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On the equitable total coloring of Loupekine snarks

Rieli Araujo[1] and **Diana Sasaki**[1]

[1]IME, State University of Rio de Janeiro, Brazil

Dedicated to Professor Jayme Szwarcfiter on the occasion of his 80th birthday

Abstract. An equitable total coloring is a total coloring such that the difference between the cardinalities of any two color classes is at most one. The Equitable Total Coloring Conjecture (ETCC) was posed by Wang, and states that the equitable total chromatic number of a graph G is at most $\Delta(G) + 2$.The ETCC was proved for cubic graphs. Consequently, the equitable total chromatic number of a cubic graph is either 4 (Type 1 graphs) or 5 (Type 2 graphs). Dantas et al. proposed the question about the existence of a Type 1 cubic graph with girth at least 5 and equitable total chromatic number 5. In this work, we establish that all members of the second infinite family of Loupekine snarks have equitable total chromatic number 4, contributing to the body of negative evidence that supports a NO answer, since they are Type 1 and have girth 5.

Keywords: Equitable total coloring, total coloring, Loupekine snarks.

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Emails: rieli.araujo@pos.ime.uerj.br, diana.sasaki@ime.uerj.br.

1 Introduction

Throughout this paper we consider simple graphs, i.e. without loops or parallel edges, and we adopt standard terminology as found in [3]. The process of coloring poses a formidable challenge, as it encapsulates various real-world situations where the representation of adjacencies embodies conflicts. A *k-total coloring* is an assignment of k colors to the edges and vertices of a graph G, so that adjacent or incident elements have different colors. The *total chromatic number* of G, denoted by $\chi''(G)$, is the smallest value of k for which G has a k-total coloring. Clearly, $\chi''(G) \geq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of G.

Conjecture 1.1 (Total Coloring Conjecture (TCC) [13, 2]). *The total chromatic number of a simple graph G is at most $\Delta(G) + 2$.*

According to Rosenfeld's work [10], the Total Coloring Conjecture (TCC) has been shown to be valid for cubic graphs, and so it has been established that the total chromatic number of such graphs is either 4 (called *Type 1* graphs) or 5 (called *Type 2* graphs).

An *equitable total coloring* of G is a total coloring such that the difference between the cardinalities of any two color classes is at most one. Analogously to the total chromatic number, the *equitable total chromatic number*, denoted $\chi''_e(G)$, is defined as the smallest k for which G admits an equitable k-total coloring.

Conjecture 1.2 (Equitable Total Coloring Conjecture (ETCC) [14]). *The equitable total chromatic number of a simple graph G is at most $\Delta\ (G) + 2$.*

In 2002, Wang [14] established that the equitable total coloring conjecture is valid for cubic graphs. Consequently, the equitable total chromatic number of a cubic graph is either 4 or 5. In 2016, Dantas et al. [7] verified that there exist an infinite number of Type 1 cubic graphs with $\chi''_e(G) = 5$,but they were small girth graphs. In the same paper, they proposed Question 1 about the existence of a Type 1 cubic graph with girth at least 5 and equitable total chromatic number 5.

Question 1 ([7]). *Does there exist a Type 1 cubic graph with girth greater than 4 and equitable total chromatic number 5?*

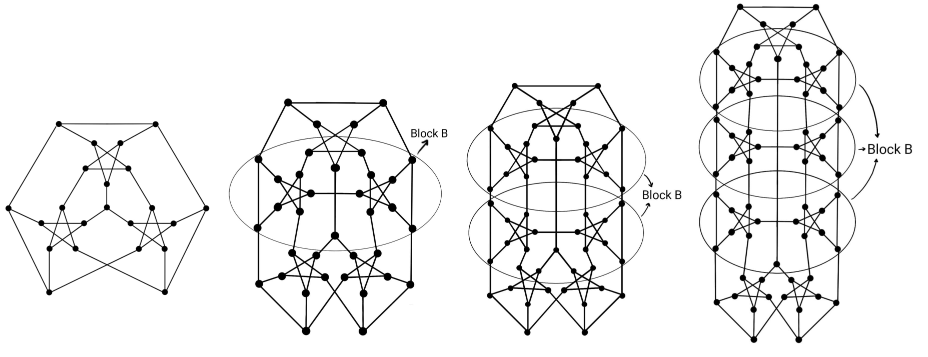
In this work, we establish that all members of an infinite family of cubic graphs have equitable total chromatic number 4, contributing to the body of negative evidence that supports a NO answer, since they are Type 1 and have girth 5.

