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An example of singular elliptic stochastic PDE

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Abstract. We illustrate (some of) the analytic and probabilistic aspects of the theory of regularity structures through the study of the elliptic singular stochastic partial differential equation $(1-\Delta)u = (a+bu)\xi$ on $\mathbb{T}^d$ in dimensions $d = 1, 2$.

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1 Introduction

In recent years, there has been a large number of works on singular stochastic partial differential equations, prompted notably by the rapid development of the theories of regularity structures [16, 5, 7, 4] (see also

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[11, 1, 2] for monographs on the theory and [25, 21, 22] for a tree-free approach) and of paracontrolled calculus [13, 14], culminating into a *complete* solution theory for all subcritical semilinear singular stochastic partial differential equations.

More recently, the authors of this article have tried to isolate some of the key analytic results of the theory of regularity structures – namely the *reconstruction theorem* and the *multi-level Schauder estimates* [6, 3] – reformulating those theorems as results in distribution theory as independently as possible from the formalism of regularity structures.

In this article, we illustrate those ideas with the following "Poisson-type" stochastic partial differential equation as motivation:

$$(1 - \Delta)u = (a + b\,u)\xi, \tag{E}$$

where $a, b \in \mathbb{R}$ are given (constant) coefficients, $u = u(x)$ with $x \in \mathbb{T}^d$, and ξ is Gaussian white noise in $\mathbb{T}^d$, $d \in \{1, 2\}$.

In dimension $d = 1$, the equation (E) can be discussed in classical Hölder spaces $\mathcal{C}^\alpha$ (we recall their definition in Appendix A.2). However, already in dimension $d = 2$, the equation is singular and even resists the Da Prato–Debussche trick introduced in [9] (see Remark 3.1 below though). Still, it is simple enough so that it does not require the full machinery of regularity structures and can in fact be renormalised "by hand". The purpose of this article is to demonstrate how to perform this renormalisation using the techniques of regularity structures [16], while keeping the presentation as elementary and pedagogical as possible: in particular, we will not formally define what a regularity structure is, although we will introduce the crucial concepts of models and modelled distributions. On the other hand, we hope to convey that the approach we present is not specific to equation (E), but is far more general.

More precisely, in the case of dimension $d = 1$ we will prove the following result.

Theorem 1.1 (Dimension $d = 1$)**.** *Let ξ be a Gaussian white noise on $\mathbb{T}$ and $a \in \mathbb{R}$. There exists a random constant $b_0(\omega) > 0$ depending* *on $\xi(\omega)$, such that for any (random) $b \in (0, b_0)$, there exists a unique solution $u \in \mathcal{C}^{\frac{3}{2}-}(\mathbb{T})$ to (E) in the distributional sense, where the product $u\,\xi$ is understood as the canonical "Young" product of Hölder distributions, defined in Theorem A.12 below.*

This is in contrast to the case of dimension $d = 2$, where there is no "classical" solution anymore, and a "renormalisation" of the equation has to be introduced. Indeed, we will show:

Theorem 1.2 (Dimension $d = 2$)**.** *Let ξ be a Gaussian white noise on $\mathbb{T}^2$, $a \in \mathbb{R}$, and $\rho \in C_c^\infty(\mathbb{R}^2)$ with $\int \rho = 1$. There exists a random constant $b_0(\omega) > 0$ depending on $\xi(\omega)$, as well as a diverging (deterministic) family of constants $(c_\epsilon)_{\epsilon>0}$ (depending on ρ) with*

$$c_\epsilon = -\frac{1}{2\pi} \log|\epsilon| + O_{\epsilon\to 0}(1),$$

such that for any (random) $b \in (0, b_0)$, there is a unique solution $u_\epsilon \in C^\infty(\mathbb{T}^2)$ to

$$(1 - \Delta)u_\epsilon = (a + b\,u_\epsilon)(\xi * \rho^\epsilon - b\,c_\epsilon), \tag{1.1}$$

and $(u_\epsilon)_{\epsilon>0}$ converges in probability in the Hölder space $\mathcal{C}^{1-}(\mathbb{T}^2)$ to a random function u which does not depend on the choice of ρ. In (1.1), we have denoted $\rho^\epsilon(\cdot) := \epsilon^{-2}\rho(\cdot/\epsilon)$ the L^1-scaling of ρ at scale ϵ.

We note that in Theorem 1.2 equation (E) has to be *modified*, or in fancier terms *renormalised*, and must be replaced with (1.1), for a limit of the smooth approximations u_ϵ to exist. We discuss in Section 5.4 below this crucial point.

This article is mostly self-contained in the sense that we will only admit the proofs of the major, well-documented, theorems (Reconstruction Theorem 4.7; multilevel Schauder estimates Theorem 4.25; Kolmogorov Theorem 5.1 for models) and of the most technical calculations when they are already performed somewhere else in the literature.

Finally, let us mention that elliptic SPDEs similar to (E) were previously studied in the context of the so-called Anderson hamiltonian, i.e. the random operator $-\Delta + \xi$: see e.g. [20] where the corresponding resolvent map $g \mapsto (-\Delta + \xi + c)^{-1} g =: f$ is constructed by solving the elliptic equation $(c - \Delta) f = g - f\xi$ for $c \in \mathbb{R}$ large enough.

Structure

This paper is organised as follows. In Section 2, we discuss the case of the dimension $d = 1$ and prove Theorem 1.1. In Section 3, we show that the approach of $d = 1$ does not generalise to $d = 2$ and discuss heuristics to repair the corresponding problems. In Section 4, we discuss the analytic aspects of the theory of regularity structures applied to (E), introducing the notions of *germs*, *models*, *modelled distributions*. In Section 5, we discuss the probabilistic aspects of the theory of regularity structures applied to (E), i.e. the question of *renormalisation*.

Finally, we have gathered in Appendix A a self-contained "toolbox", where we recall the important definitions and properties of distributions, Hölder regularity, and (Gaussian) white noise.

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2 The equation in dimension $d = 1$

In this section, we warm up by recalling some classical tools and techniques used to solve the equation (E) in dimension $d = 1$.

Let us first recall the basic analytic ingredients which we will use in the remainder of this article (the reader unfamiliar with them may consult Appendix A below, where we motivate and introduce them in more detail):

1. We will work in the space $\mathcal{D}'(\mathbb{T}^d)$ of periodic distributions, and will quantify their regularity in the Hölder scale $\left(\mathcal{C}^\alpha(\mathbb{T}^d)\right)_{\alpha\in\mathbb{R}}$. For $\alpha > 0$, those spaces contain actual functions, while for $\alpha \leq 0$ their elements are typically only distributions.

2. The Hölder scale is compatible with multiplication in the following sense: if (and only if) $\alpha + \beta > 0$, there exists a canonical continuous bilinear multiplication map $\mathcal{C}^\alpha(\mathbb{T}^d) \times \mathcal{C}^\beta(\mathbb{T}^d) \to \mathcal{C}^{\min(\alpha,\beta)}(\mathbb{T}^d)$ extending the pointwise multiplication on $C^\infty \times C^\infty$ (see Theorem A.12 of "Young multiplication").

3. The operator $1 - \Delta$ may be inverted in $\mathcal{D}'(\mathbb{T}^d)$ by taking the convolution with its fundamental solution G on $\mathbb{R}^d$, given explicitly by
$$\mathsf{G}(x) = \begin{cases} \dfrac{1}{2} e^{-|x|} & (\text{for } d = 1), \\ \dfrac{1}{2\pi} K_0(|x|) & (\text{for } d = 2), \end{cases} \tag{2.1}$$
where K_0 is explicit in terms of Bessel functions, diverges logarithmically at 0 and decays exponentially at infinity. Furthermore, the Hölder scale is compatible with convolution against G, in the sense that for all $\alpha \in \mathbb{R}$, $\mathsf{G} * \cdot \colon \mathcal{C}^\alpha(\mathbb{T}^d) \to \mathcal{C}^{\alpha+2}(\mathbb{T}^d)$ (see the "Schauder estimates", Theorem A.20).

4. Finally, our main probabilistic object is Gaussian white noise, i.e. any linear isometry $\xi\colon L^2(\mathbb{T}^d) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$. By a Kolmogorov-type continuity theorem for distributions (Theorem A.26), one can see that up to a modification, the sample paths of ξ are distributions of Hölder regularity $C^{-\frac{d}{2}-\kappa}(\mathbb{T}^d)$ for any $\kappa > 0$.

