Matemática
Contemporânea
Vol. 55, 96–104
http://doi.org/10.21711/231766362023/rmc5511

A short proof of the bijection between minimal feedback arc sets and Hamiltonian paths in tournaments

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Dedicated to Professor Jayme Szwarcfiter
on the occasion of his 80th birthday

Abstract. We present an alternative proof that in any tournament there is a bijection between its minimal feedback arc sets and its Hamiltonian paths.

Keywords: Minimal feedback arc set, Hamiltonian path, Tournament

2020 Mathematics Subject Classification: 05C20, 05C38.

1 Introduction

A digraph D is a pair $D = (V, A)$, where V is a finite set of *vertices*, and $A \subseteq V \times V$ is a set of ordered pairs of vertices, called *arcs*. For ease of

This research has been partially supported by Coordenação de Aperfeiçoamento de Pessoal de Nível Superior – Brasil – CAPES – Finance Code 001. F. Botler is supported by CNPq (423395/2018-1 and 304315/2022-2), FAPERJ (211.305/2019 and 201.334/2022), CAPES-PRINT (88887.695773/2022-00), and UFRJ (23733). CNPq is the National Council for Scientific and Technological Development of Brazil. FAPERJ is the Rio de Janeiro Research Foundation.

notation, we write uv for the arc (u,v). Given a digraph $D=(V,A)$, the *outdegree* (resp. *indegree*) of $v \in V$, denoted by $d^+(v)$ (resp. $d^-(v)$), is the number of arcs leaving (resp. entering) v, and the *minimum outdegree* of D, denoted by $\delta^+(D)$, is the minimum $\min\{d^+(u) : u \in V(D)\}$. Simmilarly, the *minimum indegree* of D is defined as $\delta^-(D) = \min\{d^-_D(u) : u \in V(D)\}$. A *path* is a digraph P that admits an ordering $v_0 \cdots v_\ell$ of its vertices for which $A(P) = \{v_i v_{i+1} : i = 0, \ldots, \ell-1\}$. We write $P = v_0 \cdots v_\ell$ to make this order explicit. The vertex v_0 (resp. v_ℓ) is called the *initial vertex* (resp. *final vertex*) of P, and the remaining vertices of P are called *internal vertices*. Given distinct vertices u, v of a digraph D, a *uv-path* is a path $P \subseteq D$ with initial vertex u and final vertex v; and a path $P \subseteq D$ is called a *Hamiltonian path* of D if $V(P) = V(D)$.

A *cycle* is a digraph C obtained from a path $v_0 \cdots v_\ell$ by adding the arc $v_\ell v_0$. Finally, a set of arcs $F \subseteq A(D)$ is a *feedback arc set*[1] of D if $D - F$, the digraph obtained from D by removing the arcs in F, does not contain cycles; and a feedback arc set F is *minimal* if there is no feedback arc set F' such that $F' \subsetneq F$. We remark that distinct minimal feedback arc sets may have distinct sizes (see Figure 1). We observe that if F is a minimal feedback arc set of a digraph D, then for every arc $uv \in F$, there is a vu-path in $D-F$. Indeed, let $uv \in F$. By the minimality of F, the set $F' = F - uv$ is not a feedback arc set, and hence $D - F' = (D-F) + uv$ contains a cycle C. Since F is a feedback arc set, $C \not\subseteq D - F$. Therefore, $uv \in A(C)$ and $C - uv$ is a vu-path in $D - F$.

In the figures throughout the paper, we use light blue (resp. orange) to highlight Hamiltonian paths (resp. minimal feedback arc sets). We refer to Bang-Jensen and Gutin [1] for undefined terms.

In particular, minimal feedback arc sets have relations to the well-known Cacceta-Häggkvist Conjecture [3] and other correlated problems, as the following conjecture, posed by Hoáng and Reed [5].

Conjecture 1 (Hoáng – Reed, 1987)**.** *Every digraph D with $\delta^+(D) \geq k$ contains k cycles $C_1, \ldots, C_k$ such that C_j intersects $\cup_{i=1}^{j-1} V(C_i)$ in at most*

[1] It is common to find in the literature the acronym *FAS* for feedback arc sets.

