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On Identifying Codes in Complementary Prisms

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Dedicated to Professor Jayme Szwarcfiter
on the occasion of his 80th birthday

Abstract. Let $G = (V, E)$ be a simple, finite, and undirected graph, and let $C \subseteq V(G)$. If for each vertex $u \in V(G)$, the intersection of C with the closed neighborhood of u are all nonempty and distinct, then we say that C is an *identifying code* (or ID code, for short). The minimum cardinality of an ID code of G is denoted by $\gamma^{ID}(G)$. In this paper, we present sharp lower and upper bounds for γ^{ID} in the complementary prism graphs and give closed formulas for the minimum ID code of the complementary prism of both the complete split and complete bipartite graphs.

Keywords: identifying codes, complementary prisms.

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1 Introduction

We consider finite, simple, and undirected graphs and use standard notation and terminology. For a graph G, the vertex set and the edge set are denoted $V(G)$ and $E(G)$, respectively. For a vertex u of G, the *neighborhood*, and the *closed neighborhood* are denoted $N_G(u)$ and $N_G[u]$, respectively.

Let $C \subseteq V(G)$. We say that C is a dominating set if, for each vertex $u \in V(G)$, $C \cap N[u] \neq \emptyset$. A vertex u of C is said to *dominate* vertex v if either $u = v$, or u is adjacent to v. Two vertices u, v are *separated* by C if $N[u] \cap C \neq N[v] \cap C$; set C is a *separating set* if every pair of distinct vertices of $V(G)$ are separated by C. A subset $C \subseteq V(G)$ is an *identifying code* (ID code, for short) if C is both a dominating set and a separating set of G. Note that a graph has an identifying code if and only if no two vertices have the same closed neighborhood. The ID code number of a graph G is the minimum cardinality of an ID code of G and is denoted $\gamma^{ID}(G)$. See, for example, the graph in Figure 2.1, where an ID code of the graph is represented by the black vertices.

ID codes were first introduced by Karpovsky et al. [9], where the authors relate the problem to fault diagnosis of multiprocessor systems. ID codes have been widely studied, and a detailed list of references on this subject can be found on Jean's webpage [8].

From a computational point of view, finding identifying codes of minimum cardinality has been proved to be NP-hard [3]. Therefore, it is natural that many researchers have restricted their study of identifying codes in some specific classes of graphs such as trees [1], cycles [4], and hypercubes [10, 7], for instance. ID Codes were also considered in some graph products, such as Cartesian products [11] and complementary prisms [2]. A concept similar to ID codes, called locating-dominating sets, was also studied in the complementary prisms [5, 6].

In this paper, we consider identifying codes in complementary prisms. We present sharp lower and upper bounds for γ^{ID} in these graphs and give closed formulas for the minimum ID code of the complementary prism of both the complete split and complete bipartite graphs.

2 Definitions and preliminary results

For a subset $S \subseteq V(G)$, we denote $N_G[S]$ the set $\bigcup_{u \in S} N_G[u]$. Given two sets A and B, we denote by $A \triangle B$ their symmetric difference, i.e., the set of elements belonging to A or B, but not to both. We denote as C_n and K_n, the cycle and the complete graph, respectively, on n vertices.

A graph $G = (K \cup S, E)$ is a *split graph* if its vertex set can be partitioned into a clique K and an independent set S. Moreover, if G is a split graph and each vertex of the clique is adjacent to each vertex of the independent set, then G is called a *complete split graph*.

The *complementary prism* $G\overline{G}$ of G is the graph formed from the disjoint union of G and the complement $\overline{G}$ by adding the edges of a perfect matching between the corresponding vertices (same label) of G and $\overline{G}$. Here, we use the following notation: for a set $X \subseteq V(G)$, let $\overline{X}$ represent the corresponding set of vertices in $V(\overline{G})$. For a vertex $v \in V(G)$, let $\overline{v}$ be the corresponding vertex in $V(\overline{G})$.

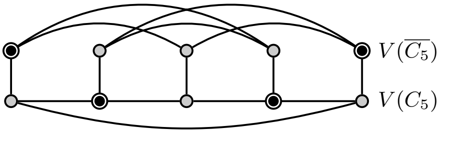

Figure 2.1: Petersen Graph, the complementary prism of C_5.