With those ingredients in mind, fix $a, b \in \mathbb{R}$ and a realisation $\xi \in \mathcal{C}^{-\frac{1}{2}-\kappa}(\mathbb{T}^{d=1})$ of a Gaussian white noise (where $\kappa > 0$ can be taken arbitrarily small), so that the equation we want to solve, $u - \Delta u = (a + bu)\xi$, is equivalent to
$$u = \mathsf{G} * ((a + bu)\xi).$$

The strategy is to perform a Picard iteration in a Hölder space of suitable exponent: consider the sequence $(u^{(n)})_{n\in\mathbb{N}_0}$ (where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$) defined by

$$u^{(0)} = 0, \qquad u^{(n+1)} = \mathsf{G} * ((a + bu^{(n)})\xi). \tag{2.2}$$

Now by the compatibility of the Hölder scale with multiplication and convolution, we hope to have defined a Cauchy sequence in some Hölder space. Explicitly, since $\mathsf{G} * \xi \in \mathcal{C}^{\frac{3}{2}-\kappa}(\mathbb{T})$, we have

$$\begin{aligned}
u^{(0)} &:= 0 && \in C^{\infty},\\
u^{(1)} &:= \mathsf{G} * ((a + bu^{(0)})\xi) = a\mathsf{G} * \xi && \in \mathcal{C}^{\frac{3}{2}-\kappa}(\mathbb{T}),\\
u^{(2)} &:= \mathsf{G} * ((a + bu^{(1)})\xi) && \in \mathcal{C}^{\frac{3}{2}-\kappa}(\mathbb{T}),
\end{aligned}$$

where the regularity of $u^{(2)}$ is guaranteed by an application of Young multiplication (Theorem A.12) to $u^{(1)} \in \mathcal{C}^{\frac{3}{2}-\kappa}(\mathbb{T})$ with the distribution $\xi \in \mathcal{C}^{-\frac{1}{2}-\kappa}(\mathbb{T})$, followed by the Schauder estimates (Theorem A.20) to the resulting $u^{(1)}\xi \in \mathcal{C}^{-\frac{1}{2}-\kappa}(\mathbb{T})$. Thus the sequence remains in the space $\mathcal{C}^{\frac{3}{2}-\kappa}(\mathbb{T})$. Let us now prove Theorem 1.1.

Proof of Theorem 1.1. Applying Theorems A.12 and A.20, the iteration map

$$\begin{aligned}
\mathcal{P}: \quad \mathcal{C}^{\frac{3}{2}-\kappa}(\mathbb{T}) &\longrightarrow \mathcal{C}^{\frac{3}{2}-\kappa}(\mathbb{T})\\
u &\longmapsto \mathsf{G} * ((a + bu)\xi),
\end{aligned} \tag{2.3}$$

is well-defined and there exists a constant $C > 0$ such that

$$\|\mathcal{P}(u) - \mathcal{P}(v)\|_{\mathcal{C}^{\frac{3}{2}-\kappa}(\mathbb{T})} \le C\, b\, \|\xi\|_{\mathcal{C}^{-1-\kappa}(\mathbb{T})} \|u - v\|_{\mathcal{C}^{\frac{3}{2}-\kappa}(\mathbb{T})},$$

thus for b small enough in function of (the random variable) $\|\xi\|_{\mathcal{C}^{-1-\kappa}(\mathbb{T})}$, the map $\mathcal{P}$ is a contraction in the Banach space $\mathcal{C}^{\frac{3}{2}-\kappa}(\mathbb{T})$, yielding the announced fixed-point. □

3 The equation in dimension $d = 2$: heuristics

We now turn to the dimension $d = 2$, and so we fix a realisation $\xi \in \mathcal{C}^{-1-\kappa}(\mathbb{T}^{d=2})$ of a Gaussian white noise (where $\kappa > 0$ can be taken arbitrarily small). Fix also $\mathsf{G} = \mathsf{K} + \mathsf{R}$ a decomposition of the fundamental solution (2.1) as the sum of a (compactly supported) 2-regularising kernel K and a Schwartz function R (see Appendix A.4 for a detailed discussion on convolution in Hölder spaces). For convenience, we will omit the remainder R in the following heuristic discussion, as it will not be problematic to treat later on. Thus, in this section we consider the fixed-point problem

$$u = \mathsf{K} * ((a + bu)\xi).$$

Let us mention at this point that similar presentations of the heuristics discussed in this section can be found in [15, Section 3.2], [8, Section 1.1.3].

3.1 Failure of Picard iteration in Hölder spaces

It is natural to first attempt the classical Picard iteration in the spaces of Hölder distributions, as we did in Section 2: we may define

$$\begin{aligned} u^{(0)} &:= 0 && \in C^\infty, \\ u^{(1)} &:= \mathsf{K} * ((a + bu^{(0)})\xi) = a\mathsf{K} * \xi && \in \mathcal{C}^{1-\kappa}(\mathbb{T}^2), \end{aligned}$$

however at this point it is not possible anymore to define

$$u^{(2)} := \mathsf{K} * ((a + bu^{(1)})\xi), \tag{3.1}$$

in a canonical way. This is because the regularities of $u^{(1)} \in \mathcal{C}^{1-\kappa}(\mathbb{T}^d)$ and $\xi \in \mathcal{C}^{-1-\kappa}(\mathbb{T}^2)$ sum to $-2\kappa < 0$, whence the product $u^{(1)}\xi$ can not be canonically defined by Theorem A.12.

Note that by exploiting the randomness in ξ we might hope to be able to give a meaning to $u^{(1)}\xi$ via stochastic techniques. Still, this is not a viable strategy because more and more complicated ill-defined products of this type will appear at every step of the Picard iteration. Thus, we look for other ideas to solve (E).

3.2 Failure of the Da Prato–Debussche trick

One possible approach, called the "Da Prato–Debussche trick" after [9], is to consider the new function $v := u - u^{(1)}$, and try to solve for v.

The hope is that all the products appearing in the Picard iteration of v are well-defined (except possibly for a finite number of them, treatable by stochastic techniques). The interpretation is that $u^{(1)}$ should be the term of "worst regularity" in the system, thus removing it from the equation might possibly improve the situation. The new function v itself satisfies a fixed-point equation: since

$$v = u - u^{(1)} = \mathsf{K} * ((a + bu)\xi - a\mathsf{K} * \xi = b\mathsf{K} * (u\xi),$$

then v satisfies

$$v = b\mathsf{K} * (u^{(1)}\xi) + b\mathsf{K} * (v\xi). \tag{3.2}$$

Of course, the product $u^{(1)}\xi$ is still classically ill-defined, but we may hope to give it a canonical meaning by stochastic techniques (since $u^{(1)}$ is given as an explicit function of ξ). Now if the Picard iteration on v could furthermore be performed in a suitable Hölder space, then the original equation could be solved, by setting $u := v + u^{(1)}$. Thus, let us try to perform the Picard iteration corresponding to (3.2): the first terms are

$$\begin{aligned} v^{(0)} &:= 0 && \in C^{\infty}, \\ v^{(1)} &:= b\mathsf{K} * (u^{(1)}\xi) + b\mathsf{K} * (v^{(0)}\xi) = b\mathsf{K} * (u^{(1)}\xi) && \in \mathcal{C}^{1-\kappa}(\mathbb{T}^2), \end{aligned}$$

but here we run into the same problem as above for u, since the multiplication Theorem A.12 does not allow the term

$$u^{(2)} := b\mathsf{K} * (u^{(1)}\xi) + b\mathsf{K} * (v^{(1)}\xi),$$

to be canonically well-defined, as the regularities of the unknown $v^{(1)} \in \mathcal{C}^{1-\kappa}(\mathbb{T}^d)$ and $\xi \in \mathcal{C}^{-1-\kappa}(\mathbb{T}^2)$ sum to $-2\kappa < 0$. It is tempting to try and iterate again and again this Da Prato–Debussche technique. However, one quickly realises that this always results in the same problem: indeed, the first step of the Picard iteration never gives a function of regularity better than $1 - \kappa$, which is not enough for the multiplication with ξ in the second step.

Remark 3.1 (Multiplicative Da Prato–Debussche trick)**.** One reason why the Da Prato–Debussche trick presented above fails here is that the equation under consideration has multiplicative noise (the term $u\xi$ appears in its right-hand side). Still, in such cases similar techniques can be implemented, see [17], but the "Ansatz" $u = v + Y$ has to be replaced by a more appropriate one. In our case, we may make the "multiplicative Ansatz" $u = ve^Y$, where $Y := b\,\mathsf{G} * \xi \in \mathcal{C}^{1-\kappa}(\mathbb{T}^2)$ is the solution of the linearised equation

$$(1 - \Delta)Y = b\,\xi,$$

then a direct calculation shows that if u solves $(1 - \Delta)u = (a + bu)\xi$, then $v := ue^{-Y}$ solves (at least formally)

$$(1 - \Delta)v = (Y + (\nabla Y)^2)v + 2\nabla Y \cdot \nabla v + a\xi e^{-Y}. \tag{3.3}$$

One observes by going through the first Picard iterates that this is still not sufficient, but we may iterate the trick because (3.3) now has additive noise: specifically, defining w by $v = w + Z$, where $Z \in \mathcal{C}^{1-\kappa}(\mathbb{T}^2)$ solves

$$(1 - \Delta)Z = a\xi e^{-Y},$$

we get the following equation on w:

$$(1 - \Delta)w = (Y + (\nabla Y)^2)w + 2\nabla Y \cdot \nabla w + \big(Z(Y + (\nabla Y)^2) + \nabla Z \cdot \nabla Y\big),$$

for which we can see that a Picard iteration should be possible in $w \in \mathcal{C}^{2-2\kappa}$: our solution u is then given by $u := (w+Z)e^Y$. Of course, the argument just sketched still requires that we should give a meaning to the few ill-defined products which appear, namely here:

$$(\nabla Y)^2, \qquad \xi e^{-Y}, \qquad \nabla Z \cdot \nabla Y.$$

Note also that this technique is very specific to the fact that our equation (E) displays a nonlinearity of the form $u\xi$: in particular it is not clear how it would generalize if we replaced it by $F(u)\xi$ for some F. On the other hand, the approach of regularity structures which we present below is more general, and it is suited to such nonlinearities (provided F is sufficiently smooth); see section 5.6 below.

