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Stochastic differential equations driven by small general Gaussian noise: Non parametric estimation

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Abstract. In this short note, we consider the problem of nonparametric estimation of the trend in a type of stochastic differential equation with small noise given by a general Gaussian noise. We consider the case of continuos and discrete time observations.

Keywords: general Gaussian, fractional processes, nonparametric estimation, small noise.

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1 Introduction

The problem of parameter estimation in stochastic differential equations (SDEs) driven by long memory processes is a relatively new and

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interesting topic of research. There has been an increasingly interest in this type of problem. For example, we can mention [3, 4, 5] among others. In there, the authors studied the problem of parameter estimation in a SDE driven by fractional Brownian motion in different contexts. Also, we can mention the works [1, 2]. In here, the authors consider the problem of drift parameter estimation in a linear SDE driven by general Gaussian process.

In this short note, motivated by the works, [1], [2] [6] and [9] we deal with the problem of nonparametric estimation in a non-linear SDE driven by small general Gaussian process. First, based on continuous time observations, we prove that our nonparametric estimator is uniformly consistent. Then, by means of an Euler type approximation of the solution of the SDE we provide a new discrete time estimator. This estimator converges, under some additional conditions on the mesh of the partitions related to the Euler scheme, to the true value of the parameter.

This paper is organized as follows. In Section 2 we give a brief introduction to the general Gaussian noise, its main properties and stochastic integration. In Section 3, we establish the main results of this paper, i.e. the consistency and the rate of convergence of the proposed estimator is shown. Finally, in Section 4 we study the convergence of the discrete version of the proposed estimator.

2 Preliminaries

We briefly recall some relevant aspects related to the noise of the equation, its main properties and stochastic integration.

2.1 General Gaussian noise

In a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we introduce the process $G = (G_t, t \in [0, T])$ as a continuous centered Gaussian process with a continuos covariance function R given by

$$E(G_tG_s) = R(s,t), \qquad t,s \in [0,T].$$

In the rest of the manuscript we will assume that the covariance of the Gaussian process G satisfies the following Hypothesis:

(H): For $\beta \in (1/2,1)$, the covariance function $E(G_tG_s) = R(s,t)$ for any $t \neq s \in [0,\infty)$ satisfies

$$\frac{\partial^2}{\partial t \partial s} R(t,s) = C_\beta |t-s|^{2\beta-2} + \Psi(t,s),$$

with

$$|\Psi(t,s)| \leq C_\beta^* |ts|^{\beta-1},$$

where the constants $C_\beta > 0$, $C_\beta^* \geq 0$ do not depend on T. Moreover, for any $t \geq 0$, $R(0,t) = 0$.

Remark 2.1. Some of the Gaussian processes that fulfill the previous hypothesis, for $H > 1/2$, are fractional Brownian motion, sub-fractional Brownian motion, bi-fractional Brownian motion and generalized sub-fractional Brownian motion (see [10, 8] for details).

An important consequence of the previous hypothesis on R is the following lemma (see [2] for the proof)

Lemma 2.2. *There exits $C_\beta > 0$ a constant independent of T, such that for all $s,t \geq 0$,*

$$E[(G_t - G_s)^2] \leq C_\beta |t-s|^{2\beta}$$

and when $s = 0$, we can obtain $E[(G_t)^2] \leq C_\beta^* t^{2\beta}$.

Let $\mathcal{E}$ denote the space of all real step function on [0,T] and $\mathcal{H}$ is defined as the closure of $\mathcal{E}$ endowed with the inner product

$$\langle 1_{[a,b)}, 1_{[c,d)} \rangle_{\mathcal{H}} = E[(G_b - G_a)(G_c - G_d)].$$

$G = \{G(h),\ h \in \mathcal{H}\}$ is an isonormal Gaussian process, indexed by the elements in the Hilbert space $\mathcal{H}$, i.e., the Gaussian family G of random variables fulfill

$$E(G(g)G(h)) = \langle g, h\rangle_{\mathcal{H}}, \qquad \forall g, h \in \mathcal{H}.$$

Now, we present a usefull result presented in [1], that we will need in the following

