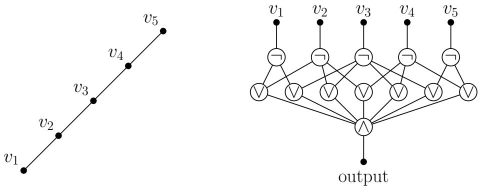

Figure 3.1: Illustration of the reduction of decision problem associated to the computation of the h-diameter to Weighted Circuit Satisfiability as used in the proof of Proposition 3.3. On the left, the graph P_5 is given, and on the right the corresponding circuit for $h = 3$.

Remark 3.4. We take the opportunity to correct an error on [2, Section 4.1], and in particular on the first bound for the graph girth $g(G)$ in terms of the h-diameter: $g(G) \leq h{\cdot}D_h(G){+}1$. Such a bound is clearly violated by a 10-cycle and $h = 4$, and several other graph families. However, it can easily be fixed by taking a minimal cycle and choose $\lfloor \frac{g(G)}{h} \rfloor$ vertices on it that have distance at least k from one another. This being lower than $D_h(G)$ is equivalent to $g(G) \leq h{\cdot}D_h(G){+}h{-}1$ (provided that $h < g(G)$, otherwise the bound is trivial).

Fiol and Garriga [[4], Section 2.4] proposed a generalization of the h-diameter. The $(s_1, s_2, \ldots, s_r)$-*diameter* of a graph G is the maximum distance between r vertex subsets $V_1, V_2, \ldots V_r, \subseteq V(G)$ of sizes $s_1, s_2, \ldots, s_r$ respectively. Note that for $s_i = 1$, for $1 \leq i \leq r$, the $(s_1, s_2, \ldots, s_r)$-diameter equals the r-diameter. Observing that for $s_i = 1$, for $1 \leq i \leq r$, the $(s_1, s_2, \ldots, s_r)$-diameter equals the r-diameter, we obtain the following complexity result.

Theorem 3.5. *The decision problem associated to the computation of the* $(s_1, s_2, \ldots, s_r)$*-diameter is NP-hard.*

Acknowledgements

The research of Aida Abiad is partially supported by the FWO grant 1285921N. The authors are thankful to Tom van der Zanden and Robin Simoens for useful discussions on the h-diameter section.

References

[1] F.R.K. Chung. Diameters of Graphs: Old Problems and New Results. *Congressus Numerantium*, 60:295–317, 1987.

[2] F.R.K. Chung, C. Delorme, and P. Solé. Multidiameters and Multiplicities. *European Journal of Combinatorics*, 20:629–640, 1999.

[3] K. Dabrowski, V. Lozin, H. Müller, and D. Rautenbach. Parameterized Algorithms for the Independent Set Problem in Some Hereditary Graph Classes. In Costas S Iliopoulos and William F Smyth, editors, *Combinatorial Algorithms*, pages 1–9, Berlin, Heidelberg, 2011. Springer Berlin Heidelberg.

[4] M.A. Fiol and E. Garriga. The alternating and adjacency polynomials, and their relation with the spectra and diameters of graphs. *Discrete Applied Mathematics*, 87:77–97, 1998.

[5] E. Hiroshi, F. Guo, and E. Miyano. Distance-d independent set problems for bipartite and chordal graphs. *Journal of Combinatorial Optimization*, 27(1):88–99, Jan 2014.

[6] J.E. Hopcroft and R.M. Karp. An n^(5/2) algorithm for maximum matchings in bipartite graphs. *SIAM Journal on Computing*, 2(4):225–231, 1973.

[7] M.C. Kong and Y. Zhao. Computing k-Independent sets for regular bipartite graphs. *Congressus numerantium*, 143:65–80, 2000.

[8] M.C. Kong and Yijun Zhao. On computing maximum k-independent sets. *Congressus numerantium*, 95:47–47, 1993.

[9] D. König. Gráfok és mátrixok. *Matematikai és Fizikai Lapok*, 38:116–119, 1931.

[10] Svatopluk Poljak. A note on stable sets and colorings of graphs. *Commentationes Mathematicae Universitatis Carolinae*, 15(2):307–309, 1974.