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Study on $(r+1)$-role assignments of complementary prisms, with $r \geq 3$

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Dedicated to Professor Jayme Szwarcfiter on the occasion of his 80th birthday

Abstract. Nowadays, a social network is a source of a huge amount of information. In this way, graphs constitute a powerful tool in which the vertices represent individuals and the edges represent relationships between them. To improve the understanding of such a graph, we study the concept of role assignment. Indeed, an *r-role assignment* of a simple graph G is an assignment of r distinct roles to the vertices of G, such that two vertices with the same role have the same set of roles in the related vertices. Furthermore, a specific r-role assignment defines a *role graph*, in which the vertices are the distinct r roles, and there is an edge between two roles whenever there are two related vertices in the graph G that correspond to these roles. We consider complementary prisms, which are graphs formed from the disjoint union of the graph with its respective complement, adding the edges of a perfect matching between their corresponding vertices. In this work, we consider $r \geq 3$, and the role graph $K'_{1,r}$ which is the bipartite graph $K_{1,r}$ with a loop at the vertex of degree r. We conclude that the problem of deciding whether a complementary prism has an $(r+1)$-role assignment, when the role graph is

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$K'_{1,r}$, is NP-complete. We conjecture that, for $r \geq 3$, the problem of deciding whether a complementary prism has an $(r+1)$-role assignment, is NP-complete.

Keywords: role assignment, complementary prism, NP-completeness.

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1 Introduction

In recent decades, social networks such as Facebook, Twitter and Instagram have become part of our daily lives. Up to study their behavior, a social network is conceptualized as a graph. In this context, it is essential to understand such a graph through a smaller graph. Hence, in 1980, Angluin introduced the concept of *covering* from which role assignment arise, as a tool for networks of processors [1]. A decade later, based on graph models for social networks, Everett and Borgatti [6] formalized role assignment under the name of *role coloring*.

A *graph* G is a pair $(V(G), E(G))$, where $V(G)$ is the set of vertices and $E(G)$ is the set of edges. The vertices u and v are *adjacent* or *neighbors* if they are joined by an edge e, also denoted by uv. A *loop* is an edge incident to only one vertex. The *neighborhood* of a vertex v, denoted by $N_G(v)$, is the set of all neighbors of v in G. A *simple graph* is a graph without loops. In a simple graph G, the *degree* of a vertex v is the cardinality of $N_G(v)$. A *leaf* is a vertex of degree 1. The neighborhood of a subset U of $V(G)$, denoted by $N_G(U)$, is the union of the neighborhoods of the vertices of U.

Given a simple graph G and a graph R, possibly with loops, *R-role assignment* of G is a surjective vertex mapping $p : V(G) \to V(R)$ such that $p(N_G(v)) = N_R(p(v))$ for all $v \in V(G)$, where $p(S)$ denotes $\{p(v) : v \in S\}$ for any vertex set S. A graph G has an *r-role assignment* if it admits an R-role assignment for some graph R, called the *role graph*, with $|V(R)| = r$. We set $1, \ldots, r$ as the vertices of R, also called *roles*. From now on, all graphs (except maybe the role graph) are simple. Note that, the role graph has no multiple edges, but permits loops since two related vertices in G

can have the same role. We emphasize that, while a social network usually gives rise to a large graph, a role assignment allows representing the same network through a smaller graph.

We consider the r-Role Assignment problem as follows:

r-Role Assignment
Instance: A simple graph G.
Question: Does G admit an r-role assignment?

Observe that, if the graph G is connected, then the role graph R of any role assignment of G is also connected. Also, if the role graph R is bipartite, then so is G. A graph G is *bipartite* if we can partition $V(G) = A \cup B$ so that if there is an edge $uv \in E(G)$, then $u \in A$ and $v \in B$, or vice versa. In this case, we say that (A, B) is a *bipartition* of G. A *clique* is a subset of vertices such that every two distinct vertices are adjacent. The *Levi graph* of H, defined as $L(H) = (V(L(H)), E(L(H)))$, is a bipartite graph, such that $V(L(H)) = V \cup E$ and $E(L(H)) = \{ue, ve \mid e \in E$ such that $e = uv\}$.