Proposition 2.3. *Let us denote $\nu_{[0,T]}$ as the set of bounded variation functions on $[0,T]$. Then, $\nu_{[0,T]}$ is dense in $\mathcal{H}$ and we have*

$$\langle f, g\rangle_{\mathcal{H}} = \int_{[0,T]^2} R(t,s)\nu_f(dt)\nu_g(ds), \qquad \forall f, g \in \nu_{[0,T]},$$

where ν_g is the Lebesgue - Stieljes signed measure associated with g^0 defined as

$$g^0(x) = \begin{cases} g(x) & if \quad x \in [0,T] \\ 0 & \qquad otherwise. \end{cases}$$

Furthermore, for the property of $R(t,s)$, we have

$$\langle f, g\rangle_{\mathcal{H}} = \int_{[0,T]^2} f(t)g(s)\frac{\partial^2}{\partial t \partial s}R(t,s)dtds, \qquad \forall f, g \in \nu_{[0,T]}.$$

2.2 The model

Let us consider the following stochastic differential equation

$$dX_t = b(X_t)dt + \varepsilon dG_t, \quad X_0 = x_0, \quad 0 \le t \le T, \tag{2.1}$$

where $b(\cdot)$ is suppose to be unknown. Also, let us assume that x is the solution of the ordinary differential equation given by

$$\frac{dx_t}{dt} = b(x_t), \quad x_{t_0} = x_0, \quad 0 \le t \le T. \tag{2.2}$$

Here, we need to estimate the function $b_t = b(x_t)$ based on observations of $\{X_t,\ 0 \le t \le T\}$. To do this we will assume the following condition

$$(H1) \quad |b(x) - b(y)| \le L|x - y|, \quad x, y \in \mathbb{R},$$

where $L > 0$. Furthermore, by assumption $(H1)$ we can obtain $|b(x)| \leq M(1+|x|)$ with $M = \max\{b(0), L\}$). Also, let assume that b is bounded above by a non random positive constant C, then

$$|b(x_t) - b(x_s)| \leq L|x_t - x_s| \leq L|\int_s^t b(x_r)dr| \leq CL|t-s|, \quad t,s \in \mathbb{R}, \tag{2.3}$$

where, we have used that x satisfies equation (2.2).

Now, we introduce the following lemma that will be needed throughout the paper.

Lemma 2.4. *Let X_t and x_t be given by (2.1) and (2.2), respectively. Additionally, assume that $b(\cdot)$ satisfies condition (H1) and G satisfies hypothesis (H). Then,*

$$|X_t - x_t| \leq e^{LT}\varepsilon|G_t|$$

and

$$\sup_{0\leq t\leq T} E|X_t - x_t|^2 \leq e^{2LT} C^*_\beta \varepsilon^2 T^{2\beta},$$

where $\beta \in (1/2, 1)$ and C^*_β is a finite constant.

Proof. Let us defined $f_t := |X_t - x_t|$, then by (2.1) and (2.2), we have

$$f_t \leq \int_0^t |b(X_r) - b(x_t)|dr + \varepsilon|G_t| \leq e^{Lt}\varepsilon|G_t|,$$

where the last inequality comes from the Gronwall's lemma. Continuing, we use the previous result and Lemma 2.2 to obtain that

$$\sup_{0\leq t\leq T} E|X_t - x_t|^2 \leq e^{2LT}\varepsilon^2 \sup_{0\leq t\leq T} E|G_t|^2 \leq e^{2LT} C^*_\beta \varepsilon^2 T^{2\beta}.$$

The last inequality is due to Lemma 2.2. Precisely, using that $E[(G_t)^2] \leq C^*_\beta t^{2\beta}$, $t \in [0,T]$ and the fact that $\beta > 0$. □

3 Consistency of the estimator

Let us define by $\Theta_0(L)$ as the class of all functions b, such that the condition $(H1)$ is fulfill and b is uniformly bounded by a constant C. In a similar way, we define $\Theta_k(L)$ as the class of all functions b which are k-differentiable ($b^{(k)}$, means k-derivative of b) with

$$|b^{(k)}(x) - b^{(k)}(y)| \leq L|x-y|, \quad x, y \in \mathbb{R},$$

as before, we also assume that b is uniformly bounded by C.
To construct our estimator of b we need to define a bounded function $K(u)$ with finite support given by $[a,b]$ satisfying the following condition

$$(H2) \quad K(u) = 0 \quad \text{for} \quad u \notin [a,b] \quad \text{and} \quad \int_a^b K(u)du = 1.$$

Additionally, suppose that the function K satisfies the following conditions (through all the work from now on)

i $\int_{\mathbb{R}} K(u)^2 du < \infty$;

ii $\int_{\mathbb{R}} u^{2(k+1)} K(u)^2 du < \infty$;

iii $\int_{\mathbb{R}} |u|^{\beta-1} K(u) du < \infty, \quad$ with $\quad \beta \in (1/2, 1)$.