3.3 The idea of germs

To summarise the discussion of the previous sections, we are confronted with the problem that multiplying distributions of regularity α and β gives (when it exists i.e. when $\alpha+\beta>0$) a distribution of regularity $\min(\alpha,\beta)$: this is too low in general. This is also a very *global* statement. On the other hand, we can make the following *local* remark: the multiplication of functions which behave at a point x at order α resp. β gives a function which behaves at x at order $\alpha+\beta$, and note that $\alpha+\beta>\min(\alpha,\beta)$ if at least one among α,β is positive. To illustrate this, take functions f,g of Hölder regularity respectively $\alpha,\beta\in(0,1)$, then

$$\begin{aligned}|f(y)g(y)-f(x)g(x)| &\lesssim |y-x|^{\min(\alpha,\beta)},\\ \text{but: } |f(y)-f(x)||g(y)-g(x)| &\lesssim |y-x|^{\alpha+\beta}.\end{aligned} \tag{3.4}$$

The heuristic idea is now that if we could exploit this improved exponent, then we may possibly "close the loop" of the Picard iteration in a satisfactory way. Of course, this will require to work with new objects, as the left-hand side of (3.4) displays the *increments* of f and g, indexed by a base point x.

Let us revisit the Picard iteration (2.2) with this idea: we know by Schauder estimates (Theorem A.20) that

$$u^{(1)}=a\mathsf{K}*\xi\in\mathcal{C}^{1-\kappa},$$

is $1-\kappa$ Hölder. Thus if we want to exploit optimally this regularity around some given point $x\in\mathbb{R}^2$, the increment $\mathsf{K}*\xi(\cdot)-\mathsf{K}*\xi(x)$ should appear: by substracting and adding $\mathsf{K}*\xi(x)$, we rewrite

$$u^{(1)}=a(\mathsf{K}*\xi-\mathsf{K}*\xi(x))+a\mathsf{K}*\xi(x), \tag{3.5}$$

where $\mathsf{K}*\xi-\mathsf{K}*\xi(x)$ vanishes at order $1-\kappa$ around x. Now the second Picard iterate (3.1) is

$$u^{(2)}=ab\mathsf{K}*((\mathsf{K}*\xi-\mathsf{K}*\xi(x))\xi)+(a+ab\mathsf{K}*\xi(x))\mathsf{K}*\xi. \tag{3.6}$$

Of course, here there is still an undefined product $(\mathsf{K}*\xi-\mathsf{K}*\xi(x))\xi$, but with stochastic techniques we might expect to be able to give a meaning to it in such a way that it should vanish around x at order $(1-\kappa)+(-1-\kappa)$ i.e. -2κ (where here we *sum* the regularities instead of taking their minimum because of the observation (3.4)). We can continue this process by convolving this term with K, substracting its Taylor expansion to exploit the regularity around x, writing the next Picard iterate, and so on. The important point is the following: iterating this process suggests that the solution u is described around x by a local expansion, of the form

$$u(y)=\sum_{i\in\mathbb{N}_0}u_i(x)\Pi_x^i(y),$$

where the Π_x^i are explicit and vanish at a known order, say α_i, around x, and where the family $(\alpha_i)_{i\in\mathbb{N}_0}$ is locally finite (in the sense that $(\alpha_i)_{i\in\mathbb{N}_0}\cap B$ is finite for any bounded set $B\subset\mathbb{R}$) and bounded from below. In fact, it will be sufficient to cut the expansion at a certain order γ (to be determined) and to describe the solution u locally around x by the *finite* expansion

$$u(y)=\sum_{\alpha_i<\gamma}u_i(x)\Pi_x^i(y)+R_x(y), \tag{3.7}$$

where R_x vanishes at order γ around x. Furthermore, it turns out that the only objects which possibly require a stochastic treatment are the $\Pi_x^i(y)$: but this family is *explicit* and *finite* (in fact only those with $\alpha_i\leq 0$ are concerned).

The conclusion of this heuristic discussion is that it should be beneficial to start working at the level of local approximations of the distributions of interests, which we will call *germs*. In the next section we come back to (3.7) in more detail and show that it can be interpreted as a generalised Taylor expansion of the solution u with respect to the family of distributions $(\Pi_x^i)_i$.

3.4 Taylor expansion

One of the main ideas of regularity structures and rough paths is to formulate a differential equation as a *local finite difference equation*. For example, the Itô SDE

$$X_t = X_0 + \int_0^t \sigma(X_s)\,\mathrm{d}B_s, \qquad t \ge 0,$$

where $(B_t)_{t\ge 0}$ is a Brownian motion in $\mathbb{R}^d$, $\sigma = \mathbb{R}^k \to \mathbb{R}^k \otimes \mathbb{R}^d$ is smooth, and $X : [0,T] \to \mathbb{R}^k$, can be reformulated as follows

$$X_t - X_s - \sigma(X_s)(B_t - B_s) - \nabla\sigma(X_s)\sigma(X_s)\int_s^t (B_r - B_s)\otimes \mathrm{d}B_r = o(t-s),$$

uniformly for $0 \le s \le t \le T$. Remarkably, the latter formulation is equivalent to the above SDE [10].

Now we want to show how a similar approach is possible for the PDE (E) on $\mathbb{R}^d$. For the remainder of this section, assume for simplicity that ξ is smooth (e.g. a smooth approximation of an actual white noise). We fix a base point $x \in \mathbb{R}^d$ and we write for $y \in \mathbb{R}^d$

$$u(y) = \mathsf{K} * ((a+bu)\xi)\,(y),$$

so that we obtain

$$\begin{aligned} u(y) &= u(x) + u(y) - u(x) \\ &= u(x) + \mathsf{K} * ((a+bu)\xi)\,(y) - u(x) \\ &= u(x) + \mathsf{K} * ((a+bu)\xi)\,(y) - \mathsf{K} * ((a+bu)\xi)\,(x). \end{aligned}$$

Now we use again $u = u(x) + u - u(x)$ and we obtain

$$\begin{aligned} u(y) =& u(x) + \mathsf{K} * ((a + b(u(x) + u - u(x)))\xi)\,(y) \\ & - \mathsf{K} * ((a + b(u(x) + u - u(x))\xi)\,(x) \\ =& u(x) + (a + bu(x))\,(\mathsf{K} * \xi(y) - \mathsf{K} * \xi(x)) + \\ & + \underbrace{b\,\mathsf{K} * ((u - u(x))\xi)\,(y)}_{f_x(y)} - \underbrace{b\,\mathsf{K} * ((u - u(x))\xi)\,(x)}_{f_x(x)}. \end{aligned}$$

Consider two symbols[1] $\mathbf{1}$ and $\stackrel{\bullet}{|}$ and set

$$\mathbf{1}_x(y) := 1, \qquad \stackrel{\bullet}{|}_x(y) := \mathsf{K} * \xi(y) - \mathsf{K} * \xi(x). \tag{3.8}$$

[1] We use the color blue for some objects viewed as abstract symbols, but only in order to enhance the readability of some expressions.

Then the above expansion can be rewritten

$$\begin{aligned} u(y) &= u(x) + (a + bu(x)) \underbrace{(\mathsf{K} * \xi(y) - \mathsf{K} * \xi(x))}_{\dot{\mathsf{I}}_x(y)} + \underbrace{f_x(y) - f_x(x)}_{R_x(y)} \\ &= u(x)\, \mathbf{1}_x(y) + (a + bu(x))\, \dot{\mathsf{I}}_x(y) + R_x(y). \end{aligned}$$

This is a generalised Taylor expansion of the solution u, where $\{\mathbf{1}_x, \dot{\mathsf{I}}_x\}$ are our *monomials*, and $\{u(x), (a+bu(x))\}$ their respective coefficients; on the other hand R_x is a *remainder*. Note that

$$R_x(y) = f_x(y) - f_x(x), \qquad f_x(y) := b\,\mathsf{K} * ((u - u(x))\xi)\,(y). \tag{3.9}$$

Since $y \mapsto f_x(y)$ is smooth, then we have the bound

$$|R_x(y)| \lesssim |x - y|,$$

uniformly for x, y in compact sets.