Figure 1: Two minimal feedback arc sets in a digraph D highlighted in orange. Drains are ilustrated by red vertices.

one vertex for $2 \leq j \leq k$*.*

Thomassen [6] proved that every digraph with minimum outdegree at least 2 has two cycles that intersect at a unique vertex, which settles the case $k = 2$ of Conjecture 1. Welhan [7] verified Conjecture 1 in the case $k = 3$ using a connectivity structure called *nearest separators*, and Havet, Thomassé and Yeo [4] checked Conjecture 1 for tournaments, where a *tournament* is a digraph obtained from a complete graph by orienting its edges. In particular, one can show that every digraph D containing a minimal feedback arc set F that has a path of length $\delta^+(D)$ in $D[F]$ satisfies Conjecture 1.

We are interested in the relation between minimal feedback arc sets and Hamiltonian paths in tournaments. In 1988, Bar-Noy and Naor [2] showed that in any tournament there is a bijection between its minimal feedback arc sets and its Hamiltonian paths. In this paper we present a short and elementary proof of this result. For that, given a digraph D, we denote by $\mathcal{P}_D$ (resp. $\mathcal{F}_D$) the set of Hamiltonian paths (resp. minimal feedback arc sets) in D.

Theorem 2 (Bar-Noy – Naor, 1988)**.** *If D is a tournament, then*

$$|\mathcal{P}_D| = |\mathcal{F}_D|$$

2 An alternative proof of Theorem 2

In this section, we present an alternative proof of Theorem 2. Let $P = v_1 \cdots v_n$ be a Hamiltonian path in a tournament D. An arc v_iv_j is

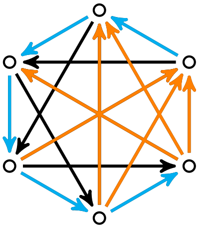

Figure 2: A minimal feedback arc set obtained from the backward arcs.

called a *backward arc* (with respect to P) if $i > j$. Otherwise, we say that $v_i v_j$ a *forward arc*. Note that, if $v_i v_j$ is a backward arc, then $P \cup v_i v_j$ contains a cycle. Therefore, the set F of backwards arcs with respect to P is a minimal feedback arc set in D (see Figure 2). In this case, we say that P *induces* the minimal feedback arc set F. This gives us the following result.

Proposition 3. *Every Hamiltonian path in a tournament D induces a minimal feedback arc set.*

Denote by i the function $\mathrm{i}\colon \mathcal{P}_\mathrm{D} \to \mathcal{F}_\mathrm{D}$ in which $\mathrm{i(P)}$ is the minimal feedback arc set induced by P. Our proof is divided into two parts. In Proposition 4 we show that i is injective, and in Theorem 7 we prove that i is surjective.

Proposition 4. *Let D be a tournament and P_1, P_2 be two Hamiltonian paths in D. If $\mathrm{i(P_1)} = \mathrm{i(P_2)}$, then $P_1 = P_2$.*

Proof. Let P_1 and P_2 be distinct Hamiltonian paths in D, and put $F_1 = \mathrm{i(P_1)}$ and $F_2 = \mathrm{i(P_2)}$. We prove that $F_1 \neq F_2$. Let v be the first vertex that differs the ordering of P_1 from the ordering of P_2, that is, if $P_1 = v_0 \cdots v_i v_{i+1} \cdots v_j \cdots v_n$ and $P_2 = v_0 \cdots v_i v_j \cdots v_l$, then, $v = v_{i+1}$. By the definition of P_1 and P_2, we have $v_i v_{i+1}, v_i v_j \in A(D)$. Now, suppose that $v_{i+1} v_j \in A(D)$. In this case, $v_{i+1} v_j$ is a forward arc with respect to P_1, and a backward arc with respect to P_2, which implies that $v_{i+1} v_j \notin F_1$ and $v_{i+1} v_j \in F_2$, and hence $F_1 \neq F_2$, as desired (see Figure 3). Thus, we may assume that $v_jv_{i+1} \in A(D)$. In this case, v_jv_{i+1} is a backward arc with respect to P_1, and a forward arc with respect to P_2, which implies that $v_{i+1}v_j \in F_1$ and $v_{i+1}v_j \notin F_2$, and hence $F_1 \neq F_2$, as desired (see Figure 3). □