Remark 3.1. Some of the kernels that fulfill conditions i to iii are the Epanechnikov, quartic and triangle kernel (see [11] for details).

Motivated by the works Mishra and Prakasa Rao [6] and Prakasa Rao [9] we define the following kernel type estimator for the function $b_t = b(x_t)$ by

$$\hat{b}_t = \frac{1}{h_\varepsilon} \int_0^T K\left(\frac{u-t}{h_\varepsilon}\right) dX_u, \tag{3.1}$$

where $h_\varepsilon \to 0$ with $\varepsilon^2 h_\varepsilon^{-1} \to 0$ as $\varepsilon \to 0$. Also, we write E_b for the expected value when $b(\cdot)$ is the trend function.

We are ready to state the main result of this section.

Theorem 3.2. *Let us assume that $b(x)$ belongs to $\Theta_0(L)$ and that $h_\varepsilon \to 0$ with $\varepsilon^2 h_\varepsilon^{2\beta-2} \to 0$ as $\varepsilon \to 0$. Also, assume that (H), $(H1)$ and $(H2)$ hold. Then, for any $0 \le c \le d \le T$, the estimator $\hat{b}_t$ given by (3.1) is uniformly consistent, i.e.*

$$\lim_{\varepsilon\to 0} \sup_{b(x)\in\Theta_0(L)} \sup_{c\le t\le d} E_b(|\hat{b}_t - b(x_t)|^2) = 0.$$

Proof. By the definition of X_t and $\hat{b}_t$, we can obtain

$$\begin{aligned} E_b[(\hat{b}_t - b(x_t))^2] &= E_b\left[\left(\frac{1}{h_\varepsilon}\int_0^T K\left(\frac{u-t}{h_\varepsilon}\right)(b(X_u) - b(x_u))du \right.\right. \\ &+ \frac{1}{h_\varepsilon}\int_0^T K\left(\frac{u-t}{h_\varepsilon}\right) b(x_u)du - b(x_t) \\ &+ \left.\left. E_b\,\frac{\varepsilon}{h_\varepsilon}\int_0^T K\left(\frac{u-t}{h_\varepsilon}\right) dG_u\right)^2\right] \\ &\le 3E_b\left[\left(\frac{1}{h_\varepsilon}\int_0^T K\left(\frac{u-t}{h_\varepsilon}\right)(b(X_u) - b(x_u))du\right)^2\right] \\ &+ 3E_b\left[\left(\frac{1}{h_\varepsilon}\int_0^T K\left(\frac{u-t}{h_\varepsilon}\right) b(x_u)du - b(x_t)\right)^2\right] \\ &+ \frac{3\varepsilon^2}{h_\varepsilon^2} E_b\left[\left(\int_0^T K\left(\frac{u-t}{h_\varepsilon}\right) dG_u\right)^2\right] = \sum_{j=1}^3 I_j. \end{aligned}$$

First, for I_1, we have

$$\begin{aligned} I_1 &= 3E_b\left[\left(\frac{1}{h_\varepsilon}\int_0^T K\left(\frac{u-t}{h_\varepsilon}\right)(b(X_u) - b(x_u))du\right)^2\right] \\ &= 3E_b\left[\left(\int_{\mathbb{R}} K(z)\,(b(X_{zh_\varepsilon+t}) - b(x_{zh_\varepsilon+t}))dz\right)^2\right] \\ &\le C\int_{\mathbb{R}} K^2(z) \sup_{0\le zh_\varepsilon+t\le T} E(X_{zh_\varepsilon+t} - x_{zh_\varepsilon+t})^2 dz \le C_{k,L,\beta,T}\,\varepsilon^2, \end{aligned}$$

here, we have used the change of variable $z = (u-t)/h_\varepsilon$, (H1) and Lemma

