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Locally identifying coloring in some chordal graphs

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Dedicated to Professor Jayme Szwarcfiter on the occasion of his 80th birthday

Abstract. A locally identifying coloring (or lid-coloring for short) in a graph is a proper vertex coloring such that, for any edge uv, if u and v have distinct closed neighborhoods, then the set of colors used on vertices of the closed neighborhoods of u and v are distinct. The lid-chromatic number of a graph G, denoted by $\chi_{lid}(G)$, is the minimum number of colors needed in any lid-coloring of G. In this work, we determine the lid-chromatic number of subclasses of both powers of paths and some split graphs, which are chordal graphs. Additionally, we present a lower bound for the lid-chromatic number in twin-free graphs.

Keywords: Lid-coloring, chordal graphs, powers of paths, split graphs.

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1 Introduction

We consider finite, simple, and undirected graphs and use standard notation and terminology. For a graph G, the vertex set and the edge set are denoted $V(G)$ and $E(G)$, respectively. For any vertex u, we denote by $N(u)$ its open neighborhood and by $N[u]$ its closed neighborhood. For a subset $S \subseteq V(G)$, we denote $N[S]$ the set $\bigcup_{u \in S} N[u]$. Two vertices in a graph G are true twins if $N[u] = N[v]$ (although they are often called true twins in the literature, we call them twins for convenience). If G has no twins, then G is twin-free.

The *vertex coloring problem* consists of assigning colors to the vertices of a graph in such a way that adjacent vertices have different colors, and such coloring is a *proper coloring*. The minimum number of colors needed to color a graph G is called its *chromatic number*, denoted by $\chi(G)$. The function $c : V \to \mathbb{N}$ is a vertex coloring of G. For any $S \subseteq V$, we define $c(S)$ as the set of colors that appear on the vertices in S.

A vertex coloring c of a graph G is a *locally identifying coloring* (lid-coloring for short) if it satisfies the following conditions: (i) c is a proper coloring of G, that is, no two adjacent vertices have the same color, and (ii) for each pair of adjacent vertices u, v with $N[u] \neq N[v]$, we have $c(N[u]) \neq c(N[v])$. The locally identifying chromatic number of graph G, denoted by $\chi_{lid}(G)$, is the smallest number of colors needed in any lid-coloring of G. A graph G is k-lid-colorable if it admits a locally identifying coloring using at most k colors. The lid-coloring of a graph is a combination between the concept of graph coloring and identifying codes [6]. The lid-coloring was introduced in 2010 [3], where several bounds on $\chi_{lid}(G)$ were proposed for different families of graphs, including planar graphs, some subclasses of perfect graphs, and graphs with bounded maximum degree. It was shown that every bipartite graph G has $\chi_{lid}(G) \leq 4$. Moreover, it was proved that deciding whether a bipartite graph is 3-lid-colorable is an NP-complete problem, while it is possible to decide in linear time whether a tree is 3-lid-colorable. Note that the lid-coloring is not hereditary. For instance, $\chi_{lid}(P_n) = 3$ for odd n at least 3, and $\chi_{lid}(P_n) = 4$ for even n.

Foucaud *et al.* [4] showed that every graph G has a lid-coloring with $2\Delta^2 - 3\Delta + 3$ colors, where $\Delta \geq 3$ is the maximum degree of G. Gonçalves *et al.* [5] proved that for any planar graph G, it holds that $\chi_{lid}(G) \leq 1280$, answering a question posed in [3]. Martins and Sampaio [7] developed linear-time algorithms to calculate the lid-chromatic number for certain classes of graphs with few P_4's, such as cographs, P_4-sparse graphs, and $(q, q-4)$-graphs, and showed that the lid-chromatic number is polynomially inapproximable by a factor of $O(n^{1-\epsilon})$ for all $\epsilon > 0$, unless $P = NP$.