Not all graphs admit an ID code. If $u, v \in V(G)$ are such that $N_G[u] = N_G[v]$, then u and v are called *twins*. A graph is *identifiable* if and only if it does not have twin vertices. If $N_G(u) = N_G(v)$, they are *false twins*.

The next two lemmas determine some vertices that must be in an ID code of G. Their proofs are omitted.

Lemma 2.1. *Let $G = (V, E)$ be an identifiable graph, C an identifying code of G, and let K be the set of vertices that are mutually false twins. Then, $|K \cap C| \geq |K| - 1$.*

Lemma 2.2. *Let G be an identifiable graph, and let C be an identifying code of G. If there exists a set of vertices $K \subseteq V(G)$ such that K induces a complete graph on n vertices, then $|(N_G[K] \setminus K) \cap C| \geq n-1$.*

The following result provides a sharp logarithmic lower bound on the size of an ID code of any graph G, if G has such a set.

Theorem 2.3. *[9] Let G be an identifiable graph on n vertices. Then, $\gamma^{ID}(G) \geq \lceil \log_2(n+1) \rceil$.*

3 Results on Complementary Prisms

In this section, we present our results on identifying codes in complementary prisms. We begin proving that for any graph G with at least two vertices, $G\overline{G}$ is identifiable.

Theorem 3.1. *Let G be a graph with order n. The graph $G\overline{G}$ is identifiable if and only if $n \geq 2$.*

Proof. If G is a trivial graph, then, $G\overline{G}$ is a complete graph on 2 vertices and it is not identifiable. Therefore, if $G\overline{G}$ is identifiable, then $n \geq 2$.

For the converse, suppose that G is a graph on $n \geq 2$ vertices. Then, for any two vertices $\overline{u}, \overline{v} \in V(\overline{G})$ in $G\overline{G}$, we have $\{u, v\} \subseteq (N_{G\overline{G}}[\overline{u}] \triangle N_{G\overline{G}}[\overline{v}])$, where $u \in N_{G\overline{G}}[\overline{u}]$ and $v \in N_{G\overline{G}}[\overline{v}]$. Analogously, the same is true for any two vertices $u, v \in V(G)$ of $G\overline{G}$.

Now consider two vertices $u \in V(G)$ and $\overline{v} \in V(\overline{G})$ of $G\overline{G}$. If $u \neq v$, these two vertices are not adjacent and $N_{G\overline{G}}[u] \neq N_{G\overline{G}}[\overline{v}]$. If $u = v$, since $n \geq 2$, there is at least one vertex that is adjacent to only one of these vertices, by the construction of $G\overline{G}$. Hence, for any two vertices $u, v \in V(G\overline{G})$, we have $N_{G\overline{G}}[u] \neq N_{G\overline{G}}[v]$ and, therefore, $G\overline{G}$ is identifiable. □

Our next result is an upper bound for $\gamma^{ID}(G\overline{G})$, given that G is connected and identifiable with order $n \geq 3$.

Theorem 3.2. *Let G be a connected and identifiable graph on $n \geq 3$ vertices. Then $V(G)$ is an identifying code of $G\overline{G}$ and $\gamma^{ID}(G\overline{G}) \leq n$.*

Proof. Since G is an identifiable graph, it follows that $V(G)$ is an identifying code of G. From Theorem 3.1, we know $G\overline{G}$ is an identifiable graph. We shall prove that $V(G)$ is also an identifying code of $G\overline{G}$. Let $C = V(G)$. The set C dominates $V(G\overline{G})$ and it separates all vertices of G in $G\overline{G}$. In addition, for any vertex $\overline{u} \in V(\overline{G})$ in $G\overline{G}$, we have $N_{G\overline{G}}[\overline{u}] \cap C = \{u\}$. Since G has no isolated vertices, $N_{G\overline{G}}[u] \cap C \neq \{u\}$. Thus C separates all vertices of $\overline{G}$ in $G\overline{G}$.

