Matemática
Contemporânea
Vol. 55, 20–29
http://doi.org/10.21711/231766362023/rmc553

A_α-spectral properties of some families of graphs with vertex connectivity equal to 1

Victor M. X. Leal[1], **Carla S. Oliveira**[1] and **André E. Brondani**[2]

[1]Escola Nacional de Ciências Estatísticas, Rua André Cavalcanti 106 , Rio de Janeiro, Brazil

[2]Departamento de Matemática, Instituto de Ciências Exatas, Universidade Federal Fluminense, Volta Redonda, RJ, Brazil

Dedicated to Professor Jayme Szwarcfiter on the occasion of his 80th birthday

Abstract. Let G be a graph with adjacency matrix $A(G)$ and let $D(G)$ be the diagonal matrix of the degrees of G. For every real $\alpha \in [0,1]$, Nikiforov [*Applicable Analysis and Discrete Mathematics,* 11(1): 81-107, 2017] defined the matrix $A_\alpha(G)$ by $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$. In this paper, we obtain the eigenvalues of some families of graphs which have vertex connectivity equals to 1.

Keywords: eigenvalues, $A_\alpha-$matrix.

2020 Mathematics Subject Classification: 12A34, 67B89.

1 Introduction

Let $G = (V(G), E(G))$ be a simple graph of order n with vertex set $V(G)$ and edge set $E(G)$. If $\{v_i, v_j\} \in E(G)$, v_i and v_j are called

The second author is partially supported by CNPq Grant 304548/2020-0, e-mail:carla.oliveira@ibge.gov.br adjacency vertices and denoted by $v_i \sim v_j$. Otherwise, we denote by $v_i \not\sim v_j$. The set of *neighbours* of a vertex v in G is denoted by $N_G(v)$ and $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of a vertex v of G, $d(v)$, is defined by $|N_G(v)|$. Two distinct vertices u and v are called *true twins* if $N_G[u] = N_G[v]$ and are called *false twins* if $N_G(u) = N_G(v)$ and u is not adjacent to v. A graph G is called r*-regular* if each vertex of G has degree r. We denote the complete graph, the path and the cycle with n vertices by K_n, P_n and C_n, respectively. The *join* $G \simeq G_1 \vee G_2$ of the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, where $V_1 \cap V_2 = \emptyset$ is the graph which is the union of G_1 and G_2 together with all the edges joining the elements of V_1 and V_2. The *vertex connectivity* of a graph G, $k(G)$, is the minimum size of a vertex subset $S \subseteq V(G)$ such that $G - S$ is disconnected or has only one vertex. The *adjacency matrix* of G, $A(G) = [a_{ij}]$, is defined by $a_{ij} = 1$ if v_i is adjacent to v_j and $a_{ij} = 0$ otherwise. The *matrix of degrees* of G, $D(G) = [d_{ij}]$, is defined by $d_{ii} = d(v_i)$ and $d_{ij} = 0$, $\forall i \neq j$. The *signless Laplacian matrix* is defined by $Q(G) = D(G) + A(G)$. In 2017, Nikiforov [5] defined for any real $\alpha \in [0, 1]$, the convex linear combinations $A_\alpha(G)$ of $A(G)$ and $D(G)$ by $A_\alpha(G) = \alpha D(G) + (1 - \alpha)A(G)$. It is easy to see that $A(G) = A_0(G)$, $D(G) = A_1(G)$ and $Q(G) = 2A_{\frac{1}{2}}(G)$. The A_α-*characteristic polynomial* of G is defined by $P_{A_\alpha(G)}(x) = \det(xI - A_\alpha(G))$ and its roots are called the α*-eigenvalues* of G. As usual, we shall index the eigenvalues of $A_\alpha(G)$ in a non-increasing order and denote them as $\lambda_1(A_\alpha(G)) \geq \lambda_2(A_\alpha(G)) \geq \cdots \geq \lambda_n(A_\alpha(G))$. The A_α-spectrum is the multi-set of the eigenvalues of $A_\alpha(G)$ denoted by $Spec(A_\alpha(G)) = \left\{\lambda_1(A_\alpha(G))^{[m(\lambda_1)]}, \lambda_2(A_\alpha(G))^{[m(\lambda_2)]}, \ldots, \lambda_r(A_\alpha(G))^{[m(\lambda_r)]}\right\}$, where $m(\lambda_j)$ is the *multiplicity* of the eigenvalue $\lambda_j(A_\alpha(G))$, for $1 \leq j \leq r$. For simplicity, we use notations A_α and $\lambda_i(A_\alpha)$ when there is no risk of ambiguity. Given a square matrix M, the classical adjoint matrix of M and the trace of M are denoted by $\operatorname{adj}(M)$ and $\operatorname{tr}(M)$, respectively. We denote the $m \times n$ all-ones matrix by $J_{m\times n}$, the all-zeros matrix by $0_{m\times n}$, and the $m \times m$ identity matrix by I_m. In particular, we denote J_m and 0_m when $m = n$. The signless Laplacian matrix, $Q(G)$, is positive semidefinite, but this is not true for A_α if α is sufficiently small. In 2017, Nikiforov [5] proved that if $\alpha \geq \frac{1}{2}$ then A_α is positive semidefinite and if $\alpha > \frac{1}{2}$ and G has no isolated vertices then A_α is positive definite. Also in 2017, Nikiforov and Rojo [6] defined $\alpha_0(G)$ as the smallest value in the interval $[0, 1]$ such that $\lambda_n(A_{\alpha_0}) \geq 0$. Then A_α is positive semidefinite if and only if $\alpha_0(G) \leq \alpha \leq 1$. In the same paper, the authors raised the following problem: "Given a graph G, find $\alpha_0(G)$" and solved this problema when G is d-regular, r-colorable and when G contains bipartite components. In 2022, Brondani et al. [2] also solved this problem for some families of graphs that contain cliques. Moreover, there are some works the explicit the eigenvalues of A_α of some classes of graphs. For more details, we suggest [1, 2] and references therein. In this paper, we solve the problem present above and explicit the eigenvalues of some families of graphs with vertex connectivity equal to 1. The paper is organized such that the preliminary results are presented in the next section and the main results are in the Section 3.