A *chordal graph* is one in which all cycles of four or more vertices have a chord, which is an edge that is not part of the cycle but connects two vertices of the cycle. Esperet *et al.* [3] conjectured that every chordal graph G has $\chi_{lid} \leq 2\chi(G)$. The authors presented some graphs satisfying this bound. In this work, we determined the exact values of the lid-chromatic number for subclasses of both powers of paths and split graphs, which are chordal graphs. Additionally, we presented a lower bound for the lid-chromatic number in twin-free graphs. Before we present our results, we present some helpful definitions.

A *clique* in a graph G is a set of vertices pairwise adjacent in G. The size of the largest clique in a graph G is denoted by $\omega(G)$. A clique K is *maximum* in G when $|K| = \omega(G)$. We use $[n]$ to denote the set $\{1, \ldots, n\}$ and $[n, m]$ to denote the set $[m] \setminus [n-1]$. The symmetric difference between sets A and B is denoted by $A \Delta B$. We denote as P_n and K_n the path graph and the complete graph, respectively, on n vertices. For basic theoretical terms not defined in this article, see [1].

2 Results

In this section, we present our results. First, we present lower bounds on the lid-coloring chromatic number for some graphs.

Proposition 2.1. *Let G be a twin-free graph containing a clique K of size $k \geq 2$. Then $c(N[K]) \geq k + \log_2 k$.*

Proof. Let G be a twin-free graph containing a clique K of size $k \geq 2$ and $K = \{v_1, \ldots, v_k\}$. Suppose that c is a lid-coloring of G. Without loss of generality, suppose that $c(v_i) = i$ for $i \in [k]$. Let $S = N[K] \setminus K$. Since G is a twin-free graph, we claim that $|S| \geq \log_2 k$; otherwise, $2^{|S|} < |K|$ and since every $N[v_i] \setminus K$ is a subset of S, this implies that at least two vertices in K are twins, contradicting the initial hypotheses. We show that $|c(S) \setminus [k]| \geq \log_2 k$. For a contradiction, suppose that $|c(S) \setminus [k]| < \log_2 k$. We now that $|\mathcal{P}(c(S))| = 2^{|c(S)|} \leq 2^{\lceil \log_2 k \rceil - 1} \leq k - 1$. Since $k - 1 \geq 1$, this implies that there exist two vertices, say u and v, in K such that $c(N[u]) = c(N[v])$. Since u and v are not twins, c is not a lid-coloring of G. Therefore, $c(N[K]) \geq k + \log_2 k$. □

Corollary 2.2. *Let G be a twin-free graph with $\omega(G) = k \geq 2$. Then $\chi_{lid}(G) \geq k + \log_2 k$.*

Observe that the bound presented in Corollary 2.2 is tight. Bipartite graphs with at least three vertices have lid-coloring at least 3. In Proposition 2.3, we improve this bound under some conditions.

Proposition 2.3. *Let G be a graph that contains a clique K of size k. If, for every vertex $v \in K$, v is adjacent to only one vertex u that does not belong to K, and u is not adjacent to any other vertex in K, then we have $\chi_{lid}(G) \geq 2k - 1$.*

Proof. Consider $|K| = k$, where the vertices of K are denoted by v_i with $1 \leq i \leq k$. According to the hypothesis, each vertex v_i in K has a neighbor u_i, with $1 \leq i \leq k$, that does not belong to K and is not adjacent to any other vertex in K. Let $S = \bigcup_{i=1}^{k} u_i$. Now, consider a coloring c of G in which, for $1 \leq i \leq k$, $c(v_i) = i$. We will show that to color the vertices of S, we need at least $k - 1$ colors that differ from those in $[k]$. Suppose that at least two vertices in S are colored with two colors from $[k]$. Without loss of generality, let $c(u_1) = 2$ and $c(u_2) = 1$, which implies that $c(N[v_1]) = c(N[v_2]) = [k]$, which is a contradiction. Therefore, we conclude that $\chi_{lid}(G) \geq 2k - 1$. □

In the following subsections, we present some results by Esperet *et al.* [3] on chordal graphs, particularly k-trees, and then discuss the locally identifying coloring when restricted to powers of paths, which is a special case of k-trees. We determined the exact values of the lid-chromatic number for some powers of paths, as well as for the complete split graph with $|S| \geq 2$ and for the split graph $G = K_n \circ K_1$.

