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Characterisation and total colouring of bipartite graphs with at most three bicliques

Gustavo L. Montanheiro [1], **Leandro M. Zatesko** [1] and **Marina E. Groshaus** [1,2]

[1]Federal University of Technology — Paraná (UTFPR), Brazil
[2]National Scientific and Technical Research Council (CONICET), Argentina

Dedicated to Professor Jayme Szwarcfiter on the occasion of his 80th birthday

Abstract. Determining the total chromatic number $\chi''(G)$ of a graph G is an NP-hard problem even for bipartite graphs. For graphs with a universal vertex, it was solved by Hilton in 1990. Using this result, in 2012, Campos et al. solved the problem for split-indifference graphs, all of which have at most three cliques. In 1991, Hilton also solved the problem for bipartite graphs with adjacent *bi-universal* vertices (a vertex is *bi-universal* if it is adjacent to all vertices in the other part of the bipartition). We conjecture that a bipartite graph G with at most three bicliques has: $\chi''(G) = \Delta(G) + 2$ if G has a Δ-subgraph H with adjacent bi-universal vertices satisfying $\chi''(H) = \Delta(H) + 2 = \Delta(G) + 2$; $\chi''(G) = \Delta(G) + 1$ otherwise. If G has at most three bicliques, we prove that either G has adjacent bi-universal vertices (and our conjecture follows from Hilton's result), or the graph obtained after successively removing twins is a P_5. For the latter case, we give a condition under which our conjecture holds.

Keywords: graph total colouring, bicliques, bipartite graphs.

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1 Introduction

All graphs[1] considered are undirected, simple, and always connected. The vertex and edge sets of a graph G are denoted $V(G)$ and $E(G)$, respectively, and the maximum degree of G is denoted $\Delta(G)$, or Δ when free of ambiguity. The set of vertices adjacent to some $u \in V(G)$ is denoted $N(u)$. Two vertices u, v are *twins* if $N(u) = N(v)$. An *independent set* (*matching*) in G is a subset of vertices (edges) no two of which are adjacent in G. The cardinality of a maximum matching in G is denoted $\mu(G)$. A vertex u of G is *universal* if it is adjacent to all other vertices of G. The graph G is *complete* if $E(G) = \{uv : u, v \in V(G)\}$.

A graph G with $|V(G)| > 1$ is *bipartite* if its vertex set can be (uniquely, since G is connected) partitioned into two independent sets A, B. The bipartition $\{A, B\}$ is said to be an *equi-bipartition*, in which case G is *equi-bipartite*, if $|A| = |B|$. A vertex u of a bipartite graph G is *bi-universal* if it is adjacent to all vertices in the other part. The *bipartite complement* of a bipartite graph G, denoted $\overline{\overline{G}}$, is the graph defined by $V(\overline{\overline{G}}) = V(G)$ and $E(\overline{\overline{G}}) = \{uv \notin E(G) : u \in A, v \in B\}$, being $\{A, B\}$ the bipartition of G. The graph G is *complete bipartite* if $E(G) = \{uv : u \in A, v \in B\}$.

A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A Δ-subgraph of G is a subgraph H of G with $\Delta(H) = \Delta(G)$. For $U \subseteq V(G)$, the subgraph of G *induced* by U, denoted $G[U]$, is the graph defined by $V(G[U]) = U$ and $E(G[U]) = \{uv \in E(G) : u, v \in U\}$. A *clique* in G is a maximal induced subgraph of G which is complete. A *biclique* in G is a maximal induced subgraph of G which is complete bipartite.

A *k-total colouring* of a graph G is a function $c : V(G) \cup E(G) \to C$, with $|C| = k$ such that, given adjacent $u, v \in V(G)$, adjacent $e_1, e_2 \in E(G)$, and an edge e incident to a vertex $w \in V(G)$, it holds that $c(u) \neq c(v)$, $c(e_1) \neq c(e_2)$, and $c(w) \neq c(e)$. The least k for which G admits a *k-total colouring* is the *total chromatic number* of G, denoted $\chi''(G)$. Clearly, $\chi''(G) \geq \Delta + 1$. The *Total Colouring Conjecture* (TCC) [1, 10] states that

[1] The reader is referred to [4] for basic concepts on Graph Theory.

$\chi''(G) \leq \Delta + 2$ for every graph G. If $\chi''(G) = \Delta + 1$ ($\chi''(G) = \Delta + 2$), then G is said to be *Type 1* (*Type 2*). The TCC was proved for some graph classes, such as complete and complete bipartite graphs [2].