Suppose now that we want to continue the expansion and obtain an even smaller remainder. At this stage it will be convenient to slightly change our notations in order to write the obtained expressions as functions of symbols $\mathbf{1}, \dot{\mathsf{I}}$, etc. Precisely, set

$$\Pi_x(\mathbf{1})(y) := \mathbf{1}_x(y) = 1, \qquad \Pi_x(\dot{\mathsf{I}})(y) := \dot{\mathsf{I}}_x(y) = \mathsf{K} * \xi(y) - \mathsf{K} * \xi(x),$$

and introduce

$$\Pi_x(X_i)(y) := y_i - x_i, \quad i = 1, 2,$$

then we want to continue the expansion:

$$\begin{aligned} u(y) = u(x)\, \Pi_x(\mathbf{1}) &+ (a + bu(x))\, \Pi_x(\dot{\mathsf{I}})(y) \\ &+ C_1 \Pi_x(X_1)(y) + C_2 \Pi_x(X_2)(y) + \overline{R}_x(y) \end{aligned}$$

with

$$\overline{R}_x(y) = f_x(y) - f_x(x) - C_1(y_1 - x_1) - C_2(y_2 - x_2).$$

In order to make $\overline{R}_x(y)$ smaller than $|y-x|^{1+\kappa}$, we are forced to choose $C_i = \partial_i f_x(x)$ so that

$$\overline{R}_x(y) = f_x(y) - f_x(x) - \sum_{i=1}^{2} \partial_i f_x(x)\,(y_i - x_i).$$

Finally we have obtained the following generalised Taylor expansion of the solution to (E)

$$\begin{aligned} u(y) = u(x)\,\Pi_x(\mathbf{1})(y) + (a + bu(x))\,\Pi_x(⫯)(y) \\ + \sum_{i=1}^{2} \partial_i f_x(x)\,\Pi_x(X_i)(y) + \overline{R}_x(y), \end{aligned} \tag{3.10}$$

where the remainder $\overline{R}_x$ is expected to satisfy $|\overline{R}_x(y)| \lesssim |y-x|^{1+\kappa}$ with $\kappa > 0$, uniformly over x, y in compact sets.

At this point, we have arrived at the following insight: solutions to (E) with smooth noise enjoy the structure of a generalized Taylor expansion

$$u(y) = \sum_k c_k(u)(x) \Pi_x^k(\xi)(y) + O(|y-x|^{1+\kappa}), \tag{3.11}$$

where the coefficients c_k are functions of u, while the monomials Π_x^k are explicit functions only of the noise ξ: as we have seen, they can be deduced from (E) by a recursive procedure. The interest of (3.11) is that we may hope for this formulation to be "stable" in some sense at the limit where the smooth noise ξ goes to an actual white noise. The programme of solving (E) can now be pursued in two steps:

1. Giving a suitable well-posedness theory for equations whose solutions are further constrained to satisfy a local expansion of the form (3.11). The key point will be the continuity of the solution map $(\Pi^k(\xi))_k \mapsto u$ in an appropriate distributional topology. This is the *analytic* part of the theory, which we discuss in Section 4.

2. Giving a suitable sense to the the family of Π_x^k at the limit where ξ is an actual white noise: this requires a stochastic correction, called renormalisation. It is the *probabilistic* part of the theory, which we discuss in Section 5.

As it is often the case with Taylor expansions, in the following we ignore the remainder and we concentrate on the other terms of the sum. Regularity structures are based on the two following approaches to such sums:

1. One can consider a *germ*, namely a family of distributions (generalised functions) on $\mathbb{R}^2$ and indexed by $x \in \mathbb{R}^2$, which in this case takes the form $F_x = F_x(y)$ with

$$F_x := u(x)\,\Pi_x(\mathbf{1}) + (a + bu(x))\,\Pi_x(\dot{\vert}) + \sum_{i=1}^{2} \partial_i f_x(x)\,\Pi_x(X_i), \quad (3.12)$$

see Section 4.1.

2. One can separate the family of *monomials* $\{\Pi_x(\mathbf{1}), \Pi_x(\dot{\vert}), \Pi_x(X_i), \dots\}$ from the respective *coefficients* $\{u(x), (a + bu(x)), \partial_i f_x(x), \dots\}$, and define

$$U_x := u(x)\,\mathbf{1} + (a + bu(x))\,\dot{\vert} + \sum_{i=1}^{2} \partial_i f_x(x)\,X_i, \quad (3.13)$$

which is an element of the vector space generated by $\{\mathbf{1}, \dot{\vert}, X_i, \dots\}$, see Section 4.3. The relation between the two objects is given by the formula $F_x = \Pi_x(U_x)$.

4 The equation in dimension $d = 2$: analytic aspects

4.1 Germs of distributions and their properties

We have introduced in (3.12) a first example of germ, a notion that we now make precise.

Definition 4.1 (Germ)**.** A *germ*[2] is a family of distributions $(F_x)_{x \in \mathbb{R}^d} \subset \mathcal{D}'(\mathbb{R}^d)$.

[2]There is another notion of "germ" in topology, which does not coincide with this one. Still, we will use this terminology in the remainder of this article.

For technical reasons (which appear in the proofs of Theorems 4.7 and 4.12), we will also impose the (always satisfied in practice) constraint that for each $\varphi \in C_c^\infty$, the map $x \mapsto F_x(\varphi)$ should be measurable.

Definition 4.2 (Periodic germ)**.** We say that a germ $(F_x)_{x\in\mathbb{R}^d}$ is *perodic of period* $a \in \mathbb{R}^d$ if for all $x \in \mathbb{R}^d$, $F_{x+a} \circ \tau_a = F_x$.

Example 4.3. We see a germ as a family of local approximations. Let us collect some first examples:

1. If $f \in \mathcal{D}'(\mathbb{R}^d)$ is a distribution, one can of course define a *constant germ* $F_x := f$, which is periodic of period $a \in \mathbb{R}^d$ if and only if f is.

2. For $k \in \mathbb{N}_0^d$ one can define the corresponding *monomial germ* on $x, y \in \mathbb{R}^d$ by $F_x(y) := (y-x)^k$, which is periodic of period a for every $a \in \mathbb{R}^d$.

3. If f is a locally γ-Hölder function for some $\gamma \in \mathbb{R}_+ \setminus \mathbb{N}_0$, one can define its *Taylor germ* on $x, y \in \mathbb{R}^d$ by

$$F_x(y) := \sum_{|k|\le\gamma} \frac{f^{(k)}(x)}{k!}(y-x)^k,$$

   which is periodic of period $a \in \mathbb{R}^d$ as soon as f is.

4. In Section 3.4 we have introduced, for $x, y \in \mathbb{R}^2$,

$$\Pi_x(\mathbf{1}) := 1, \quad \Pi_x(X_i)(y) := y_i - x_i, \quad \Pi_x(\uparrow)(y) := \mathsf{K} * \xi(y) - \mathsf{K} * \xi(x).$$

   Note that the first two are monomial germs, while the third one is the Taylor germ of the $\mathcal{C}^{1-\kappa}(\mathbb{T}^d)$ function $\mathsf{K} * \xi$. We will see later that the equation (E) gives rise to other "basis germs" $\Pi(\tau)$ which are neither monomial germs nor Taylor germs, see e.g. Table 4.1.

5. A last example from Section 3.4 has been introduced in (3.12): if u is solution to (E) with a smooth ξ, then we saw that the germ

$$F_x := u(x)\,\Pi_x(\mathbf{1}) + (a + bu(x))\,\Pi_x(\uparrow) + \sum_{i=1}^{2} \partial_i f_x(x)\,\Pi_x(X_i),$$

produces a local approximation of u, at order $1+\kappa$ in the sense of (3.10).

Of course, if we want to lift the problem of solving (E) to the space of germs, the following question should be adressed: when does a germ really correspond to a family of local approximations? In other words, under which conditions on a given germ F is it possible to produce a (ideally canonical) distribution f which is suitably well approximated by F_x around each point x?

To answer this question, we will need to quantify the local behaviour of $F_x(\varphi)$, both in x and φ. For this purpose, let we define two key properties. Below, $\mathscr{B}^r$ denotes the family of smooth functions φ supported in $B(0,1) \coloneqq \{x \in \mathbb{R}^d, |x| \coloneqq (\sum_i x_i^2)^{1/2} \leq 1\}$ with $\|\varphi\|_{C^r} \leq 1$, see Definition A.9, and φ_x^λ is defined as in Definition A.8.