2 Preliminaries

In this section we fix some notations and review some important results for the development of the next section.

Proposition 2.1 ([4]). *Let M be a matrix of the form $\begin{bmatrix} A & x \\ y^T & a \end{bmatrix}$, where $a \in \mathbb{R}$, $x, y \in \mathbb{R}^n$ and A is a square matrix of order $(n-1)$. Then* $\det \begin{bmatrix} A & x \\ y^T & a \end{bmatrix} = a \det(A) - y^{\mathrm{T}}(\mathrm{adj(A)})x$.

Proposition 2.2 ([4]). *Let M be a symmetric matrix of order n defined by* $M = \begin{bmatrix} M_{1,1} & M_{1,2} & \cdots & M_{1,k} \\ M_{2,1} & M_{2,2} & \cdots & M_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ M_{k,1} & M_{k,2} & \cdots & M_{k,k} \end{bmatrix}$, *where $M_{i,j}$, $1 \leq i, j \leq k$, is a submatrix of order $n_i \times n_j$ such that the sum of each of its rows is equal* *to $c_{i,j}$. If $\overline{M} = [c_{i,j}]_{k\times k}$, then the eigenvalues of $\overline{M}$ are also eigenvalues of M.*

Let $A = [a_{ij}]$ be a $m \times n$ matrix and $B = [b_{ij}]$ be a $p \times q$ matrix. The *Kronecker product* of A and B, denoted by $A \otimes B$, is the $mp \times nq$ matrix obtained by replacing each entry a_{ij} of A by $a_{ij}B$.

Proposition 2.3 ([4]). *Let A and B be squares matrices of order n and m, respectively. If $\mathrm{Spec}(A) = \{\lambda_i : 1 \le i \le n\}$ and $\mathrm{Spec}(B) = \{\mu_j : 1 \le j \le m\}$, then $\mathrm{Spec}(A \otimes B) = \{\lambda_i\mu_j : 1 \le i \le n$ and $1 \le j \le m\}$.*

Proposition 2.4 ([2]). *Let G be a graph on $n \ge 2$ vertices with twin vertices v_i and v_{j_p}, for some $1 \le p \le r < n$.*

(i) If v_i and v_{j_p} are false twins, then $\lambda = \alpha d(v_i) \in Spec(A_\alpha(G))$.