The *Total Colouring Problem* consists of, given a graph G, determining $\chi''(G)$. This problem is NP-hard [9] even when restricted to regular bipartite graphs [8]. In 1990, the problem was solved (i.e. a polynomial-time algorithm for the problem was shown) by Hilton [6] for graphs with universal vertices. In 1991, Hilton [7] showed the analogous result for bipartite graphs with adjacent bi-universal vertices, as transcribed in Lemma 1.

Lemma 1 ([7])**.** *A bipartite graph G with adjacent bi-universal vertices is: Type 2 if G is equi-bipartite and $|E(\overline{\overline{G}})| + \mu(\overline{\overline{G}}) \leq n-1$, being n the number of vertices in each part of the bipartition of G; Type 1 otherwise.* $\square$

Observation 1 ([2])**.** *If G is a bipartite graph whose vertices of degree Δ are all in the same part of the bipartition, then G is Type 1 and a $(\Delta + 1)$-total colouring of G can be easily constructed.* $\square$

In 2012, Campos et al. [3] solved the Total Colouring Problem for split-indifference graphs, proving that such a graph G is: Type 2 if it has some Type 2 Δ-*subgraph* with universal vertices; Type 1 otherwise. Split-indifference graphs have at most three cliques. We conjecture below the analogous for bipartite graphs with at most three bicliques, relying on Hilton's result for bipartite graphs with adjacent bi-universal vertices.

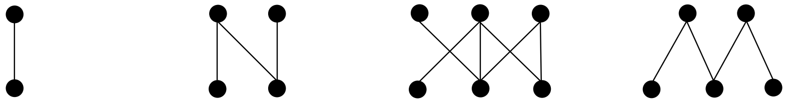
Conjecture 1. *A bipartite graph G with at most three bicliques is Type 2 if and only if it has a Type 2 Δ-subgraph with adjacent bi-universal vertices.*

In Sect. 2 we characterise the structure of bipartite graphs with at most three bicliques. In Sect. 3 we prove Conjecture 1 for a subclass of them.

2 Characterisation of the graphs in the class

We characterise the structure of the bipartite graphs G with at most three bicliques by exhausting all the possibilities, listed in Theorem 1, for the graph obtained from G after successively removing twins.

Theorem 1. *Let G be a connected bipartite graph with at most three bicliques with no twins. Then G is isomorphic to one of the following graphs: K_2, P_4, A, P_5 (Figs 2.1a, 2.1b, 2.1c, 2.1d, respectively).*



(a) K_2: 1 biclique (b) P_4: 2 bicliques (c) A: 3 bicliques (d) P_5: 3 bicliques

Figure 2.1: Graphs with at most three bicliques and no twins

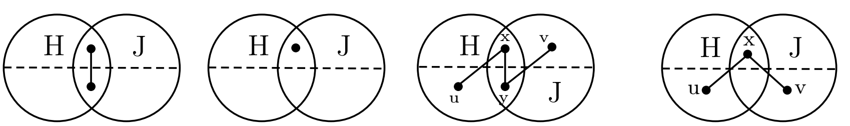
Proof. First, observe that if two vertices belonging to the same set of bicliques of G are twins, then they are in the same part of the bipartition. Second, if U, V are the vertex sets of two bicliques such that some $u \in U \setminus V$ and some $v \in V \setminus U$ are not in any other biclique, then $uv \notin E(G)$, otherwise there would be another biclique containing both u and v.

We consider the cases in which G has 1, 2, or 3 bicliques separately.

Case G has exactly one biclique. In this case, G is complete bipartite. As G has no twins, each of the parts has a single vertex, thus $G \simeq K_2$.

Case G has exactly two bicliques: H and J. Since G is connected, $V(H) \cap V(J) \neq \emptyset$. Let $H' = G[V(H) \setminus V(J)]$ and $J' = G[V(J) \setminus V(H)]$.

(i) Consider that $E(H) \cap E(J) \neq \emptyset$ (Fig. 2.2a).



(a) Subcase (i) (b) Subcase (ii) (c) Solution for (i) (d) Contradiction in (ii)

Figure 2.2: Subcases when G has exactly two bicliques

We show that this subcase is possible. Let $xy \in E(H) \cap E(J)$. Clearly, H' and J' are nonempty. As per the opening remarks to this proof, and seeing that G has no twins, H' and J' contain a single vertex each, not mutually adjacent, but each adjacent to exactly one of x, y. Without loss of generality, let $u \in V(H')$ be adjacent to x, and $v \in V(J')$ be adjacent to y. Therefore, $G \simeq P_4$ (see Fig. 2.2c).