Definition 4.4 (Coherence and homogeneity)**.** Let $\bar{\alpha}, \alpha, \gamma \in \mathbb{R}$, with $\alpha \leq \gamma$. A germ $(F_x)_{x\in\mathbb{R}^d}$ is said to be

1. *(locally) $\bar{\alpha}$-homogeneous* if there exists $r \in \mathbb{N}_0$ such that for every compact $K \subset \mathbb{R}^d$,
$$\sup_{x\in K, \lambda\in(0,1], \varphi\in\mathscr{B}^r} \frac{|F_x(\varphi_x^\lambda)|}{\lambda^{\bar{\alpha}}} < +\infty; \tag{4.1}$$
2. *(locally) (α,γ)-coherent* if there exists $r \in \mathbb{N}_0$ such that for every compact $K \subset \mathbb{R}^d$,
$$\sup_{x,y\in K, \lambda\in(0,1], \varphi\in\mathscr{B}^r} \frac{|(F_y - F_x)(\varphi_x^\lambda)|}{\lambda^{\alpha}(|y-x|+\lambda)^{\gamma-\alpha}} < +\infty. \tag{4.2}$$

The interpretation is as follows: the property of homogeneity quantifies the behaviour of F_x around x, while the property of coherence quantifies the relative proximity of F_y and F_x around x, at the relevant scales.

Remark 4.5 (Choice of r)**.** When a germ is both (locally) $\bar{\alpha}$-homogeneous and (α,γ)-coherent, it can be shown that the estimates (4.1)-(4.2) do not depend on r as long as $r > \max(-\bar{\alpha}, -\alpha)$, see e.g. [3, Appendix B].

It will then be convenient to use the following notation:

Notation 4.6. We will note:

1. $\mathcal{G}^{\bar{\alpha};\alpha,\gamma}(\mathbb{R}^d)$ the space of (locally) $\bar{\alpha}$-homogeneous and (α,γ)-coherent germs, which is a Fréchet space when endowed with the seminorms (4.1)-(4.2) for any (equivalent) choice of $r > \max(-\bar{\alpha}, -\alpha)$ and over any (countable) exhaustion of $\mathbb{R}^d$ by compacts.

2. $\mathcal{G}^{\bar{\alpha};\alpha,\gamma}(\mathbb{T}^d)$ the subspace of $\mathcal{G}^{\bar{\alpha};\alpha,\gamma}(\mathbb{R}^d)$ consisting of 1-periodic germs. This is also a Fréchet space for the same collection of seminorms (4.1)-(4.2).

The following celebrated *reconstruction theorem* states that the properties of homogeneity and coherence are precisely those required on a germ F in order to exhibit a (unique when $\gamma > 0$) distribution $\mathcal{R}(F)$ which is approximated by F_x at order γ around each x. In fact only the assumption of coherence is required to construct $\mathcal{R}(F)$, while that of homogeneity guarantees its regularity, see [29]. Because we will only consider periodic germs in the remainder of this article, we state the result on the torus (the cited references work on the whole space but the case of the torus is a straightforward adaptation).

Theorem 4.7 (Reconstruction, see [16, 6, 29])**.** *Let $\bar{\alpha}, \alpha, \gamma \in \mathbb{R}$ with $\alpha \leq \gamma$, $\gamma \neq 0$. There exists a (unique if and only if $\gamma > 0$) continuous linear map*

$$\begin{array}{rccc} \mathcal{R}: & \mathcal{G}^{\bar{\alpha};\alpha,\gamma}(\mathbb{T}^d) & \longrightarrow & \mathcal{C}^{\bar{\alpha}}(\mathbb{T}^d) \\ & F & \longmapsto & \mathcal{R}(F), \end{array}$$

making $\mathcal{R}(F)$ well-approximated by F_x at order γ around x in the sense that the remainder $(F_x - \mathcal{R}(F))_{x\in\mathbb{R}^d}$ is a γ-homogeneous germ. In fact, the map

$$\begin{array}{ccc} \mathcal{G}^{\bar{\alpha};\alpha,\gamma}(\mathbb{T}^d) & \longrightarrow & \mathcal{G}^{\gamma;\alpha,\gamma}(\mathbb{T}^d) \\ F & \longmapsto & (F_x - \mathcal{R}(F))_{x\in\mathbb{R}^d}, \end{array}$$

is continuous.

Remark 4.8. There is a statement of reconstruction when $\gamma = 0$, but logarithms appear in the estimates, whence we refrain from discussing this case here, see [6] for a precise statement.

Example 4.9. Let us come back to the germs from Example 4.3: the following points follow from straightforward calculations.

1. If $f \in \mathcal{C}^{\bar{\alpha}}(\mathbb{T}^d)$ and $F_x = f$, then $F \in \mathcal{G}^{\min(\bar{\alpha},0);\alpha,\gamma}(\mathbb{T}^d)$ for any choice of α and for $\gamma > 0$, $\mathcal{R}(F) = f$.
2. If $F_x(y) = (y-x)^k$ for some $k \in \mathbb{N}_0^d$, then $F \in \mathcal{G}^{|k|;0,|k|}(\mathbb{T}^d)$ and $\mathcal{R}(F) = 0$ except for $k = 0$ in which case $\mathcal{R}(F) = 1$.
3. If F is the "Taylor germ" of some periodic function $f \in \mathcal{C}^{\gamma}(\mathbb{T}^d)$, $\gamma \in \mathbb{R}_+ \setminus \mathbb{N}_0$, then $F \in \mathcal{G}^{0;0,\gamma}(\mathbb{T}^d)$, and $\mathcal{R}(F) = f$.

In general, checking the assumptions of coherence and homogeneity on germs of the form $F_x = \sum_k f_k(x)\Pi_x^k$ requires good assumptions on both families of Π^k and f_k: this will motivate the notion of *modelled distributions*, which we discuss in Section 4.3 below.

Example 4.10 (Continuous germs)**.** The construction performed in the proof of Theorem 4.7 ensures that if a continuous function $(x,y) \mapsto F_x(y)$ is also an (α,γ)-coherent germ for some $\gamma > 0$, then one can explicitly express its reconstruction as

$$\mathcal{R}(F)(x) = F_x(x), \qquad x \in \mathbb{R}^d.$$

Example 4.11 (Young multiplication, see [6, Section 14])**.** Let $\beta \leq 0 < \alpha$ with $\alpha + \beta > 0$, and $f \in \mathcal{C}^{\alpha}(\mathbb{T}^d), g \in \mathcal{C}^{\beta}(\mathbb{T}^d)$. Setting $P_x \coloneqq gF_x$ where F_x is the Taylor germ of f from Example 4.3 (the multiplication is well-defined since F_x is a smooth function), one can prove that $P \in \mathcal{G}^{\beta;\beta,\alpha+\beta}(\mathbb{T}^d)$, and that one retrieves Theorem A.12 by defining $f \cdot g \coloneqq \mathcal{R}(P)$.

It turns out that the operation of convolution can be adapted at the level of germs, in such a way that it is compatible with the operation of reconstruction. In fact, this convolution map is simply obtained from the pointwise convolution $\mathsf{K} * F_x$ subtracting a suitable explicit polynomial germ, as the following result shows. The notion of β-regularising kernel is given in Definition A.18.

Theorem 4.12 (Schauder estimates for germs, see [3])**.** *Let* $\bar{\alpha}, \alpha, \gamma \in \mathbb{R}$, $\beta > 0$ *be such that* $\bar{\alpha}, \alpha \leq \gamma$, $\gamma \neq 0$, $\bar{\alpha} + \beta \neq 0$, $\alpha + \beta \neq 0$, $\gamma + \beta \notin \mathbb{N}_0$. *Let* K *be a* β*-regularising kernel. Then the map*

$$\begin{aligned} \mathcal{K}: \quad \mathcal{G}^{\bar{\alpha};\alpha,\gamma}(\mathbb{T}^d) \quad &\longrightarrow \quad \mathcal{G}^{(\bar{\alpha}+\beta)\wedge 0;(\alpha+\beta)\wedge 0,\gamma+\beta}(\mathbb{T}^d) \\ F \quad &\longmapsto \quad \mathsf{K} * F_x - \sum_{|k|<\gamma+\beta} \partial^k (\mathsf{K} * (F_x - \mathcal{R}F))(x) \frac{(\cdot - x)^k}{k!}, \end{aligned} \tag{4.3}$$

is well-defined (i.e. the derivatives appearing in (4.3) *have a canonical definition), it is linear, continuous, and compatible with reconstruction in the sense that for* $F \in \mathcal{G}^{\bar{\alpha};\alpha,\gamma}(\mathbb{T}^d)$,

$$\mathcal{R}(\mathcal{K}F) = \mathsf{K} * \mathcal{R}F. \tag{4.4}$$

Remark 4.13. Recall from Theorem 4.7 that $\mathcal{R}$ is only canonically defined for positive coherence exponents. Still Theorem 4.12 remains valid even for negative exponents with the following convention: when $\gamma < 0$, Theorem 4.12 is valid for any choice of reconstruction $\mathcal{R}(F)$, and when $\gamma + \beta < 0$, (4.4) means that $\mathsf{K} * \mathcal{R}F$ is *one* reconstruction of $\mathcal{K}F$.