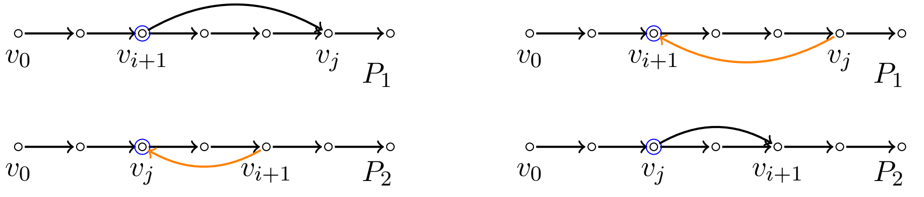

Figure 3: Left: the case $v_{i+1}v_j \in A(D)$, in which $v_{i+1}v_j \in F_2$ and $v_{i+1}v_j \notin F_1$. Right: the case $v_jv_{i+1} \in A(D)$, in which $v_jv_{i+1} \in F_1$ and $v_jv_{i+1} \notin F_2$.

From now on, given a vertex v in a tournament D we denote by $A^+(v)$ (resp. $A^-(v)$) the set of arcs of D leaving (resp. entering) v. In the rest of the paper we prove that i is surjective. For that, we need two auxiliary results. The following proposition shows that for every minimal feedback arc set F in a tournament there is a vertex v with the following property: The set of arcs in F that are incident to v is precisely $A^-(v)$. Formally, we say that a vertex $v \in V(D)$ is a *drain*[2] of F in D if $F \cap (A^+(v) \cup A^-(v)) = A^-(v)$ (see Figure 1).

Proposition 5. *Let F be a minimal feedback arc set in a tournament D. Then there is precisely one drain of F in D.*

Proof. Let F and D be as in the statement, and let $S = \{v \in V(D) : A^-(v) \subseteq F\}$. First, we prove that if $v \in S$ and $vu \in A^+(v)$, then $vu \notin F$. This implies that every vertex in S is a drain of F. Suppose $vu \in F$, and let $F' = F - vu$. By the minimality of F, there is a cycle C in $D - F'$, but

[2]The word "drain" was chosen to avoid confusions with the usual definition "sink", used for vertices with outdegree 0.

then C must contain vu, otherwise F would not be a minimal feedback arc set of D. Thus, $D - F'$ contains an arc entering v, a contradiction since $A^-(v) \subseteq F$.

It remains to prove that $|S| = 1$. If $S = \emptyset$, then every vertex has an incoming arc that does not belong to F, and hence $\delta^-(D - F) \geq 1$. Then there is a cycle in $D - F$, a contradiction to the definition of feedback arc set. Now, suppose that there are two distinct vertices in S, say v and u. Suppose, without loss of generality, that $vu \in A(D)$. Since $u \in S$, we have $vu \in F$. On the other hand, as proved above, since $v \in S$ and $vu \in A^+(v)$, we have $vu \notin F$, a contradiction. □

The next proposition is an important step in our proof of Theorem 2.

Proposition 6. *Let F be a minimal feedback arc set in a tournament D, and let v be the drain of F. Then F induces a minimal feedback arc set on $D - v$.*

Proof. Let F, D, and v be as in the statement. Put $D' = D - v$ and let $F' = F \cap A(D')$. If F' is not a feedback arc set in D', then there is a cycle in $D' - F' \subseteq D - F$, a contradiction to F being a feedback arc set. Now, suppose that F' is not a minimal feedback arc set of D'. Then there is an arc $xy \in F'$ such that $F' - xy$ is a feedback arc set of D'. By the minimality of F, $F - xy$ is not feedback arc set. Thus, there is a yx-path P in $D - F$, while there is no yx-path in $D' - F'$. Therefore P must contain v as an internal vertex, a contradiction because $D - F$ has no arcs in $A^-(v)$. □

Recall that given a Hamiltonian path P in a tournament D, the set of backward arcs of P, $\mathrm{i(P)}$, is the minimal feedback arc set induced by P. The following result says that i is a surjective function, concluding our proof of Theorem 2.