(ii) Now, consider that $E(H) \cap E(J) = \emptyset$ (Fig. 2.2b).

We show that this subcase is not possible. Since G is connected, there must be $x \in V(H) \cup V(J)$ and $xu, xv \in E(G)$ with $v \in V(H')$ an $u \in V(J')$. However, the subgraph induced by $\{x, u, v\}$ is a complete bipartite graph that is neither a subgraph of H, nor of J, which is a contradiction (see Fig. 2.2d).

Case G has exactly three bicliques: H, J, K. There are only two subcases:

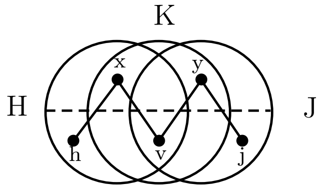
(iii) $V(H) \cap V(J) \cap V(K) \neq \emptyset, E(H) \cap E(J) \cap E(K) = \emptyset$;

(iv) $E(H) \cap E(J) \cap E(K) \neq \emptyset$.

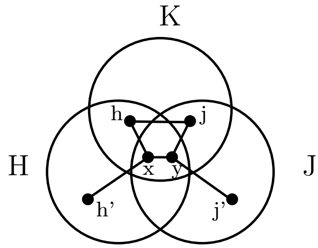
All other subcases can be shown to be impossible, as they clearly lead to a complete bipartite graph that is not a subgraph of any of H, J, K.

Proof for (iii). We show that this subcase is possible. As there are no edges in the total intersection of the bicliques and G has no twins, the total intersection contains only a single vertex v. Therefore, one of the three bicliques, say K, is equal to $G[\{v\} \cup N(v)]$. This implies that the vertices exclusive to bicliques H and J are in the same part as v. As per the opening remarks to this proof, we conclude that $V(H) \setminus V(K)$ and $V(J) \setminus V(K)$ contain a single vertex each. Let those vertices be h and j, respectively. Because the vertices belonging to the same set of bicliques are all twins, we conclude that there is a single vertex $x \in V(H) \cap V(K) \setminus V(J)$ that is adjacent to h, and symmetrically, there is a single vertex $y \in V(J) \cap V(K) \setminus V(H)$ that is adjacent to j. Thus, the only possible graph that satisfies these conditions is isomorphic to P_5 (see Fig. 2.3a).

Proof for (iv). We show that this subcase is possible. Let xy be the edge in $E(H \cap J \cap K)$, and let V_1 and V_2 be the parts of G. We note that x and y are bi-universal in G, and thus $V_1 = N(x)$, $V_2 = N(y)$, without loss of



(a) Solution for (iii): P_5



(b) Solution for (iii): graph A

Figure 2.3: Solutions for when G has exactly three bicliques

generality. Also, since $G[\{x\} \cup N(x)]$ and $G[\{y\} \cup N(y)]$ induce bipartite subgraphs included in some biclique, we assume that $G[\{x\} \cup N(x)]$ is a subgraph of H, and $G[\{y\} \cup N(y)]$ is a subgraph of J. This implies that there is no vertex in $V(K)$ that is not also in either $V(H)$ or $V(J)$. There cannot be any vertex in $V(H) \cap V(J) \setminus V(K)$, otherwise that vertex would be bi-universal in $G[V(H) \cup V(J)]$, thus bi-universal in G, and then a twin vertex of either x or y, contradicting the fact that G has no twins. Furthermore, both $(V(H) \cap V(K)) \setminus V(J)$ and $(V(J) \cap V(K)) \setminus V(H)$ are nonempty, otherwise K would be entirely contained in either H or J. Lastly, $V(H) \setminus (V(J) \cup V(K))$ and $V(J) \setminus (V(H) \cup V(K))$ must be both nonempty, because H and J are distinct from K. According to the opening remarks to this proof, and since all edges of G are contained either in H or in J, we obtain that G is isomorphic to the graph A (Fig. 2.3b). □

3 Total colouring some graphs in the class

Let G be a bipartite graph with at most three bicliques and let G' be the graph obtained from G after the successive removal of twins. Observe that each maximal set of twins in G corresponds to a single vertex in G'. Amongst the graphs characterised in Theorem 1, P_5 is the only one which does not have adjacent bi-universal vertices. Hence, if $G' \not\simeq P_5$, then the Total Colouring Problem is solved for G by Lemma 1. So, the only case remaining to prove Conjecture 1 is when $G' \simeq P_5$.