Applications of role assignment are highlighted in several contexts such as social networks, see Everett and Borgatti [6], Roberts and Sheng [12], and distributed computing, see Chalopin, Métivier, and Zielonka [4].

In 2001, Roberts and Sheng [12] proved the NP-completeness of 2-Role Assignment. That result was extended by Fiala and Paulusma [7], in 2005, by showing that r-Role Assignment is NP-complete for any fixed $r \geq 3$. In this context, Purcell and Rombach [11], established that r-Role Assignment, with $r \geq 2$, remains NP-complete for planar graphs, while the cographs always have r-role assignment, which makes the complexity of the problem constant for this class.

Considering chordal and split graphs, a dichotomy for the complexity of r-Role Assignment arises. While for chordal graphs, the problem is solvable in linear time for $r = 2$ and NP-complete for $r \geq 3$, given by van 't Hof *et al.* [13]; for split graphs, the problem is trivial, with true answer, for $r = 2$, solvable in polynomial time for $r = 3$ and NP-complete for any fixed $r \geq 4$, set by Dourado [5].

The complementary prism is linked to the notion of complementary product, introduced in the literature in 2007, by Haynes *et al.* [9] as a generalization of the Cartesian product. The authors give special attention to the particular case of the operation called the *complementary prism* of a graph, which can be seen as a variant of a *prism*, the Cartesian product of a graph with K_2.

The *complementary prism* of G, denoted by $G\overline{G}$, is the graph formed from the disjoint union of G and its complement $\overline{G}$, adding the edges of a perfect matching between the corresponding vertices of G and $\overline{G}$. For sake of simplicity, we designated by G and $\overline{G}$, the copies of G and $\overline{G}$, respectively, in $G\overline{G}$. Furthermore, for a vertex v of G, we denote by $\overline{v}$ the corresponding vertex in $\overline{G}$ and, for a set $X \subseteq V(G)$, $\overline{X}$ denote the corresponding set of vertices in $V(\overline{G})$.

In 2018, Castonguay *et al.* [2] characterized complementary prisms that admit a 2-role assignment, by showing that only the complementary prism of a P_3 does not have one. In 2022, Mesquita [10] gives a proof for complementary prisms with a 3-role assignment, and a list of those without one, composed of complementary prism of some disconnected bipartite graphs, concluding that the problem 3-Role Assignment for complementary prisms is linear-time solvable.

In this paper, we conjectured that, for $r \geq 3$, $(r+1)$-Role Assignment for complementary prisms is NP-complete. In this sense, we considered the role graph $K'_{1,r}$ which is the bipartite graph $K_{1,r}$ with a loop at the vertex of degree r and we showed that the problem of deciding whether a complementary prism has an $(r+1)$-role assignment, when the role graph is $K'_{1,r}$, is NP-complete.

2 Results

We recall that $K'_{1,r}$ is the bipartite graph $K_{1,r}$ with a loop at the vertex of degree r. Motivated by the constructions given by van 't Hof *et al.* [13], Dourado [5] and Castonguay *et al.* [3], we propose a new construction to show that the decision problem related to finding a $K'_{1,r}$-role assignment, with $r \geq 3$, for complementary prism is NP-complete. For that, we will use the k-coloring problem which is NP-complete, for $k \geq 3$ [8].

k-COLORING
Instance: A graph G and a positive integer k.
Question: Does the graph G admits a mapping $c\colon V(G) \rightarrow \{1, \ldots, k\}$ such that $c(u) \neq c(v)$, for every edge $uv \in E(G)$?

Starting from an instance of r-COLORING, that is, a graph H and a positive integer r, we present Construction 1, to obtain a simple graph denoted by $G_{r+1}(H)$. The graph $G_{r+1}(H)$ will be used to build the complementary prism $G_{r+1}(H)\overline{G_{r+1}(H)}$.

Construction 1: Given a graph $H = (V(H), E(H))$ and $r \geq 3$. Consider $V = V(H)$ and $E = E(H)$. We construct $G_{r+1}(H)$, from the Levi graph $L(H)$. Observe that (V, E) is a bipartition of $L(H)$.