Now, we show that C separates each vertex in $V(G)$ from each vertex in $V(\overline{G})$. We know that $N_{G\overline{G}}[v] \cap C = N_G[v] \cap C = N_G[v]$ and, since G is connected and has at least 3 vertices, then $N_G[v] \geq 2$. Thus, $(N_{G\overline{G}}[v] \cap C) \neq (N_{G\overline{G}}[\overline{v}] \cap C)$. Furthermore, if $u \in V(G)$, $\overline{v} \in V(\overline{G})$ with $u \neq v$, then $(N_{G\overline{G}}[u] \cap C) \neq (N_{G\overline{G}}[\overline{v}] \cap C)$. □

We define some sets we shall use in the following results. Let S be an ID code of $G\overline{G}$ with $S_1 = S \cap V(G)$ and $\overline{S_2} = S \cap V(\overline{G})$. Let S_2 be the set of corresponding vertices of $\overline{S_2}$ in $V(G)$, $\overline{S}_1$ be the set of corresponding vertices of S_1 in $V(\overline{G})$, D be the set $V(G) \setminus (S_1 \cup S_2)$ and $\overline{D}$ be the corresponding vertices of D in $V(\overline{G})$, with $S_1 \cap S_2 = X$ and $|X| = x$. The reader is referred to Figure 3.1.

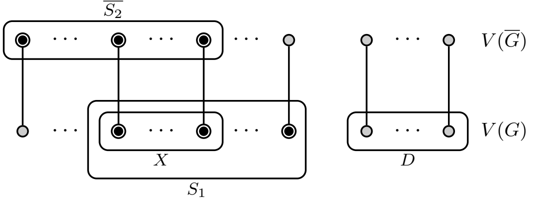

Figure 3.1: An illustration of the sets S_1, $\overline{S_2}$, X and D where $S_1 \cup \overline{S_2}$ is an ID code of $G\overline{G}$. The edges of G and $\overline{G}$ were omitted.

The next two lemmas provide bounds on the size of the subset $D = V(G) \setminus (S_1 \cup S_2)$ of $G\overline{G}$, considering that $S = S_1 \cup \overline{S_2}$ is an ID code of $G\overline{G}$.

Lemma 3.3. *Let G be a graph of order $n \geq 3$ and S an ID code of $G\overline{G}$. If $x = 0$, then $n - |S_1| - |\overline{S_2}| \leqslant 2^r - r - 1$, where $r = \min\{|S_1|, |\overline{S_2}|\}$.*

Proof. The vertices in S_2 are separated by vertices in $S_1 \cup \overline{S_2}$. By construction, $N_{G\overline{G}}(D \cup S_1) \cap \overline{S_2} = \emptyset$. For every distinct pair $u, v \in S$, we have that $(N_G[u] \cap S_1) \neq (N_G[v] \cap S_1)$. So, the size of $D \cup S_1$ is bounded by the number of nonempty subsets of S_1. So we can conclude that $|D| + |S_1| \leq 2^{|S_1|} - 1$, which implies $|D| \leq 2^{|S_1|} - |S_1| - 1$. Analogously, $|\overline{D}| \leq 2^{|\overline{S_2}|} - |\overline{S_2}| - 1$. Since $|D| = |\overline{D}|$ and $r = \min\{|S_1|, |\overline{S_2}|\}$, it follows that $|D| = n - |S_1| - |\overline{S_2}|$ is bounded by $2^r - r - 1$. □

Lemma 3.4. *Let G be a graph of order $n \geq 3$ and S an ID code of $G\overline{G}$. If $x \neq 0$, then $n - |S_1 \cup S_2| \leqslant 2^r - r + x - 2$, where $r = \min\{|S_1|, |\overline{S_2}|\}$.*