Thus, it is possible to lift the convolution with K at the level of germs; however, there is no canonical way of multiplying two germs in general, just as there is no canonical way of multiplying two distributions. Of course, recalling (3.7), the germs that appear in the context of solving (E) are not arbitrary, but rather enjoy a specific structure: they correspond to finite linear combinations of basis germs Π_x^i. Such germs will be examples of *modelled distributions*, and it will be possible to multiply them in a suitable way.

But before discussing modelled distributions, let us first describe the collection of germs Π_x^i which appear in the process described in Section 3.3.

4.2 The model for equation (E)

We now turn to a precise discussion of equation (E) in dimension $d = 2$. The first step is to regularize the equation i.e. replace it by a version where the noise has been smoothly approximated. Rigorously, we fix from now on:

1. A *mollifier* $\rho \in C_c^\infty$ i.e. a symmetric test-function with $\int \rho = 1$. For $\epsilon > 0$ we denote $\xi_\epsilon := \xi * \rho^\epsilon \in C^\infty$, so that (by a standard result in distribution theory) ξ_ϵ converges to ξ in $\mathcal{D}'$ as $\epsilon \to 0$.

2. A decomposition $\mathsf{G} = \mathsf{K} + \mathsf{R}$ of the kernel defined in (2.1), where K is 2-regularising as per Definition A.18, and R belongs to the Schwartz class. Without loss of generality, we will also assume that K coincides with G, say, on $[-1, 1]^2$.

This section is concerned with the equation driven by ξ^ϵ. On the other hand, the question of convergence of the solution as $\epsilon \to 0$ will be addressed in Section 5.

4.2.1 The basis germs Π

Let us go back to the context described in Section 3.3. Recall the expressions (3.5)-(3.6) of the first (recentered) Picard iterates. In order to analyse those expressions in a systematic manner, it is convenient to adopt a symbolic notation, as we have already begun to do in (3.8). The first symbol, a blue dot $\bullet$, represents the noise, and we note

$$\Pi_x^\epsilon(\bullet)(y) := \xi_\epsilon(y).$$

Furthermore, putting a bar under a symbol denotes convolution with K: for instance, the symbol $\uparrow$ represents the convolution of the noise by K, to which a suitable Taylor expansion is substracted. More precisely, we define analogously to (3.8)

$$\Pi_x^\epsilon(\uparrow)(y) := \mathsf{K} * \xi_\epsilon(y) - \mathsf{K} * \xi_\epsilon(x).$$

Note that the substraction of the Taylor expansion ensures that the resulting germ is *homogeneous* in the sense of Definition A.7. For instance, since $\xi \in \mathcal{C}^{-1-\kappa}$ and $\mathsf{K} * \xi \in \mathcal{C}^{1-\kappa}$, the germs $\Pi^\epsilon(\bullet)$ resp. $\Pi^\epsilon(\uparrow)$ are homogeneous of exponent $\alpha_\bullet = -1-\kappa$ resp. $\alpha_\uparrow = 1-\kappa$ (uniformly in ϵ).

We also add symbols $(X^k = X_1^{k_1} X_2^{k_2})_{k_1,k_2 \in \mathbb{N}_0}$ representing polynomials, with

$$\Pi^\epsilon_x(X_1^{k_1} X_2^{k_2})(y) \coloneqq (y_1 - x_1)^{k_1} (y_2 - x_2)^{k_2},$$

giving germs of homogeneity $k_1 + k_2$. Note that with this notation, one can now rewrite the first Picard iterate (3.5) as

$$\begin{aligned} u_\epsilon^{(1)}(y) &= a\Pi^\epsilon_x(\uparrow)(y) + a\mathsf{K} * \xi_\epsilon(x)\Pi^\epsilon_x(\mathbf{1})(y), \\ &= \Pi^\epsilon_x(a\uparrow + a\mathsf{K} * \xi_\epsilon(x)\mathbf{1})(y), \end{aligned}$$

where in the second line we naturally extended Π^ϵ by linearity on the free vector space spanned by the symbols. Finally, we encode the operation of pointwise multiplication formally by the concatenation of the corresponding symbols. For instance, $\overset{\uparrow}{\bullet}$ corresponds to the multiplication of $\bullet$ and $\uparrow$, the corresponding germ being the pointwise multiplication of smooth functions

$$\Pi^\epsilon_x(\overset{\uparrow}{\bullet})(y) \coloneqq \Pi^\epsilon_x(\bullet)(y)\Pi^\epsilon_x(\uparrow)(y) = (\mathsf{K} * \xi_\epsilon(y) - \mathsf{K} * \xi_\epsilon(x))\xi_\epsilon(y),$$

and the resulting homogeneity is the sum of those of the multiplied symbols. Continuing the process described in Sections 3.3 and 3.4, we see that a number of such germs Π^ϵ appear in the calculations. It will turn out that only those with homogeneity less than or equal to 1 will play a role in what follows; we list them in Table 4.1.

Notation 4.14. We will denote

$$I \coloneqq \{\bullet, \overset{\uparrow}{\bullet}, \bullet X_1, \bullet X_2, \mathbf{1}, \uparrow, X_1, X_2\}, \qquad T \coloneqq \mathrm{Span}(I). \tag{4.5}$$

the set of the symbols thus formed, resp. the vector space of their formal linear combinations. We will refer to I as the *index set* or *set of symbols*.

In the literature, T is called *model space* [16] or *structure space* [11].

Germ $\Pi_x^\epsilon(\tau)(\cdot) =$	**Homogeneity** α_τ	**Symbol**
$\xi_\epsilon(\cdot)$	$-1-\kappa$	$\bullet$
$(\mathsf{K} * \xi_\epsilon(\cdot) - \mathsf{K} * \xi_\epsilon(x))\xi_\epsilon(\cdot)$	-2κ	$\overset{\bullet}{\underset{\bullet}{\vert}}$
$\xi_\epsilon(\cdot)(\cdot_1 - x_1)$	$-\kappa$	$\bullet X_1$
$\xi_\epsilon(\cdot)(\cdot_2 - x_2)$	$-\kappa$	$\bullet X_2$
1	0	$\mathbf{1}$
$\mathsf{K} * \xi_\epsilon(\cdot) - \mathsf{K} * \xi_\epsilon(x)$	$1-\kappa$	$\overset{\bullet}{\vert}$
$(\cdot_1 - x_1)$	1	X_1
$(\cdot_2 - x_2)$	1	X_2

Table 4.1: The first basis germs, by increasing homogeneity

Remark 4.15 (Multiplicativity of Π^ϵ)**.** By construction, for all (applicable) symbols $\tau, \tau' \in I$,

$$\Pi_x^\epsilon(\tau\tau') = \Pi_x^\epsilon(\tau)\Pi_x^\epsilon(\tau') \qquad \text{(as a product of smooth functions).}$$

Remark 4.16 (Divergence)**.** The family Π^ϵ has a divergent behaviour as $\epsilon \to 0$. As an example, observe from the definition of (periodic) white-noise that for $x, y \in \mathbb{R}^d$,

$$\begin{aligned} \mathbb{E}[\Pi_x^\epsilon(\overset{\bullet}{\underset{\bullet}{\vert}})(y)] &= \mathbb{E}[(\mathsf{K} * \xi_\epsilon)(y)\xi_\epsilon(y)] \\ &= \mathbb{E}[\xi((\mathsf{K} * \rho^\epsilon)_y)\, \xi(\rho_y^\epsilon)] \\ &= \langle (\mathsf{K} * \rho^\epsilon)^{\mathrm{per}}, \rho^{\epsilon,\mathrm{per}} \rangle, \end{aligned}$$

which can be shown to diverge (logarithmically for $d = 2$) to ∞ as $\epsilon \to 0$, because K diverges (logarithmically) at the origin, recall (2.1). We will quantify the divergence more precisely in Section 5, where we will also implement a *renormalisation procedure* in order to "cure" this divergence.

4.2.2 The reexpansion operator Γ

One very useful characteristic of the family of basis germs $(\Pi^\epsilon(\tau))_{\tau\in I}$ defined above is that it admits a *reexpansion operator* $\Gamma^\epsilon : T \to T$, i.e. a linear map satisfying

$$\Pi^\epsilon_y(\tau) = \Pi^\epsilon_x(\Gamma^\epsilon_{x,y}(\tau)), \qquad \Gamma^\epsilon_{x,y}(\tau) = \sum_{\sigma\in I} \Gamma^{\epsilon,\sigma,\tau}_{x,y}\sigma. \tag{4.6}$$

In fact, such a Γ^ϵ can even be explicitly constructed, as we highlight in the following calculations:

Example 4.17 (Some calculations for $\Gamma^\epsilon(\tau)$)**.** Let us explore some (representative) examples.