Theorem 7. *Let F be a minimal feedback arc set in a tournament D. Then there is a Hamiltonian path P in D whose starting vertex is the drain of F, and such that* $\mathrm{i(P) = F}$.

Proof. The proof follows by induction on the number n of vertices of D. If $n = 1$ the result is trivial, so we assume $n \geq 2$. By Proposition 5, there is precisely one drain of F in D, say v_0. Put $D' = D - v_0$ and $F' = F \cap A(D')$. By Proposition 6, F' is a minimal feedback arc set of D'. Now, let $v_1 \in V(D')$ be the drain of F', which (again) exists by Proposition 5. By the induction hypothesis, there is a Hamiltonian path P' in D' whose starting vertex is v_1 and, moreover, F' is the minimal feedback arc set induced by P'. We claim that $v_0v_1 \in A(D)$. Indeed, suppose that $v_0v_1 \notin A(D)$. Since D is a tournament, we have $v_1v_0 \in A(D)$, and, in particular, $A^-_{D'}(v_1) = A^-_D(v_1)$. In this case, as $A^-(v_0) \subseteq F$ and $v_1v_0 \in A^-(v_0)$, we have $v_1v_0 \in F$. By the minimality of F, there is a v_0v_1-path in $D - F$, a contradiction because $A^-(v_1) \subseteq F' \subseteq F$. Now, since P' is a Hamiltonian path in D', $P = v_0P'$ is a Hamiltonian path in D (see Figure 4). Moreover, since v_0 is the starting vertex of P, every arc in $A^-(v_0)$ is a backward arc of P, and since $F - A^-(v_0) = F'$, every other arc in F is a backward arc of P', and hence is a backward arc of P. Therefore, F is the minimal feedback arc set induced by P as desired. □

3 Conclusion

In this paper we present an alternative and short proof of the bijection between the minimal feedback arc sets and the Hamiltonian paths in a tournament. This proof yields a polynomial time algorithm that, given a feedback arc set F, returns a Hamiltonian path by repeatedly finding and removing a drain. We wonder whether drains can be used to find other interesting structures in tournaments as, for example, a minimal feedback arc set that contains a sufficiently long path, which would yield a strengthening of the result in [4].

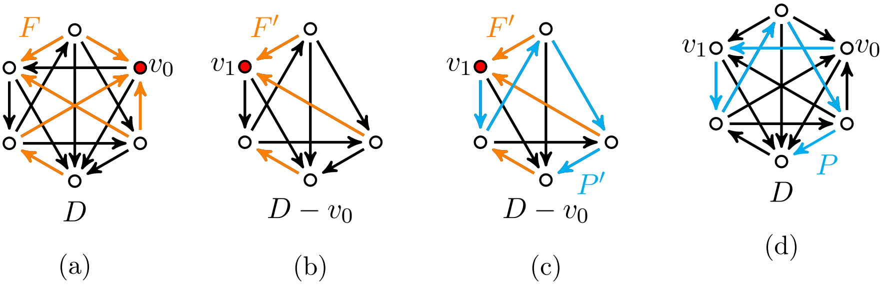

Figure 4: The procedure described in the proof of Theorem 7 on a tournament D. Drains are depicted as red vertices. (a) A minimal feedback arc set F and its drain v_0; (b) A minimal feedback arc set F' in $D - v_0$ and its drain v_1; (c) A Hamiltonian path P' in $D - v_0$ whose starting vertex is v_1. (d) A Hamiltonian path P in D obtained form P' by adding v_0v_1.

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