2.1 Lid-coloring in powers of paths P_n^k

A *k-tree* is a graph whose vertices can be ordered $v_1, v_2, \ldots, v_n$ in such a way that the vertices v_1 to v_{k+1} induce a $(k+1)$-clique, and for each $k+2 \leq i \leq n$, the neighbors of v_i in $\{v_j \mid j < i\}$ induce a k-clique. By definition, for all $k+1 \leq i \leq n$, the graph G_i induced by $\{v_j \mid j \leq i\}$ is a k-tree, and every k-clique in a k-tree is contained in a $(k+1)$-clique. In Figure 2.1, we present the construction of a 3-tree.

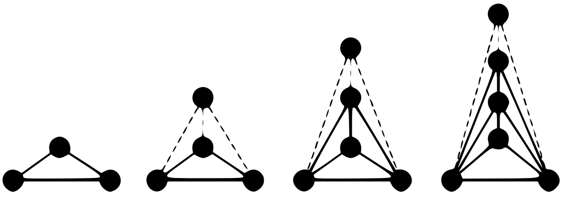

Figure 2.1: Example of construction of a 3-tree.

Theorem 2.4. *[3] If G is a k-tree, then $\chi_{lid}(G) \leq 2k+2$.*

If G is a k-tree with at least $k+1$ vertices, then $\chi(G) = k+1$ [2]. By Theorem 2.4, we have $\chi_{lid}(G) \leq 2k+2$. Thus, $\chi_{lid}(G) \leq 2\chi(G)$, which implies that k-trees satisfy the conjecture proposed by Esperet *et al.* [3].

A *power of path*, denoted by P_n^k, is a graph where $V(P_n^k) = \{v_1, \ldots, v_n\}$ and there exists an edge v_iv_j if and only if $|i-j| \leq k$, $1 \leq i, j \leq n$. In Figure 2.2, we present a power of path P_7^3.

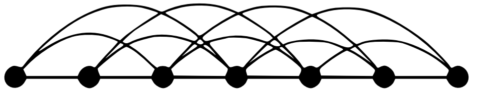

Figure 2.2: Power of path P_7^3.

The graph P_n^k is a k-tree, and can be constructed starting from a clique formed by the vertices $v_1, \ldots, v_{k+1}$ and adding at each step, for $k+2 \leq i \leq n$, a vertex v_i adjacent to $v_{i-k}, \ldots, v_{i-1}$.

Lemma 2.5. *If G is a power of the path P_n^k with $k \geq 1$ and $n \geq 2k+1$, then $|c(v_1, \ldots, v_{2k+1})| = 2k+1$.*

Proof. Let c be a lid-coloring of P_n^k. Without loss of generality, we have $c(v_i) = i$ for each $1 \leq i \leq k+1$. Note that for any $1 \leq i \leq k$, the symmetric difference $N[v_i] \Delta N[v_{i+1}] = \{v_{i+k+1}\}$. Furthermore, $N[v_i] = \{v_1, \ldots, v_{i+k}\}$ and thus $c(N[v_i])$ contains the colors 1 to $k+1$. Therefore, $c(v_i) > k+1$ whenever $k+2 \leq i \leq 2k+1$. Since the vertices $v_{k+2}, \ldots, v_{2k+1}$ induces a complete graph, they have distinct colors. Hence, $|c(v_1, \ldots, v_{2k+1})| = 2k+1$. □

From Theorem 2.4 and Lemma 2.5, we have the following bounds for $\chi_{lid}(P_n^k)$.