2.4. Now, for I_2

$$\begin{aligned}
I_2 &= 3E_b\left[\left(\frac{1}{h_\varepsilon}\int_0^T K\left(\frac{u-t}{h_\varepsilon}\right)b(x_u)du - b(x_t)\right)^2\right] \\
&= 3L^2E_b\left[\left(\int_{\mathbb{R}} K(z)\left(b(x_{zh_{\varepsilon+t}}) - b(x_t)\right)dz\right)^2\right] \\
&\leq C\left(\int_{\mathbb{R}} K(z)\, zh_\varepsilon dz\right)^2 \leq Ch_\varepsilon^2,
\end{aligned}$$

where, we have used the same change of variables as before, inequality (2.3) and the assumptions over K. Finally, for I_3 we use Proposition 2.3, Lemma 2.2 and, again, the assumptions over K

$$\begin{aligned}
I_3 &= \frac{3\varepsilon^2}{h_\varepsilon^2}E_b\left[\left(\int_0^T K\left(\frac{u-t}{h_\varepsilon}\right)dG_u\right)^2\right] \\
&= C_\beta\frac{3\varepsilon^2}{h_\varepsilon^2}\int_0^T\int_0^T K\left(\frac{u-t}{h_\varepsilon}\right)K\left(\frac{v-t}{h_\varepsilon}\right)|u-v|^{2\beta-2}dvdu \\
&+ C_\beta^*\frac{3\varepsilon^2}{h_\varepsilon^2}\int_0^T\int_0^T K\left(\frac{u-t}{h_\varepsilon}\right)K\left(\frac{v-t}{h_\varepsilon}\right)|uv|^{\beta-1}dvdu \\
&= I_{3.1} + I_{3.2}.
\end{aligned}$$

First, for $I_{3.1}$

$$I_{3.1} \leq 3C_\beta\varepsilon^2h_\varepsilon^{2\beta-2}\int_{\mathbb{R}}\int_{\mathbb{R}} K(u)K(v)|u-v|^{2\beta-2}dvdu$$

Now, for $I_{3.2}$ using the change of variable $\tilde{u} = (u-t)/h_\varepsilon$ and $\tilde{v} = (v-t)/h_\varepsilon$

$$\begin{aligned}
I_{3.2} &\leq 3C_\beta^*\varepsilon^2h_\varepsilon^{2\beta-2}\int_{\mathbb{R}}\int_{\mathbb{R}} K(u)K(v)|uv|^{\beta-1}dvdu \\
&+ C\varepsilon^2h_\varepsilon\left(\int_{\mathbb{R}} K(u)|u|^{\beta-1}du + 1\right).
\end{aligned}$$

Consequently, inequalities for I_1 to I_3 allow us to obtain the result. □

Now, we consider the rate of convergence of $\hat{b}$ in the following theorem

Theorem 3.3. *Let us assume that* $b(x)$ *belongs to* $\Theta_k(L)$ *and that* $h_\varepsilon = \varepsilon^{\frac{1}{k-\beta+2}}$. *Also, assume that (H) and* $(H2)$ *hold, and*

$$(H3) \int_{\mathbb{R}} u^j K(u)du = 0, \quad \text{for} \quad j = 1, 2, \dots k \quad \text{and} \quad \int_{\mathbb{R}} u^{k+1} K(u)du < \infty.$$

Then, for any $0 \leq c \leq d \leq T$, *we have*

$$\limsup_{\varepsilon \to 0} \sup_{b(x) \in \Theta_{k+1}(L)} \sup_{c \leq t \leq d} E_b(|\hat{b}_t - b(x_t)|^2) \varepsilon^{\frac{-2(k+1)}{k-\beta+2}} < \infty.$$

Proof. To prove the result, we need to refine some of the computations in the previous theorem. Precisely, the computation of the upper bound for I_2. To do this, we will use the Taylor's formula, i.e.