(ii) If v_i and v_{j_p} are true twins, then $\alpha(d(v_i) + 1) - 1 \in Spec(A_\alpha(G))$.

In both cases, the multiplicity of the eigenvalue is at least r.

Proposition 2.5 ([3]). *Let K_t be a complete graph on t vertices. Then, $Spec(A(K_t)) = \left\{(t-1), (-1)^{[t-1]}\right\}$.*

Proposition 2.6 ([5]). *Let G_1 and G_2 be graphs on n_1 and n_2 vertices, respectively. If G_1 is an r_1-regular graph, G_2 is an r_2-regular graph then, for $\alpha \in [0, 1]$, the largest and smallest eigenvalues of $A_\alpha(G_1 \vee G_2)$ are the eigenvalues of the matrix $\begin{pmatrix} r_1 + \alpha n_2 & (1-\alpha)^2 n_1 n_2 \\ 1 & r_2 + \alpha n_1 \end{pmatrix}$. The others $n_1 + n_2 - 2$ eigenvalues of $A_\alpha(G_1 \vee G_2)$ are $\alpha r_1 + (1-\alpha)\lambda_i(A(G_1))$ and $\alpha r_2 + (1-\alpha)\lambda_j(A(G_2))$, where $2 \le i \le n_1$ and $2 \le j \le n_2$.*

Proposition 2.7 ([6]). *A connected graph G is bipartite if and only if $\alpha_0 = \frac{1}{2}$.*

Now, we present the definitions the families of graphs studied in this paper.

Definition 2.8. Given the integers $r, t \ge 1$, let $L_{r,t}$ be the graph obtained from join of K_1 and r copies of the K_t, that is, $L_{r,t} \simeq K_1 \vee rK_t$.

Definition 2.9. Given the integers $a_1 \geq a_2 \geq \cdots \geq a_k \geq 3$ and $k \geq 2$, let $F_{a_1,a_2,\ldots,a_k}$ be a graph that consists of k cycles of order $a_1, a_2, \ldots, a_k$, respectively, all of them sharing a single vertex.

Definition 2.10. Given the integers $n \geq 3$ and $k \geq 2$, the graph $G_k(C_n)$ consists of k cycles $C_n^1, \ldots, C_n^k$ of the same size n and an extra vertex, s, adjacent to exactly one vertex of each cycle, that is, $V(G_k(C_n)) = (\bigcup_{i=1}^{k} V(C_n^i)) \cup \{s\}$ and $E(G_k(C_n)) = (\bigcup_{i=1}^{k} E(C_n^i)) \cup (\bigcup_{i=1}^{k} \{u_{i,n}, s\})$, where $u_{i,n} \in V(C_n^i)$ for each $1 \leq i \leq k$.

Figure 2.1 shows graphs corresponding to definitions 2.8, 2.9 and 2.10.

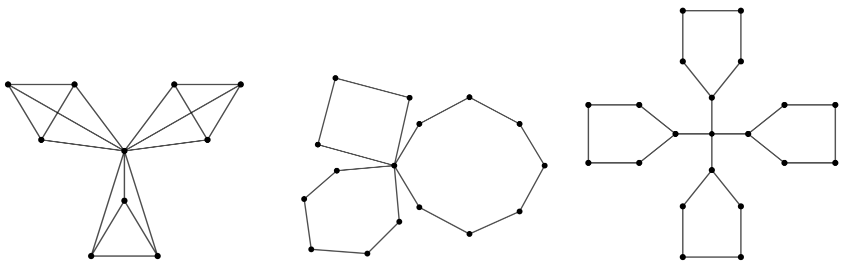

Figure 2.1: Graphs $L_{3,3}$, $F_{4,6,8}$ and $G_4(C_5)$.

3 Main Results

In this section, we present the results involving the eigenvalues of A_α matrix for families of graphs defined in Section 2 and we determine the smallest value of α for which the matrix A_α is positive semidefinite.