2 Snarks

Snarks are non-trivial cubic graphs which cannot be 3-edge coloured (for a precisely definition of non-trivial, we refer to [4]). The name snark was first used in 1976 by Martin Gardner, in reference to the Lewis Carroll's poem "The Hunting of the Snark" [8]. The name reflects the difficulty of finding these graphs in the context of the famous Four Color Theorem [1]. In fact, although they were first named snarks in 1976, these graphs were introduced in 1880 by Tait [12], about the Four Color Theorem. Snarks are very important in Graph Theory, since they are counterexamples to various important conjectures.

Loupekine snarks Loupekine snarks were discovered by Loupekine and described by Isaacs in [9], and they consist of two distinct infinite families. Both of the families can be generated iteratively from two distinct initial graphs, each of which has 22 vertices and 33 edges. The second Loupekine family can be constructed iteratively as follows: starting from the initial graph L_0 shown in Figure 2.1a, extend the family infinitely by adding the connecting block B, while preserving the properties of snarks. The snark L_n, $n \geq 1$, is obtained by adding n blocks B to L_0. In Figure 2.1, we present the first four members of this family, where the blocks B are highlighted.

In 2014, 4-total colorings for all members of the two infinite families of Loupekine snarks were determined by Sasaki et al. [11], but these total colorings were not equitable. In 2017, Cordeiro et al.[6] proved that all members of the first family of Loupekine snarks have equitable 4-total colorings.



(a) L_0. (b) L_1. (c) L_2. (d) L_3.

Figure 2.1: The first four members of the second Loupekine family.

3 Our Main Result

In this section, we prove that all members of the second infinite family of Loupekine snarks have equitable total chromatic number 4, by presenting equitable 4-total colorings for all members of this family.

Theorem 3.1. *All members of the second infinite family of Loupekine snarks have equitable total chromatic number 4.*

Sketch of proof Initially, we begin with an equitable 4-total coloring for graph L_0, such as that presented in Figure 3.1. This coloring is preserved in all colorings of the members of the second infinite family of Loupekine snarks.

The process of obtaining an equitable 4-total coloring for each snark requires the use of four different 4-total colorings for the connection block. This was done to ensure that each block added to the graph will not have a conflict of colors with the previous colorings and that the equitable property will be preserved in the resulting colorings of the subsequent

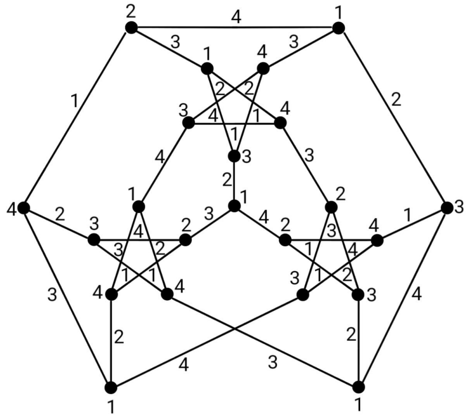
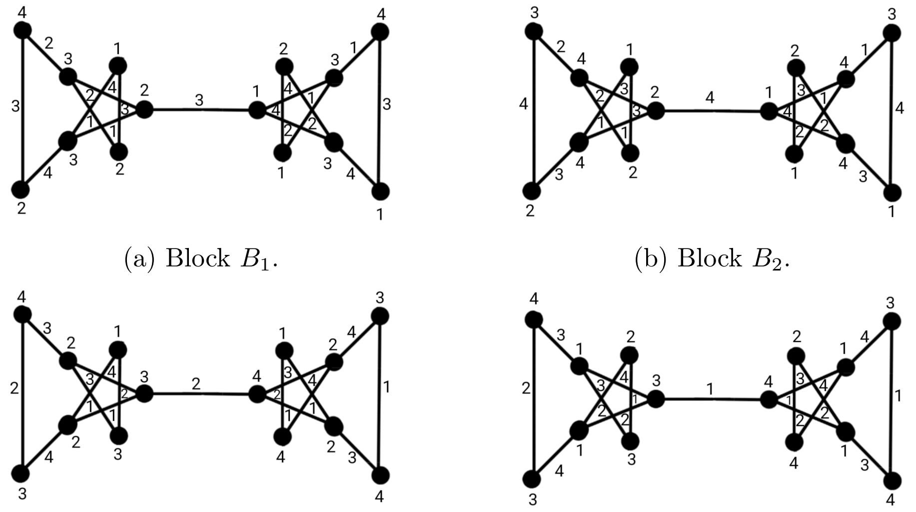


Figure 3.1: An equitable 4-total coloring of Loupekine snark L_0. Remark that the number in the small triangle formed by one vertex and two intersections of edges in the subgraphs isomorphic to C_5 that look like a five-pointed star, represents the color of the edges which is not one of two incidents edges of the vertex.

graphs. The 4-total colorings of the block B, denoted B_1, B_2, B_3 and B_4, are presented in Figures 3.2a, 3.2b, 3.2c, 3.2d.