$$b(y) = b(x) + \sum_{j=1}^{k} b^{(j)}(x) \frac{(y-x)^j}{j!} + \left[b^{(k)}(z) - b^{(k)}(x) \right] \frac{(y-x)^k}{k!},$$

where $x \in \mathbb{R}$ with $|z - x| \leq |y - x|$. Using Taylor's formula, the expression for x and the upper bound in the previous theorem for I_2 , we have

$$\begin{aligned} I_2 &\leq C \left[\left(\int_{\mathbb{R}} K(z) \left(b(x_{zh_\varepsilon + t}) - b(x_t) \right) dz \right)^2 \right] \\ &= C \left[\sum_{j=1}^{k} b^{(j)}(x_t) \left(\int_{\mathbb{R}} K(u) u^j du \right) h_\varepsilon^j (j!)^{-1} \right. \\ &+ \left. \left(\int_{\mathbb{R}} K(u) u^k (b^{(k)}(z_u) - b^{(k)}(x_t)) du \right) h_\varepsilon^k (k!)^{-1} \right]^2, \end{aligned}$$

for z_u such that $|x_t - z_u| \leq |x_{t+h_\varepsilon u} - x_t| \leq C|h_\varepsilon u|$. Therefore,

$$\begin{aligned} I_2 &\leq C \left[\left(\int_{\mathbb{R}} |K(u) u^{k+1}| h_\varepsilon^{k+1} (k!)^2 du \right)^2 \right] \\ &\leq C h_\varepsilon^{2(k+1)} \int_{\mathbb{R}} K^2(u) u^{2(k+1)} du \leq C h_\varepsilon^{2(k+1)}. \end{aligned}$$

This and the estimates in the previous proof (upper bounds for I_1 and I_3) allow us to obtain

$$\sup_{c \leq t \leq d} E_b(|\hat{b}_t - b(x_t)|^2) \leq C \left(\varepsilon^2 + h_\varepsilon^{2(k+1)} + \varepsilon^2 h_\varepsilon^{2\beta - 2} \right).$$

Now, taking $h_\varepsilon = \varepsilon^{1/(k-\beta+2)}$, we get

$$\limsup_{\varepsilon\to 0} \sup_{b(x)\in\Theta_{k+1}(L)} \sup_{c\le t\le d} \varepsilon^{-2(k+1)/(k-\beta+2)} E_b(|\hat{b}_t - b(x_t)|^2) < \infty.$$

□

Remark 3.4. Clearly, in order to obtain the previous result Theorem 3.3 we need to assume more regularity for the function b. If we take $\varepsilon^{1/(2-\beta)}$ and only assume that $b \in \Theta_0(L)$. Then,

$$\limsup_{\varepsilon\to 0} \sup_{b(x)\in\Theta_0(L)} \sup_{c\le t\le d} \varepsilon^{-2/(2-\beta)} E_b(|\hat{b}_t - b(x_t)|^2) < \infty.$$

Remark 3.5. For the case of fractional Brownian motion ($\beta = H$) we can see that the rate of convergence is faster as H goes to one.

4 Discrete time version of the estimator

In this section, we present a discrete time version of the non parametric estimator given by (3.1). To do this, we construct an Euler type numerical approximation to the unique solution of (2.1). In fact, we define the method as follows

$$Z_{t_{i+1}} = Z_{t_i} + b(Z_{t_i})\frac{T}{m} + \varepsilon\Delta G_{t_i}, \quad Z_{t_0} = Z_0 = x_0,$$

where $t_i = iT/m$, $\Delta G_{t_i} = G_{t_{i+1}} - G_{t_i}$, using $c_m(s) = \lfloor ms \rfloor/m$, we can write

$$Z_t = x_0 + \int_0^t b(Z_{c_m(s)})ds + \varepsilon G_t, \qquad t \in [0,T].$$

Under the assumptions on b the convergence of the Euler scheme follows by some standard arguments (see [7] among others).

Now, using the Euler approximation we can construct a discrete type counterpart to the estimator $\hat{b}$. We have the following discrete time estimator for b

$$\tilde{b}^m = \frac{1}{h_\varepsilon}\sum_{i=1}^{m} K\left(\frac{u_i - t}{h_\varepsilon}\right)\Delta Z_{u_i} \tag{4.1}$$

We obtain the following result concerning the convergence of $\tilde{b}^m$ to b.

Theorem 4.1. *Let $\tilde{b}^m$ be given by (4.1). Assume conditions of Theorem 1, b bounded and that K is a γ-Holder continuos function with $0 < \gamma \leq 1$. Then,*

$$\tilde{b}^m \to b \qquad (in\ L^2)$$

as $h_\varepsilon^{-2-2\gamma}\Delta_m^{2\gamma} \to 0$ as $m \to \infty$ and $\varepsilon \to 0$.