Proposition 3.1. *Let r and t be positive integers. If $G \simeq L_{r,t}$ and $\alpha \in [0,1]$, then $P_{A_\alpha(G)}(x) = f(x)(x-(t-1))^{r-1}(x-(\alpha t-1))^{r(t-1)}$, where $f(x) = x^2 + (1 - rt\alpha - t - \alpha)x + (rt\alpha + rt^2\alpha - rt)$.*

Proof. By definition $G \simeq L_{r,t} \simeq K_1 \vee rK_t$. From Proposition 2.6, the largest and smallest eigenvalues of $A_\alpha(G)$ are the roots of polynomial

$f(x) = x^2 + (1 - rt\alpha - t - \alpha)x + (rt\alpha + rt^2\alpha - rt)$. Moreover, from Proposition 2.5, the others eigenvalues are $t-1$ and $\alpha t - 1$ with multiplicities $r-1$ and $r(t-1)$, respectively, and the result follows. $\square$

Corollary 3.2 determines the smallest α for which $A_\alpha(L_{r,t})$ is positive semidefinite.

Corollary 3.2. *Let $G \simeq L_{r,t}$ with $r, t \geq 1$. The matrix $A_\alpha(L_{r,t})$ is positive semidefinite if and only if $\alpha \in [\alpha_0, 1)$, where $\alpha_0 = \frac{1}{t+1}$.*

Proof. From Proposition 3.1 the smallest eigenvalue of $A_\alpha(L_{r,t})$ is $\lambda = \frac{-1+rt\alpha+t+\alpha-\sqrt{\Delta}}{2}$, where $\Delta = (1 - rt\alpha - t - \alpha)^2 - 4(rt\alpha + rt^2\alpha - rt)$. Therefore, $\lambda \geq 0$ if and only if $\alpha \geq \frac{1}{1+t}$, which proves the result. $\square$

Remark 3.3. If the order of cycles, $a_1, a_2, \ldots, a_k$, are even the graphs $F_{a_1,a_2,\ldots,a_k}$ are bipartite and consequently from Proposition 2.7 the matrices $A_\alpha(F_{a_1,a_2,\ldots,a_k})$ are positive semidefinite for $\alpha \geq \frac{1}{2}$. For the other cases, the problem of finding $\alpha_0(G)$ is open.

Now, we present the results for graphs of the family of graphs $F_{4,4,\ldots,4} \simeq F_{4[q]}$ present in Figure 3.1 with the labeling of its vertices.

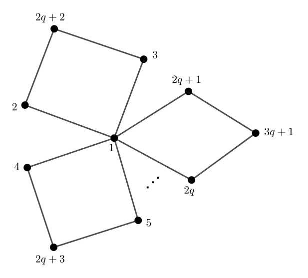

Figure 3.1: Family $F_{4[q]}$.

From this labeling of vertices, the matrix $A_\alpha(F_{4[q]})$ can be written as

$$A_\alpha(F_{4[q]}) = \begin{bmatrix} 2q\alpha & (1-\alpha)J_{1\times 2q} & \mathbf{0}_{1\times \mathrm{q}} \\ (1-\alpha)J_{2q\times 1} & 2\alpha I_{2q} & B_{2q\times q} \\ \mathbf{0}_{q\times 1} & (B_{2q\times q})^T & 2\alpha I_q \end{bmatrix}, \tag{3.1}$$

where q is the number of cycles C_4 and $B = (1-\alpha)I_q \otimes J_{2\times 1}$

Lemma 3.4. *If $G \simeq F_{4[q]}$, $q \geq 1$, and $\alpha \in [0,1]$, then $t_1 = (2-\sqrt{2})\alpha + \sqrt{2}$ and $t_2 = (2+\sqrt{2})\alpha - \sqrt{2}$ are eigenvalues of $A_\alpha(G)$, both with multiplicity at least $q-1$.*