1. (polynomials: $\tau = X^k$) We want $\Pi^\epsilon_x(\Gamma^\epsilon_{x,y}(X^k)) = \Pi^\epsilon_y(X^k)$, but from the binomial formula

$$\begin{aligned}\Pi^\epsilon_y(X^k) &= (\cdot - y)^k \\ &= \sum_{0\le l\le k} \binom{k}{l} (x-y)^{l-k} (\cdot - x)^l \\ &= \Pi^\epsilon_x\Bigg(\sum_{0\le l\le k} \binom{k}{l} (x-y)^{l-k} X^l \Bigg).\end{aligned}$$

whence it suffices to set $\Gamma^\epsilon_{x,y}(X^k) = \sum_{0\le l\le k} \binom{k}{l} (x-y)^{l-k} X^l$.

2. (noise: $\tau = \bullet$) We want $\Pi^\epsilon_x(\Gamma^\epsilon_{x,y}(\bullet)) = \Pi^\epsilon_y(\bullet)$, but we know

$$\Pi^\epsilon_y(\bullet) = \xi_\epsilon = \Pi^\epsilon_x(\bullet),$$

whence it suffices to set $\Gamma^\epsilon_{x,y}(\bullet) = \bullet$.

3. (convolution: $\tau = \mathord{\uparrow}$) We want $\Pi^\epsilon_x(\Gamma^\epsilon_{x,y}(\mathord{\uparrow})) = \Pi^\epsilon_y(\mathord{\uparrow})$ but we know

$$\begin{aligned}\Pi^\epsilon_y(\mathord{\uparrow}) &= \mathsf{K} * \xi_\epsilon - \mathsf{K} * \xi_\epsilon(y) \\ &= \mathsf{K} * \xi_\epsilon - \mathsf{K} * \xi_\epsilon(x) + \mathsf{K} * \xi_\epsilon(x) - \mathsf{K} * \xi_\epsilon(y) \\ &= \Pi^\epsilon_x(\mathord{\uparrow} + (\mathsf{K} * \xi_\epsilon(x) - \mathsf{K} * \xi_\epsilon(y))\mathbf{1}),\end{aligned}$$

whence it suffices to set $\Gamma^\epsilon_{x,y}(\mathord{\uparrow}) = \mathord{\uparrow} + (\mathsf{K} * \xi_\epsilon(x) - \mathsf{K} * \xi_\epsilon(y))\mathbf{1}$.

4. (multiplication: $\tau = \mathbf{I}$) We want $\Pi^\epsilon_x(\Gamma^\epsilon_{x,y}(\mathbf{I})) = \Pi^\epsilon_y(\mathbf{I})$ but we know

$$\begin{aligned}\Pi^\epsilon_y(\mathbf{I}) &= (\mathsf{K} * \xi_\epsilon - \mathsf{K} * \xi_\epsilon(y))\xi_\epsilon \\ &= (\mathsf{K} * \xi_\epsilon - \mathsf{K} * \xi_\epsilon(x))\xi_\epsilon + (\mathsf{K} * \xi_\epsilon(x) - \mathsf{K} * \xi_\epsilon(y))\xi_\epsilon \\ &= \Pi^\epsilon_x(\mathbf{I} + (\mathsf{K} * \xi_\epsilon(x) - \mathsf{K} * \xi_\epsilon(y))\bullet),\end{aligned}$$

whence it suffices to set $\Gamma^\epsilon_{x,y}(\mathbf{I}) = \mathbf{I} + (\mathsf{K} * \xi_\epsilon(x) - \mathsf{K} * \xi_\epsilon(y))\bullet$.

All calculations done, we can write the operator Γ^ϵ explicitly in matrix form:

$$\Gamma^\epsilon_{x,y} = \begin{array}{c|cccccccc} \sigma\backslash\tau & \bullet & \mathbf{I} & \bullet X_1 & \bullet X_2 & \mathbf{1} & \uparrow & X_1 & X_2 \\ \hline \bullet & 1 & \mathsf{K}*\xi_\epsilon(x) - \mathsf{K}*\xi_\epsilon(y) & x_1 - y_1 & x_2 - y_2 & 0 & 0 & 0 & 0 \\ \mathbf{I} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \bullet X_1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \bullet X_2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \mathbf{1} & 0 & 0 & 0 & 0 & 1 & \mathsf{K}*\xi_\epsilon(x) - \mathsf{K}*\xi_\epsilon(y) & x_1 - y_1 & x_2 - y_2 \\ \uparrow & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ X_1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ X_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}. \tag{4.7}$$

It follows from the expressions of Table 4.1 and (4.7) that the pair $M^\epsilon := (\Pi^\epsilon, \Gamma^\epsilon)$ forms what is called a (periodic) *model* as per the following definition:

Definition 4.18 (Periodic model)**.** Let I be a finite set and $(\alpha_\tau)_{\tau\in I}$ a collection of real numbers. A *periodic model* on I is any pair $M = (\Pi, \Gamma)$ such that

1. $\Pi = (\Pi_x(\tau))_{x\in\mathbb{R}^d,\tau\in I}$ is a family of 1-periodic germs which are locally α_τ-homogeneous,

2. $\Gamma = (\Gamma^{\sigma,\tau}_{x,y})_{x,y\in\mathbb{R}^d,\sigma,\tau\in I}$ is a family of real numbers satisfying the reexpansion property (4.6), and are furthermore 1-periodic in the sense that for $i \in \{1, \dots, d\}$, $\Gamma^{\sigma,\tau}_{x+e_i,y+e_i} = \Gamma^{\sigma,\tau}_{x,y}$,

and where Γ also enjoys the further additional properties:

3. (triangular structure): $\Gamma^{\tau,\tau}_{x,y} = 1$, and $\Gamma^{\sigma,\tau}_{x,y} = 0$ if $\sigma \neq \tau$ and $\alpha_\sigma \geq \alpha_\tau$,

4. (group property): $\Gamma_{x,y}\Gamma_{y,z} = \Gamma_{x,z}$,
5. (analytic bound): $|\Gamma^{\sigma,\tau}_{x,y}| \lesssim |y-x|^{\alpha_\tau - \alpha_\sigma}$.

As a consequence of those three properties and (4.6), the germs $\Pi_x(\tau)$ are automatically $(\bar{\alpha}, \alpha_\tau)$-coherent for $\bar{\alpha} := \min_{\tau \in I} \alpha_\tau$. We will denote $\mathcal{M}$ the set of periodic models, and set for compact sets $K \subset \mathbb{R}^d$:

$$\|\Pi\|_{\mathcal{M}(K)} := \max_{\tau \in I} \|\Pi(\tau)\|_{\mathcal{G}^{\alpha_\tau;\bar{\alpha},\alpha_\tau}(K)}, \quad \|\Gamma\|_{\mathcal{M}(K)} := \sup_{\substack{\sigma,\tau \in I \\ x,y \in K}} \frac{|\Gamma^{\sigma,\tau}_{x,y}|}{|y-x|^{\alpha_\tau - \alpha_\sigma}},$$

and $\|M\|_{\mathcal{M}(K)} := \|\Pi\|_{\mathcal{M}(K)} + \|\Gamma\|_{\mathcal{M}(K)}$.

In the following, the index set I will be either arbitrary or the one defined by (4.5), depending on the context. We now comment on some of the further properties of the model $M^\epsilon = (\Pi^\epsilon, \Gamma^\epsilon)$ defined above

Remark 4.19 (Multiplicativity of Γ^ϵ)**.** For all (applicable) symbols $\tau, \tau' \in I$,

$$\Gamma^\epsilon_{x,y}(\tau\tau') = \Gamma^\epsilon_{x,y}(\tau)\Gamma^\epsilon_{x,y}(\tau').$$

As an example:

$$\begin{aligned} \Gamma^\epsilon_{x,y}\Big(\substack{\bullet\\ |\\ \bullet}\Big) &= \substack{\bullet\\ |\\ \bullet} + (\mathsf{K} * \xi_\epsilon(x) - \mathsf{K} * \xi_\epsilon(y))\,\bullet \\ &= \Big(\substack{\bullet\\ |} + (\mathsf{K} * \xi_\epsilon(x) - \mathsf{K} * \xi_\epsilon(y))\mathbf{1}\Big)\,\bullet \\ &= \Gamma^\epsilon_{x,y}\Big(\substack{\bullet\\ |}\Big)\Gamma^\epsilon_{x,y}(\bullet). \end{aligned}$$

This property of multiplicativity is not a coincidence: for general equations, it is in fact guaranteed by the algebraic part of the theory.