Proof. To prove the convergence of (4.1) to b, we will decompose $\hat{b}^m - b$ as

$$\tilde{b}^m - b(x_t) = \tilde{b}^m - \hat{b}_t + \hat{b}_t - b(x_t),$$

now

$$\begin{aligned} E[(\tilde{b}^m - b(x_t))^2] &\leq 2E[(\tilde{b}^m - \hat{b}_t)^2] + 2E[(\hat{b}_t - b(x_t))^2] \\ &= J_1 + J_2. \end{aligned}$$

The convergence to zero of J_2 is ensure by Theorem 1. Therefore, we will concentrate in the study of the term J_1

$$\begin{aligned} J_1 &= 2E[(\tilde{b}^m - \hat{b}_t)^2] \\ &\leq 4E\left[\left(\frac{1}{h_\varepsilon}\frac{1}{m}\sum_{i=0}^{m-1} K\left(\frac{u_i - t}{h_\varepsilon}\right)\Delta Z_{u_i} - \frac{1}{h_\varepsilon}\frac{1}{m}\sum_{i=0}^{m-1} K\left(\frac{u_i - t}{h_\varepsilon}\right)\Delta X_{u_i}\right)^2\right] \\ &\quad + 4E\left[\left(\frac{1}{h_\varepsilon}\frac{1}{m}\sum_{i=0}^{m-1} K\left(\frac{u_i - t}{h_\varepsilon}\right)\Delta X_{u_i} - \frac{1}{h_\varepsilon}\int_0^T K\left(\frac{u - t}{h_\varepsilon}\right) dX_u\right)^2\right] \\ &= J_{1.1} + J_{1.2}. \end{aligned}$$

The convergence of $J_{1.1}$ follows by the continuous mapping theorem and the fact the Euler scheme converges. For $J_{1.2}$, w.l.o.g, let assume that

$T = 1$

$$\begin{aligned}\frac{1}{h_\varepsilon}\sum_{i=0}^{m-1} K\left(\frac{u_i - t}{h_\varepsilon}\right)\Delta X_{u_i} &= \frac{1}{h_\varepsilon}\sum_{i=0}^{m-1}\int_{u_i}^{u_{i+1}} K\left(\frac{u_i - t}{h_\varepsilon}\right) dX_u \\ &= \frac{1}{h_\varepsilon}\sum_{i=0}^{m-1}\int_{u_i}^{u_{i+1}} K\left(\frac{u_i - t}{h_\varepsilon}\right) dX_u \\ &= \frac{1}{h_\varepsilon}\int_0^1 K\left(\frac{\frac{[mu]}{m} - t}{h_\varepsilon}\right) dX_u.\end{aligned}$$

This implies,

$$\begin{aligned}J_{1.2} &= 4E\left[\left(\frac{1}{h_\varepsilon}\int_0^1 K\left(\frac{\frac{[mu]}{m} - t}{h_\varepsilon}\right) dX_u - \frac{1}{h_\varepsilon}\int_0^1 K\left(\frac{u - t}{h_\varepsilon}\right) dX_u\right)^2\right] \\ &= 4E\left[\left(\frac{1}{h_\varepsilon}\int_0^1 \left[K\left(\frac{\frac{[mu]}{m} - t}{h_\varepsilon}\right) - K\left(\frac{u - t}{h_\varepsilon}\right)\right] dX_u\right)^2\right] \\ &= J_{1.2.1} + J_{1.2.2},\end{aligned}$$

where

$$J_{1.2.1} = 4E\left[\left(\frac{1}{h_\varepsilon}\int_0^1 \left[K\left(\frac{\frac{[mu]}{m} - t}{h_\varepsilon}\right) - K\left(\frac{u - t}{h_\varepsilon}\right)\right] b(X_u)d_u\right)^2\right],$$

assuming b is bounded and since K is γ-Holder continuos function with $0 < \gamma \leq 1$, we get

$$J_{1.2.1} \leq \frac{C}{h_\varepsilon^{2+2\gamma}}\left(\frac{[mu]}{m} - u\right)^{2\gamma} \leq \frac{C}{h_\varepsilon^{2+2\gamma}}\Delta_m^{2\gamma}.$$