Proof. The vertices in $X \cup S_2$ are separated by $S_1 \cup \overline{S_2}$. The vertices in $(S_1 \setminus X) \cup D$ are separated by vertices in S_1. By construction, $N_{G\overline{G}}(D) \cap \overline{S_2} = \emptyset$. For every distinct pair $u, v \in S$, we have that $(N_G[u] \cap S_1) \neq (N_G[v] \cap S_1)$. So, the size of $D \cup (S_1 \setminus X)$ is bounded by the number of nonempty proper subsets of S_1 (in case that $X = S_1$, if a vertex in D is adjacent to all vertices in S_1, then the corresponding vertex in $\overline{G}$ has no neighbor in $\overline{S_2}$, and it is not dominated by S). Hence, we can conclude that $|D \cup (S_1 \setminus X)| \leq 2^{|S_1|} - 2$, which implies $|D| \leq 2^{|S_1|} - |S_1| + x - 2$. Analogously, $|\overline{D}| \leq 2^{|\overline{S_2}|} - |\overline{S_2}| + x - 2$. Since $|D| = |\overline{D}|$ and $r = \min\{|S_1|, |\overline{S_2}|\}$, it follows that $|D| = n - |S_1 \cup S_2|$ is bounded by $2^r - r + x - 2$. □

The bound presented in Theorem 2.3 is now sharpened for complementary prism graphs.

Theorem 3.5. *Let G be a graph of order $n \geq 4$ and S an ID code of $G\overline{G}$. Then, $|S| \geq 2\lceil \log_2(n + 1)\rceil - 2$.*

Proof. Assume $|S_1| = k_1$ and $|\overline{S_2}| = k_2$. We consider the cases where (i) $x = 0$ and, and (ii) $x \neq 0$. By symmetry, for each of (i) and (ii), we can assume that $k_1 \leq k_2$.

Case (i): By Lemma 3.3, $|D| = n - k_1 - k_2 \leq 2^{k_1} - k_1 - 1$. Thus $n \leq 2^{k_1} - k_1 - 1 + k_1 + k_2 = 2^{k_1} + k_2 - 1$. If $k_2 \leq 2^{k_1}$, then $n \leq 2^{k_1+1} - 1$, which implies $k_1 \geq \log_2(n + 1) - 1$. If $2^{k_1} \leq k_2$, then $n \leq 2k_2 - 1$, which implies $k_2 \geq \log_2(k_2) \geq \log_2(n + 1) - 1$.

Case (ii): By Lemma 3.4, $|D| = n - (k_1 + k_2 - x) \leq 2^{k_1} - k_1 + x - 2$. Thus $n \leq 2^{k_1} - k_1 + x - 2 + k_1 + k_2 - x = 2^{k_1} + k_2 - 2$. If $k_2 \leq 2^{k_1}$, then $n \leq 2^{k_1+1} - 2$, which implies $k_1 \geq \log_2(n+1) - 2$. If $2^{k_1} \leq k_2$, then $n \leq 2k_2 - 2$, which implies $k_2 \geq \log_2(k_2) \geq \log_2(n+1) - 2$.

By Cases (i) and (ii), and since k_1 and k_2 are both integers, we can conclude that $|S| = k_1 + k_2 \geq 2\lceil \log_2(n+1) \rceil - 2$. □

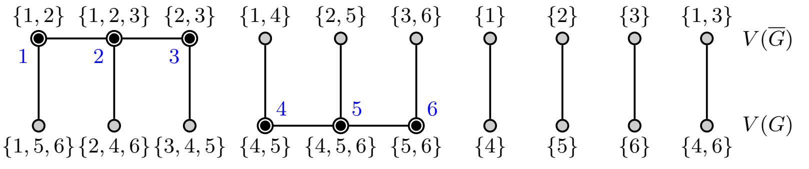

Figure 3.2: $S = \{1, 2, 3, 4, 5, 6\}$ is an ID code of $G\overline{G}$. The set in each vertex v is $N_{G\overline{G}}[v] \cap S$. For simplicity, most of edges of G and $\overline{G}$ were omitted.

The lower bound in Theorem 3.5 is sharp. Let $G\overline{G}$ be the infinite family of graphs where the order of G is $n = 2^p + p - 1$, with $p \geq 2$. We do $|S_1| = |\overline{S_2}| = p$ and $|S| = 2p$. Thus, $|D| = n - |S| = 2^p - p - 1$. An illustration for the case $p = 3$ can be seen in Figure 3.2. From Theorems 3.2 and 3.5, we conclude the following result.