Proof. We take the vector $\mathbf{y} = [y_1, \dots, y_q] \in \mathbb{R}^q$, such that $\sum_{k=1}^{q} \mathbf{y}_k = 0$ and we consider the vectors, in $\mathbb{R}^{3q+1}$, $\mathbf{x} = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}}\mathbf{y} \otimes J_{2\times 1} & \mathbf{y} \end{bmatrix}^T$ and $\mathbf{z} = \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}}\mathbf{y} \otimes J_{2\times 1} & \mathbf{y} \end{bmatrix}^T$. Note that

$$\begin{aligned} A_\alpha(F_{4[q]})\mathbf{x} &= \begin{bmatrix} 0 & \frac{2\alpha}{\sqrt{2}}\mathbf{y} \otimes J_{2\times 1} + B_{2q\times q}\mathbf{y} & (B_{2q\times q})^T\left(\frac{1}{\sqrt{2}}\mathbf{y} \otimes J_{2\times 1}\right) + 2\alpha\mathbf{y} \end{bmatrix}^T \\ &= \begin{bmatrix} 0 & (\sqrt{2}\alpha + 1 - \alpha)(\mathbf{y} \otimes J_{2\times 1}) & (\sqrt{2} - \sqrt{2}\alpha + 2\alpha)\mathbf{y} \end{bmatrix}^T \\ &= \begin{bmatrix} (2-\sqrt{2})\alpha + \sqrt{2} \end{bmatrix} \mathbf{x}. \end{aligned}$$

Analogously we can show that $\mathbf{z}$ is an eigenvector of $A_\alpha(G)$ associated with the eigenvalue $t_2 = (2+\sqrt{2})\alpha - \sqrt{2}$. Now, it is simple to verify that the eigenspace associated with t_1 (respectively, t_2) has dimension $q-1$ because the dimension of $\left\{(y_1, y_2, \dots, y_q) \in \mathbb{R}^q; \sum_{k=1}^{q} y_k = 0\right\}$ is equal to $q-1$ and the result follows. ☐

Proposition 3.5. *If $G \simeq F_{4[q]}$, $q \geq 1$, and $\alpha \in [0,1]$, then $P_{A_\alpha(G)}(x) = (x-2\alpha)^q[x^2 - 4\alpha x + 2(\alpha^2 + 2\alpha - 1)]^{q-1}g(x)$, where $g(x) = x^3 - 2\alpha(q+2)x^2 + 2[2\alpha^2 q + (\alpha^2 + 2\alpha - 1)(q+1)]x - 8\alpha(2\alpha - 1)q$.*

Proof. Let $G \simeq F_{4[q]}$. Applying the Propositions 2.4 for each cycle of G, we obtain that 2α is an eigenvalue of $A_\alpha(G)$ with multiplicity at least q. Applying the Proposition 2.2 in (3.1), the spectrum of matrix $M = \begin{bmatrix} 2q\alpha & 2q(1-\alpha) & 0 \\ 1-\alpha & 2\alpha & 1-\alpha \\ 0 & 2(1-\alpha) & 2\alpha \end{bmatrix}$, whose characteristic polynomial is $g(x)$, is contained in the spectrum of $A_\alpha(G)$. The other eigenvalues of $A_\alpha(G)$, given in Lemma 3.4, are roots of the polynomial $h(x) = x^2 - 4\alpha x + 2(\alpha^2 + 2\alpha - 1)$ and the result follows. ☐

Remark 3.6. If the order of cycle, C_n, is even the graphs $G_q(C_n)$ are bipartite and consequently from Proposition 2.7 the matrices $A_\alpha(G_q(C_n))$ are positive semidefinite for $\alpha \geq \frac{1}{2}$. For the other cases, the problem of finding $\alpha_0(G)$ is open.

Figure 3.2 shows the family of graphs $G_q(C_4)$.

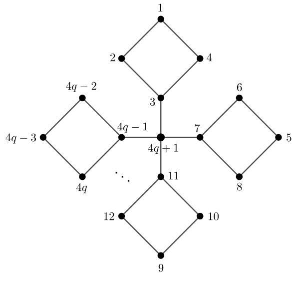

Figure 3.2: Family $G_q(C_4)$.