Theorem 2. *Let G' be the graph obtained from a bipartite graph G after the successive removal of twins such that $G' \simeq P_5$. Let A, B, C, D, E be the maximal sets of twins in G as in Fig. 3.1a, with cardinalities a, b, c, d, e, respectively. Without loss of generality, suppose $a \geq e$. Then:*

(i) if $b + d \neq c + a$, then G is Type 1.

(ii) if $b + d = c + a > ad + min(a, d)$, then G has a Type 2 Δ-subgraph with adjacent bi-universal vertices (thus G is also Type 2).

(iii) if $b + d = c + a \leq ad + min(a, d)$ and $a > max(d, e)$, then G is Type 1.

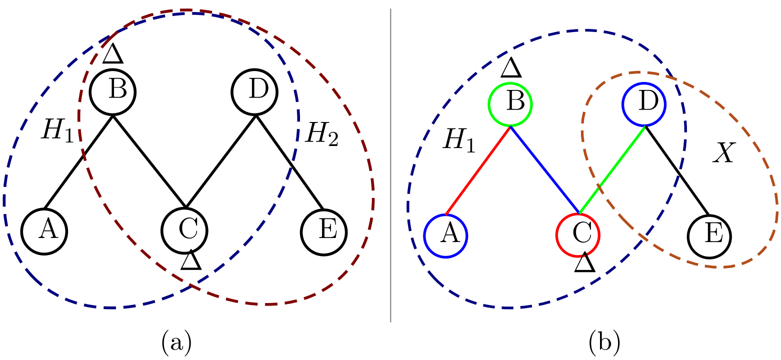


Figure 3.1: Illustrations for Theorem 2

Proof. First, observe that the Δ-vertices of G may only occur in sets B, C, D, since the degree of vertices in C is $b + d$, which is strictly greater than the degree of vertices in A (E), which is b (d). We begin by proving (i). Since $a \geq e$, if $b + d \neq c + a$, then the Δ-vertices of G occur only in $B \cup D$, and the theorem follows from Observation 1.

In (ii) and (iii), remark that $b + d = c + a = \Delta$, implying, since $a \geq e$, that Δ-vertices occur in B and C, occurring also in D if $a = e$. Let $H_1 = G[A \cup B \cup C \cup D]$ and $H_2 = G[B \cup C \cup D \cup E]$ (see Fig. 3.1a). Note that H_1 and H_2 are Δ-subgraphs of G. Note further that H_1 is equi-bipartite and has adjacent bi-universal vertices, with $E(\overline{\overline{G}}) = ad$ and $\mu(\overline{\overline{G}}) = min(a, d)$.

Therefore, from Lemma 1, if $ad + min(a, d) < \Delta(H_1) = \Delta(G) = \Delta$, then H_1 is Type 2, and so is G, and this concludes the proof of (ii). By the way, remark that H_2 can only be Type 2 if $a = e$, in which case we can swap the roles of A and E and of B and D, and then we are reduced to (ii).

It remains to prove (iii). In this case, as $b+d = c+a \leq ad+min(a, d)$ holds, H_1 is Type 1. Furthermore, because $a > d$, we have $ad+min(a, d) = ad + d$, and thus $ad \geq b$. It remains now to extend the $(\Delta + 1)$-total colouring ϕ of H_1 to a $(\Delta + 1)$-total colouring of G. Let $X = G[D \cup E]$ (see Fig. 3.1b). For each $v \in D$, the set $\{\phi(uv) : v \in D\} \cup \{\phi(v)\}$ is the set of the $c+1$ colours not available to colour the edges of X incident to v. We can colour the vertices in E by choosing an arbitrary colour used to colour some vertex in C and assigning it to all vertices in E. Now, for every vertex $v \in D$, we have a set $L(v)$ with at least $\Delta+1-(c+1)-1 = a-1 \geq max(d, e)$ colours that may be used to colour the edges of X incident to v. Assigning the list $L(v)$ to all edges of X incident with v, each such list has $a - 1 \geq max(d, e) = \Delta(X)$ colours. Then, we obtain a colouring of all edges of X from *Galvin's theorem on edge choosability* [5]. □

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