- For each vertex $e \in E$ in $L(H)$, we add a set V_e of $r - 3$ vertices and add edges ue, for every vertex of V_e. We denote $V_E = \bigcup_{e \in E} V_e$. We observe that the vertices of V_E are leaves in $G_{r+1}(H)$ and when $r = 3$, we have that $V_E = \emptyset$;
- We denote by $I = V \cup V_E$ and $C = E$ and add all edges to make C a clique.

An example of Construction 1 with $r = 4$, follows in Figure 2.1, where $H = K_4$. In order to have a better understanding of the graph, some edges are omitted. In this sense, the double line between two highlighted subgraphs means that we have all possible edges between them. On the other hand, double crossed lines between $\overline{E}$ and $\overline{V_E}$ represent all possible edges between the subgraphs without the perfect matching given between E and V_E. Also, three spaced lines between the highlighted subgraphs $G_5(K_4)$ and $\overline{G_5(K_4)}$ designate the perfect matching between these subgraphs. Finally, the subgraphs highlighted by a square are cliques. The labels inside the vertices stand for the roles in a $K'_{1,4}$-role assignment for $G_5(K_4)\overline{G_5(K_4)}$.

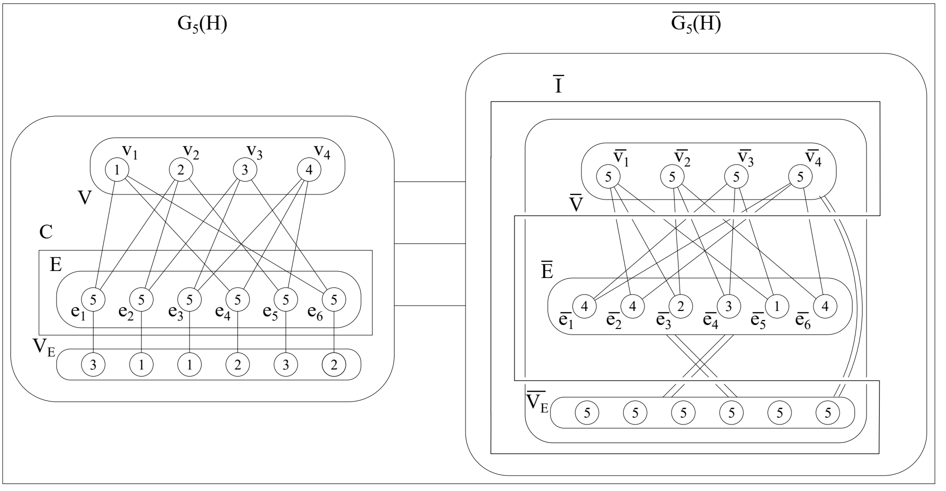

Figure 2.1: The complementary prism of $G_5(K_4)$.

First, we exhibit, in the following theorem, how to obtain a $K'_{1,r}$-role assignment of $G_{r+1}(H)\overline{G_{r+1}(H)}$, with $r \geq 3$, when H admits an r-coloring.

Theorem 2.1. *Let H be a graph, with $r \geq 3$, and $G_{r+1}(H)$ the graph obtained from Construction 1. If H has an r-coloring, then the complementary prism of $G_{r+1}(H)$ has a $K'_{1,r}$-role assignment.*

Proof. Let $c : V(H) \to \{1, \ldots, r\}$ an r-coloring of H. As before, consider $V = V(H)$ and $E = E(H)$. To facilitate the definition of a role assignment, we define $\ell : E \cup V_E \to \{1, \ldots, r\}$ as follows. For each $e \in E$, with $e = uv$, we have that $c(u) \neq c(v)$, since c is a coloring. So $\{1, \ldots, r\} - \{c(u), c(v)\}$ has cardinality $r - 2$. Since $\{e\} \cup V_e$ also has cardinality $r - 2$, we can randomly choose $\ell(e)$ and $\ell(x)$, for all $x \in V_e$, such that $\{\ell(e)\} \cup \{\ell(x) \mid x \in V_e\} = \{1, \ldots, r\} - \{c(u), c(v)\}$. For $e \in E$, we must be aware that, while $\ell(x)$ is used to define the roles of the vertices of V_e, $\ell(e)$ defines the role of $\overline{e}$. Remember that $I = V \cup V_E$ and $C = E$. We define $p : V(G_{r+1}(H)\overline{G_{r+1}(H)}) \to \{1, \ldots, r\}$, for $x \in V(G_{r+1}(H)\overline{G_{r+1}(H)})$, as follows:

$$p(x) = \begin{cases} c(x), & \text{if } x \in V; \\ \ell(x), & \text{if } x \in V_E \cup \overline{E}; \\ r+1, & \text{if } x \in C \cup \overline{I}. \end{cases}$$

It is easy to see that p is a $K'_{1,r}$-role assignment, with $r \geq 3$. □

In the next theorem, we obtain an r-coloring of H from a $K'_{1,r}$-role assignment of the complementary prism of $G_{r+1}(H)$, with $r \geq 3$.

Theorem 2.2. *Let H be a graph, with $r \geq 3$ and $G_{r+1}(H)$ the graph obtained from Construction 1. If the complementary prism of $G_{r+1}(H)$ has a $K'_{1,r}$-role assignment, then H has an r-coloring.*

Proof. Let p be a $K'_{1,r}$-role assignment of the complementary prism of $G_{r+1}(H)$, with $r \geq 3$. We may suppose, without loss of generality, that all pendant vertices in $V(K'_{1,r})$ are numbered as $1, \ldots, r$ and the vertex with a loop numbered as $r+1$.

We show that $p(E) = \{r+1\}$. Although, we only need to show for $r = 4$, we will do this in a general way. We can assume, by contradiction, that there is $e \in E$ with $e = uv$, such that $p(e) \neq r+1$. We get that $p(u) = p(v) = r+1$. Since E forms a clique on $G_{r+1}(H)$, then $p(E - \{e\}) = \{r+1\}$, if $|E| \geq 1$, or $p(E - \{e\}) = \emptyset$, otherwise. In both cases, we conclude that $p(N(u)) \subseteq \{p(\overline{u}), p(e), r+1\}$, as $r \geq 3$, we have a contradiction, since $|N_R(r+1)| = r+1 \geq 4$.

Let $v \in V$. Since $p(N(v)) = \{p(\overline{v}), r+1\}$, we have that $p(v) \in \{1, \ldots, r\}$. We define $c : V(H) \to \{1, \ldots, r\}$ by $c(x) = p(x)$ for all $x \in V$. We show that c is an r-coloring of H. Let $e = uv$ be an edge of H. Since $p(N(e)) = \{p(\overline{e}), p(u), p(v), r+1\} \cup p(V_e)$ and $|\{\overline{e}, u, v\} \cup V_e| = r$, we have that $p(u) \neq p(v)$, in other words $c(u) \neq c(v)$. □

The above results directly imply the NP-completeness of deciding whether a complementary prism admits a $K'_{1,r}$-role assignment, with $r \geq 3$.

Theorem 2.3. *For $r \geq 3$, $(r+1)$-*Role Assignment *with fixed role graph $K'_{1,r}$ is* NP*-complete for complementary prisms.*

Proof. The problem is clearly in NP, see Roberts and Sheng [12]. To show the NP-completeness, we use a reduction from r-Coloring with $r \geq 3$, which is NP-complete by Garey and Johnson [8]. The result follows from Theorems 2.1 and 2.2. □

We observe that, with $r \geq 3$, Construction 1 cannot be used to demonstrate that $(r+1)$-Role Assignment for complementary prisms is NP-complete, for more details see Mesquita [10]. As seen in Introduction, for all studied classes of graphs in the literature, with the exception of cographs, $(r+1)$-Role Assignment in the respective class is NP-complete with $r \geq 3$. From this fact, and the previous result, we set the following conjecture:

Conjecture. *For $r \geq 3$, $(r+1)$-*Role Assignment *for complementary prisms is* NP*-complete.*

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