To ensure the infinite extension of the family with equitable 4-total colorings, the block B with the four different 4-total colorings was inserted in a specific manner, that is, $B_{n(mod\ 4)}$. For instance, to obtain the colored graph L_1 shown in Figure 3.3, we add the block B_1 presented in Figure 3.2a to L_0 in Figure 3.1.

Following the construction of the family, to obtain the colored graph L_2, we add the block B_2 presented in Figure 3.2b to the graph L_1 in Figure 3.3, and so on. Useful information of the equitable 4-total colorings of the the first six Loupekine snarks, i.e., up to L_5, are presented in Table 3.1. For the general case, we have that the coloring of graph L_n is obtained as $L_{n-1} + B_{n(mod\ 4)}$.



(a) Block B_1.

(b) Block B_2.

(c) Block B_3.

(d) Block B_4.

Figure 3.2: Block B with four 4-total colorings used in the construction of equitable 4-total colorings of the members of the second Loupekine family.

On the colorings of the first six graphs of the second Loupekine family.						
Graphs	Coloring	Elements	Color 1	Color 2	Color 3	Color 4
L_0	L_0	55	14	13	14	14
L_1	$L_0 + B_1$	90	23	22	23	22
L_2	$L_1 + B_2$	125	32	31	31	31
L_3	$L_2 + B_3$	160	40	40	40	40
L_4	$L_3 + B_4$	195	49	48	49	49
L_5	$L_4 + B_1$	230	58	57	58	57

Table 3.1: Overview of the equitable 4-total colorings obtained for the first six graphs of the second Loupekine family.

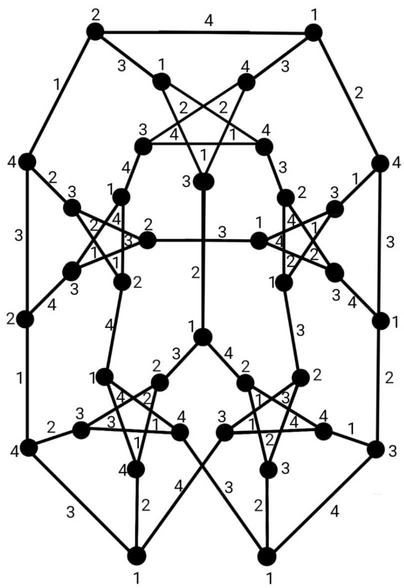


Figure 3.3: Snark L_1 with an equitable 4-total coloring.

4 Final Remarks

In this paper, we proved that all members of the second family of Loupekine snarks have equitable total chromatic number 4. This result contributes to the body of negative evidence that supports a NO answer to the question posed by Dantas et al. [7]. In this section, we provide a summary of the total coloring theory of some well-known snarks in Table 4.1 and as a future work, we will investigate the remaining open problems presented in the table.

* $2n + 1$ when $n < 3$; 6 when $n \geq 3$

Total coloring theory of snarks				
Snark	Order	Girth	Coloring	Is it equitable?
Petersen	10	5	Type 1	Yes
Celmins-Swart 1	26	5	Type 1	Yes
Celmins-Swart 2	26	5	Type 1	Yes
Double star	30	6	Type 1	*(open)*
Flower F_{2n+1}	$8n+4$	*	Type 1	Yes [5]
Goldberg G_{2n+1}	$16n+8$	5	Type 1	Yes [7]
Loupekine 1	$14n+22$	5	Type 1	Yes [6]
Loupekine 2	$14n+22$	5	Type 1 [11]	**Yes [This work]**
Blanuša 1	$18+8n$	5	Type 1	Yes [11]
Blanuša 2	$18+8n$	5	Type 1	Yes [11]
Zamfirescu	$18+8n$	5	*(open)*	*(open)*

Table 4.1: On the total coloring theory of the well known snarks.

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