Corollary 2.6. *If G is a power of the path P_n^k with $k \geq 1$ and $n \geq 2k+1$, then $2k+1 \leq \chi_{lid}(G) \leq 2k+2$.*

We present values for n such that $\chi_{lid}(P_n^k) = 2k+1$ in Theorem 2.7 and for $\chi_{lid}(P_n^k) = 2k+2$ in Theorem 2.8.

Theorem 2.7. *For $k \geq 1$, $n \geq 2k+1$, and $n \equiv k \pmod{k+1}$, we have $\chi_{lid}(P_n^k) = 2k+1$.*

Proof. By Corollary 2.6, $\chi_{\text{lid}}(P_n^k) \geq 2k+1$. Now we need to show that $\chi_{\text{lid}}(P_n^k) \leq 2k+1$. We construct a lid-coloring c with this cardinality: $c(v_i) = k+1$ if $i \equiv 0 \pmod{2k+2}$, and $c(v_i) = i \pmod{2k+2}$, otherwise.

It is easy to see that c is a proper coloring of P_n^k. To show that for each pair of adjacent vertices v_i and v_j, $c(N[v_i]) \neq c(N[v_j])$, we present the set $c(N[v_i])$ for every $1 \leq i \leq n$. If $1 \leq i \leq k-1$, we have $c(N[v_i]) = [k+i]$. If $k \leq i \leq n-k+1$, we have the following sets:

$$c(N[v_i]) = \begin{cases} [2k+1], & \text{if } c(v_i) = k+1, \\ [2k+1] \setminus \{j-1\}, & \text{if } c(v_i) = k+j,\ 2 \leq j \leq k+1, \\ [2k+1] \setminus \{j+k+1\}, & \text{if } c(v_i) = j,\ 1 \leq j \leq k. \end{cases}$$

For $n-k+1 \leq i \leq n$, we have the following sets:

$$c(N[v_i]) = \begin{cases} [k+1] \cup [c(v_i)+k+2, 2k+1], & \text{if } c(v_i) \leq k, \\ [c(v_i)-k, 2k+1], & \text{if } c(v_i) > k. \end{cases}$$

Since there are distinct sets of colors for vertices with different colors, c is a lid-coloring. □

Theorem 2.8. *For $k \geq 1$ and $2k+2 \leq n \leq 3k+1$, we have $\chi_{lid}(P_n^k) = 2k+2$.*

Proof. From Corollary 2.6, we have $\chi_{lid}(P_n^k) \leq 2k+2$. Now, we show that $\chi_{lid}(P_n^k) \geq 2k+2$. By Lemma 2.3, $|c(v_1, \ldots, v_{2k+1})| = 2k+1$. Symmetrically, $|c(v_{n-2k}, \ldots, v_n)| = 2k+1$. Hence, $c(N[v_{k+1}]) = c(N[v_{n-k}]) = [2k+1]$. Since $n \leq 3k+1$, we have $n-k-(k+1) \leq 3k+1-k-(k+1) \leq k$, which implies that v_{k+1} and v_{n-k} are adjacent, which is a contradiction to the lid-coloring. Thus, we conclude that $\chi_{lid}(P_n^k) \geq 2k+2$. □

For the remaining cases, we leave the following conjecture.

Conjecture 2.9. *For $k \geq 1$, $n \geq 3k+3$, and $n \equiv 0, 1, \ldots, k-1 \pmod{k+1}$, $\chi_{lid}(P_n^k) = 2k+2$.*

2.2 Lid-coloring on split graphs

The *split graph* $G = (K \cup S, E)$ is a graph whose set of vertices can be partitioned into a clique K of size $|K| = k$ and an independent set S

of size $|S| = s$. The split graph is a chordal graph where the maximum clique size and its chromatic number are equal. Esperet *et. al* [3] proved that if G is a split graph $\chi_{\text{lid}}(G) \leq 2\omega - 1$.