For $J_{1.2.2}$, we get

$$\begin{aligned}&J_{1.2.2} = 4\frac{\varepsilon^2}{h_\varepsilon^2}E\left[\left(\int_0^1 \left[K\left(\frac{\frac{[mu]}{m} - t}{h_\varepsilon}\right) - K\left(\frac{u - t}{h_\varepsilon}\right)\right] dG_u\right)^2\right] \\ &\leq 4\frac{\varepsilon^2}{h_\varepsilon^2}\int_0^1\int_0^1 \left|K\left(\frac{\frac{[mu]}{m} - t}{h_\varepsilon}\right) - K\left(\frac{u - t}{h_\varepsilon}\right)\right|\left|K\left(\frac{\frac{[mv]}{m} - t}{h_\varepsilon}\right) - K\left(\frac{v - t}{h_\varepsilon}\right)\right| \\ &\qquad\times\left|\frac{\partial^2}{\partial u\partial v}R(u,v)\right| dudv,\end{aligned}$$

the Holder continuity of K implies

$$J_{1.2.2} \leq 4\frac{\varepsilon^2}{h_\varepsilon^{2+2\gamma}}\Delta_m^{2\gamma}\int_0^1\int_0^1\left|\frac{\partial^2}{\partial u\partial v}R(u,v)\right|dudv \leq C\frac{\varepsilon^2}{h_\varepsilon^{2+2\gamma}}\Delta_m^{2\gamma}.$$

Taking into account the previous computations, we can get

$$J_{1.2} \leq C\frac{1}{h_\varepsilon^{2+2\gamma}}\Delta_m^{2\gamma}.$$

Clearly, under the assumption of Theorem 4.1 we have that J_2 converges to zero as $m \to \infty$ and $\varepsilon \to 0$. Therefore, the result is achieved. □

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References

[1] Y. Chen, H. Zhou, Parameter Estimation for an Ornstein-Uhlenbeck Process Driven by a General Gaussian Noise, Acta Math Sci. **41**(2021), 573-595.

[2] Y. Chen, Y. Li, P. Pei, Parameter estimation for Vasicek model driven by a general Gaussian noise, Communications in Statistics - Theory and Methods. **52**(2021), No. 9, 3132-3148.

[3] K. Kubilius, Y. Mishura, K. Ralchenko, *Parameter Estimation in Fractional Diffusion Models*, Bocconi Springer Ser, Vol. 8, Springer-Verlag, New York, 2018.

[4] K. Kubilius, V. Skorniakov, On some estimators of the hurst index of the solution of SDE driven by a fractional Brownian motion, Stat. Probab. Lett. **109** 2016, 159-167.

[5] K. Kubilius, V. Skorniakov, D. Melichov, Estimation of parameters of SDE driven by fractional Brownian motion with polynomial drift, J. Stat. Comput. Simulation, **86**(2016), No. 10, 1954-1969.

[6] N. N. Mishra, B. L. S. Prakasa Rao, Nonparametric estimation of trend for stochastic differential equations driven by fractional Brownian motion, Sta Inference Stoch Process. **14**(2011), 101-109.

[7] Y. Mishura, G. Shevchenko, The rate of convergence for euler approximations of solutions of stochastic differential equations driven by fractional brownian motion. Stochastics An International Journal of Probability and Stochastic Processes. **80**(2008), No. 5, 489-511.

[8] Z. Mounir, Generalized fractional Brownian motion, Modern Stoch. Theory Appl. **4**(2017), No. 1,15-24.

[9] B. L. S. Prakasa Rao, Nonparametric estimation of trend for stochastic differential equations driven by mixed fractional Brownian motion. Stochastic Analysis and Applications. **37**(2019), No. 2, 271-280.

[10] C. A. Tudor, *Analysis of Variations for Self-Similar Processes. A Stochastic Calculus Approach. Probability and Its Applications.* Springer, Cham, New York, 2013.

[11] A. B. Tsybakov, *Introduction a l' estimation non - parametrique*, Springer, 2004.

[12] Y. Qian, Statistical inference for Vasicek-type model driven by self-similar Gaussian processes. Communications in Statistics - Theory and Methods. **49**(2020), No. 2, 471-484.