Proposition 3.7. *If $G \simeq G_q(C_4)$, then $P_{A_\alpha(G)}(x)$ is given by*

$$(x-2\alpha)^q f(x)^{q-1}\left[(x-\alpha q)f(x) - q(1-\alpha)^2(x^2-4\alpha x+2\alpha^2+4\alpha-2)\right],$$

where $f(x) = x^3 - 7\alpha x^2 + 4(\alpha+1)(3\alpha-1)x - 2\alpha\left(\alpha^2+10\alpha-5\right)$.

Proof. Let $G \simeq G_q(C_4)$ be the graph whose vertices are rotulated according to the Figure 3.2. So $A_\alpha(G)$ can be written the following way

$$A_\alpha(G) = \begin{bmatrix} A_\alpha(C_4)+\alpha D_3 & \mathbf{0} & \mathbf{0} & (1-\alpha)Z \\ \mathbf{0} & \ddots & \mathbf{0} & \vdots \\ \mathbf{0} & \mathbf{0} & A_\alpha(C_4)+\alpha D_3 & (1-\alpha)Z \\ (1-\alpha)Z^T & \dots & (1-\alpha)Z^T & \alpha q \end{bmatrix},$$

where $D_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $Z^T = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$. So the matrix $A_\alpha(G)$ can be written as

$$A_\alpha(G) = \begin{bmatrix} B & Y \\ Y^T & \alpha q \end{bmatrix},$$

where $B = I_q \otimes [A_\alpha(C_4) + \alpha D_3]$ and $Y^{\mathrm{T}} = (1-\alpha)\begin{bmatrix} Z^T & Z^T & \dots & Z^T \end{bmatrix} \in \mathbb{R}^{4q}$. From Proposition 2.1, we can write $P_{A_\alpha(G)}$ as

$$P_{A_\alpha(G)}(x) = (x - \alpha q)P_B(x) - Y^{\mathrm{T}}\,\mathrm{adj}(xI - B)Y.$$

From Proposition 2.3, $P_B(x) = (x-2\alpha)^q f(x)^q$, where $f(x) = x^3 - 7\alpha x^2 + 4(\alpha+1)(3\alpha-1)x - 2\alpha(\alpha^2+10\alpha-5)$. Since $\mathrm{adj}(xI-B) = P_B(x)(xI-B)^{-1}$, we have

$$\begin{aligned} P_{A_\alpha(G)}(x) &= (x-\alpha q)P_B(x) - Y^T P_B(x)(xI-B)^{-1}Y \\ &= P_B(x)[x - \alpha q - Y^T(xI-B)^{-1}Y]. \end{aligned}$$

As B is a block diagonal matrix the form $A_\alpha(C_4) + \alpha D_3$ we have

$$\begin{aligned} Y^{\mathrm{T}}(xI-B)^{-1}Y &= qZ^{\mathrm{T}}(xI-F)^{-1}Z \\ &= \frac{q(x^2 - 4\alpha x + 2\alpha^2 + 4\alpha - 2)}{f(x)}, \end{aligned}$$

where $F = A_\alpha(C_4) + \alpha D_3$. So the result follows. □

Acknowledgements

The research of C. S. Oliveira is supported by CNPq Grant 304548/2020− 0.

References

[1] A. E. Brondani, F. A. M. França, and C. S. Oliveira. Positive semidefiniteness of $A_\alpha(G)$ on some families of graphs. *Discrete Applied Mathematics*, 323:113–123, 2022.

[2] A. E. Brondani, F. A. M. França, C. S. Oliveira, and L. Lima. A_α-spectrum of a firefly graphs. *Electronic Notes in Theoretical Computer Science*, 346:209–219, 2019.

[3] D. Cvetković, P. Rowlinson, and S. Simić. *An Introduction to the Theory of Graph Spectra*. London Mathematical Society Student Texts. Cambridge University Press, Cambridge, 2010.

[4] R. A. Horn and C. R. Johnson. *Matrix Analysis.* Cambridge University Press, New York, 1992.

[5] V. Nikiforov. Merging the A- and Q-Spectral Theories. *Applicable Analysis and Discrete Mathematics*, 11(1):81–107, 2017.

[6] V. Nikiforov and O. Rojo. A note on the positive semidefiniteness of $A_\alpha(G)$. *Linear Algebra and its Applications*, 519:156–163, 2017.