Theorem 2.10. *[3] Let $G = (K \cup S, E)$ be a split graph. If $\omega(G) \geq 3$ or if G is a star, then $\chi_{lid}(G) \leq 2\omega - 1$.*

The *corona product* $G \circ H$ of two graphs G and H. It is defined as the graph obtained by taking one copy of G and $|V(G)|$ copies of H and joining the i-th vertex of G to every vertex in the i-th copy of H. The graph $K_n \circ K_1$ is also a split graph. In the following result, we determine the lid-coloring chromatic number in these graphs.

Proposition 2.11. *If $G = K_n \circ K_1$, with $n \geq 3$, then $\chi_{lid}(G) = 2\omega - 1$.*

Proof. Observe that $n = \omega(G) \geq 3$. By Theorem 2.10, we have that $\chi_{lid}(G) \leq 2\omega - 1$. Moreover, by Proposition 2.3, we have that $\chi_{lid}(G) \geq 2\omega - 1$. Therefore, we conclude that $\chi_{lid}(G) = 2\omega - 1 = 2n - 1$. □

The graph G is a *complete split graph* if each vertex in the clique K is adjacent to every vertex in the independent set S. We prove the following result for the complete split graphs.

Theorem 2.12. *Let $G = (K \cup S, E)$ be a complete split graph. If $|S| \geq 2$, then $\chi_{lid}(G) = |K| + 2$.*

Proof. Let $K = \{v_1, \ldots, v_k\}$, $S = \{u_1, \ldots, u_s\}$, $|K| = k$ and $|S| = s$. We construct the following coloring c of G. For $i \in [k]$, $c(v_i) = i$, $c(u_1) = k+2$ and for $u_j \in S$, $c(u_j) = k+1$ with $j \geq 2$. Note that c is a proper coloring of G. Since the vertices of the clique K are universal, we only need to verify the colors in $N[v_i]$ and $N[u_j]$, for every $i \in [k]$ and $j \in [s]$. Thus, we have $c(N[v_i]) = [k+2]$, while $c(N[u_1]) = [k] \cup \{k+2\}$ and $c(N[u_j]) = [k+1]$ for $j \geq 2$. Therefore, $c(N[v_i]) \neq c(N[u_j])$, which implies that c is a lid-coloring of G. Hence, $\chi_{lid}(G) \leq |K| + 2$.

For the lower bound, as the complete split graph has a clique of size $|K| + 1$, we have $\chi_{lid}(G) \geq |K| + 1$. Suppose that $\chi_{lid}(G) = |K| + 1$. So, we can color $c(u_j) = k+1$, for every $j \in [s]$. However, $c(u_1) = k+1$ implies $c(N[v_1]) = c(N[u_1]) = [k+1]$, which is a contradiction. Therefore, $\chi_{lid}(G) \geq |K|+2$. Hence, $\chi_{lid}(G) = |K|+2$. □

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References

[1] J. A. Bondy and U. S. R. Murty. *Graph theory with applications*, volume 290. Macmillan London, 1976.

[2] Paulo Renato da Costa Pereira. *Códigos para subfamílias dos grafos cordais*. PhD thesis, Universidade Federal do Rio de Janeiro, 2007.

[3] L. Esperet, S. Gravier, M. Montassier, P. Ochem, and A. Parreau. Locally identifying coloring of graphs. *Electron. J. Comb.*, 19:40, 2010.

[4] F. Foucaud, I. Honkala, T. Laihonen, A. Parreau, and G. Perarnau. Locally identifying colourings for graphs with given maximum degree. *Discrete Mathematics*, 312(10):1832–1837, 2012.

[5] D. Gonçalves, A. Parreau, and A. Pinlou. Locally identifying coloring in bounded expansion classes of graphs. *Discrete Applied Mathematics*, 161(18):2946–2951, 2013.

[6] M. G. Karpovsky, K. Chakrabarty, and L. B Levitin. On a new class of codes for identifying vertices in graphs. *IEEE transactions on information theory*, 44(2):599–611, 1998.

[7] N. Martins and R. Sampaio. Locally identifying coloring of graphs with few p4s. *Theoretical Computer Science*, 707:69–76, 2018.