Corollary 3.6. *If G is a connected and identifiable graph on $n \geq 3$ vertices, then $2\lceil \log_2(n+1) \rceil - 2 \leq \gamma^{ID}(G\overline{G}) \leq n$, and these bounds are sharp.*

Observe that the upper bound in Corollary 3.6 is not valid for $\gamma^{ID}(G\overline{G})$ when G is not an identifiable graph. For instance, $\gamma^{ID}(K_n\overline{K_n}) = n+1$. In the next result, we consider ID codes in complete bipartite graphs, which attain the upper bound of Corollary 3.6.

Theorem 3.7. *Let $G = K_{r,s}$ be a complete bipartite graph, with $r, s \geq 3$ and $n = r + s$. Then, $\gamma^{ID}(G\overline{G}) = n$.*

Proof. Let $V(G) = (A \cup B)$ with $A = \{u_1, \ldots, u_r\}$ and $B = \{v_1, \ldots, v_s\}$. Let C be a minimum ID Code of G.

The set $\overline{A}$ induces a complete subgraph on r vertices and, according to Lemma 2.2, $|(N_{G\overline{G}}[\overline{A}] \setminus \overline{A}) \cap C| \geq (r-1)$. Since $(N_{G\overline{G}}[\overline{A}] \setminus \overline{A}) = A$, then $|A \cap C| \geq (r-1)$. Suppose that $|A \cap C| = r-1$. Since one of the vertices of A, for instance, u_i does not belong to C, then $\overline{u}_i$ must belong to C or it must be adjacent to a vertex that belongs to C so that C dominates all vertices of $\overline{A}$. Hence, $|(A \cup \overline{A}) \cap C| \geq r$. Analogously, $|(B \cup \overline{B}) \cap C| \geq s$. Thus, $\gamma^{ID}(G\overline{G}) \geq r+s$. However, by Theorem 3.2, $C = (A \cup B)$ is an identifying code of $G\overline{G}$ with $r+s$ vertices. Hence, $\gamma^{ID}(G\overline{G}) = r+s = n$. □

Finally, we consider the complementary prism of complete split graphs. A complete split graph $G = (K \cup S, E)$ with $|K| \geq 2$ is not identifiable. This fact, however, changes when we analyze their complementary prisms.

Theorem 3.8. *Let $G = (K \cup S, E)$ be a complete split graph, $|K| = k, |S| = s$, with $k, s \geq 2$ and $n = k+s$. Then, $\gamma^{ID}(G\overline{G}) = n$.*

Proof. Let $K = \{u_1, \ldots, u_k\}$ and $S = \{v_1, \ldots, v_s\}$. Let C be an ID code in G. We know that $(N_{G\overline{G}}[\overline{S}] \setminus \overline{S}) = S$. The set $\overline{S}$ induces a complete subgraph and, by Lemma 2.2, we have $|(S \cap C)| \geq s-1$. However, if C is an ID code and $|S \cap C| = s-1$, then there exists a vertex $\overline{v_i} \in C$, otherwise $\overline{v_i}$ would not be dominated. Therefore, $|(S \cup \overline{S}) \cap C| \geq s$. Moreover, given two distinct vertices $u_i, u_j \in V_{G\overline{G}}(K)$, we have $N_{G\overline{G}}[u_i] \triangle N_{G\overline{G}}[u_j] = \{\overline{u}_i, \overline{u}_j\}$ and, therefore, $|\overline{K} \cap C| \geq k-1$. However, if $|\overline{K} \cap C| = k-1$, then there exists a vertex $u_i \in C$, otherwise $\overline{u_i}$ is not dominated. Hence, $|(K \cup \overline{K}) \cap C| \geq k$. Therefore, $|C| \geq k+s$. The set $C = \{u_1\} \cup \{\overline{u}_2, \ldots, \overline{u}_k\} \cup \{v_1, \ldots, v_s\}$ is an ID code in $G\overline{G}$ of cardinality $k+s = n$. □

An open question is whether finding minimum ID Codes in complementary prisms remains NP-hard.

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