Remark 4.20 (Stationarity)**.** For any given symbol τ, test-function $\varphi \in C^\infty_c$ and point $h \in \mathbb{R}^d$, the random processes

$$x \mapsto \Pi^\epsilon_x(\tau)(\varphi_x), \qquad x \mapsto \Gamma^\epsilon_{x,x+h}(\tau),$$

are *stationary* in the sense that their distribution do not depend on x. As an example, observe that

$$\Pi^\epsilon_x\Big(\substack{\bullet\\ |}\Big)(\varphi_x) = \mathsf{K} * \xi_\epsilon(\varphi_x) - \mathsf{K} * \xi_\epsilon(x) = \xi\big((\mathsf{K} * \rho^\epsilon * \varphi - \mathsf{K} * \rho^\epsilon)_x\big),$$

so that the claimed stationarity follows from that of the white noise ξ. A useful consequence which will be exploited in Section 5 below is the identity in expectation

$$\mathbb{E}[|\Pi_x^\epsilon(\tau)(\varphi_x^\lambda)|^2] = \mathbb{E}[|\Pi_0^\epsilon(\tau)(\varphi^\lambda)|^2].$$

Remark 4.21 (Admissibility)**.** Observe that the model $M^\epsilon = (\Pi^\epsilon, \Gamma^\epsilon)$ built above is *admissible*, which we define to mean that

$$\begin{aligned} \Pi_x^\epsilon(\uparrow) &= \mathsf{K} * \Pi_x^\epsilon(\bullet)(\cdot) - \mathsf{K} * \Pi_x^\epsilon(\bullet)(x), \\ \Gamma_{x,y}^\epsilon(\uparrow) &= \uparrow - \Pi_x^\epsilon(\uparrow)(y)\mathbf{1}, \\ \Gamma_{x,y}^\epsilon(\tau\tau') &= \Gamma_{x,y}^\epsilon(\tau)\Gamma_{x,y}^\epsilon(\tau') \quad \text{ for all } \tau, \tau' \in T. \end{aligned}$$

(This notion makes sense for general equations, in which case the first two identities above should be replaced by a more general expression of the action of the model on abstract integration, which come from the Schauder estimates).

The importance of this property is that it implies that Γ^ϵ is completely determined by Π^ϵ. In particular, it justifies that one can *identify* the model M^ϵ and the basis germs Π^ϵ. In what follows, we will denote $\mathcal{M}_{\mathrm{adm}}$ the set of admissible models, and when we consider such a model we will indifferently refer to it as M or Π.

Finally, note that admissibility requires the multiplicativity property on Γ^ϵ but not on Π^ϵ, and indeed the approximating renormalised model constructed in Section 5.4 will be admissible but not multiplicative.

4.3 Modelled distributions

Recall from the heuristics of Sections 3.3 and 3.4 and in the example (3.13) that we look for solutions that are locally approximated by germs of the form

$$F_x = \Pi_x^\epsilon\big(a(x)\mathbf{1} + b(x)\uparrow + c_1(x)X_1 + c_2(x)X_2\big), \tag{4.8}$$

i.e. explicitly

$$
\begin{aligned}
F_x(y) = f_1(x) + f_2(x)\big(\mathsf{K} * \xi_\epsilon(y) - \mathsf{K} * \xi_\epsilon(x)\big) \\
+ f_3(x)(y_1 - x_1) + f_4(x)(y_2 - x_2),
\end{aligned} \tag{4.9}
$$

for some functions f_1, f_2, f_3, f_4 to be determined. Of course, the end goal is to reconstruct the germ F into an actual solution $f \in \mathcal{D}'$, with the help of the reconstruction Theorem 4.7: thus we need to guarantee that (4.9) indeed corresponds to a coherent germ. In particular, we should be able to estimate the difference $F_y - F_x$: for this purpose, observe that for any model (Π, Γ), recall Definition 4.18, and any germ of the form $F_x = \sum_{\tau \in I} f^\tau(x) \Pi_x(\tau)$, one may use the reexpansion operator Γ to express

$$
F_y - F_x = \sum_{\tau \in I} \Big(\sum_{\sigma \in I} \Gamma^{\tau,\sigma}_{x,y} f^\sigma(y) - f^\tau(x) \Big) \Pi_x(\tau), \tag{4.10}
$$

This expression justifies the following definition of a *modelled distribution.*

Definition 4.22 (Modelled distributions)**.** Let $M \in \mathcal{M}_{\mathrm{adm}}$ be a model and $\gamma \in \mathbb{R}$. A *periodic modelled distribution* on M is a 1-periodic $T = \mathrm{Span}_{\mathbb{R}}(I)$-valued function $f(x) = \sum_{\tau \in I, \alpha_\tau < \gamma} f^\tau(x) \tau$ with real coefficients $f^\tau(x) \in \mathbb{R}$, such that for every compact set $K \subset \mathbb{R}^d$,

$$
\|f\|_{\mathcal{D}^\gamma(K)} \coloneqq \sup_{x \in K, \tau \in I} |f^\tau(x)| + \sup_{\substack{x,y \in K, \tau \in I, \\ |x-y| \le 1}} \frac{\Big| f^\tau(x) - \sum_{\sigma \in I} \Gamma^{\tau,\sigma}_{x,y} f^\sigma(y) \Big|}{|y-x|^{\gamma - \alpha_\tau}} < +\infty. \tag{4.11}
$$

We will note $\mathcal{D}^\gamma = \mathcal{D}^\gamma_M(\mathbb{T}^d)$ the Banach space of periodic modelled distributions on M with norm given by (4.11) on $K = [0,1]^d$.

Let $f \in \mathcal{D}^\gamma_M$ be some modelled distribution on some model $M = (\Pi, \Gamma)$. We will denote by $\langle f, \Pi \rangle$ the germ

$$
\langle f, \Pi \rangle_x \coloneqq \sum_{\tau \in I} f^\tau(x) \Pi_x(\tau) = \Pi_x(f(x)).
$$

so that from (4.10)-(4.11) (see also [6, Example 4.10]), and noting $\bar{\alpha} \coloneqq \min_{\tau \in I} \alpha_\tau$,

$$\langle f, \Pi \rangle \in \mathcal{G}^{\bar{\alpha};\bar{\alpha},\gamma}(\mathbb{T}^d),$$

and we will define the reconstruction of f as that of this germ:

$$\mathcal{R}_M(f) \coloneqq \mathcal{R}(\langle f, \Pi \rangle).$$

When the model M is clear from the context, we will omit it from the subscript.

As noted in (4.8), the modelled distributions that are of interest for us are spanned only by a strict subset of I, namely the symbols $\{\mathbf{1}, \dot{|}, X_1, X_2\}$. This motivates the following:

Notation 4.23. For a model M on the index set I defined in (4.5), we will note $\mathcal{D}^\gamma_{M;\mathrm{fp}}$ the subspace of $\mathcal{D}^\gamma_M$ containing modelled distibutions of the form $f = f_1\mathbf{1} + f_2\dot{|} + f_3X_1 + f_4X_2$ for functions f_1, f_2, f_3, f_4.

Note that $\mathcal{D}^\gamma_{M;\mathrm{fp}}$ is again a Banach space, and that we have added the subscript fp to indicate that this is the space in which we want to set up a *fixed-point.*

In the remainder of this section, we discuss the different operations (truncation, convolution, multiplication) that can be performed at the level of modelled distributions in a way that is compatible with reconstruction: the goal is to express and solve (E) (with mollified noise) at the level of modelled distributions.

4.3.1 Truncation

Truncation will play an important part in the iteration: indeed as motivated in Section 3.3 a truncation will need to be performed at each step in order to get rid of the terms of "too high" order.

Definition 4.24 (Truncation of modelled distributions)**.** Let $\gamma' < \gamma$ and let $f = \sum_{\tau \in I} f^\tau \tau \in \mathcal{D}^\gamma_M$ be a modelled distribution on some model $M \in \mathcal{M}$.

We truncate f at level γ' by removing the terms corresponding to symbols with homogeneity larger than γ', i.e. we set

$$f^{\leq \gamma'} := \sum_{\tau \in I, \alpha_\tau \leq \gamma'} f^\tau \tau.$$

Then $f^{\leq \gamma'} \in \mathcal{D}_M^{\gamma'}$: this is a consequence of the analytic property of Γ. Furthermore, one has the following bound: there exists a constant $C > 0$ and a compact $K \subset \mathbb{R}^d$ such that for all $f \in \mathcal{D}^\gamma$

$$\|f^{\leq \gamma'}\|_{\mathcal{D}^{\gamma'}} \leq C\big(1 + \|\Gamma\|_{\mathcal{M}(K)}\big)\|f\|_{\mathcal{D}^\gamma}.$$

Furthermore,

$$\mathcal{R}(f^{\leq \gamma'}) = \mathcal{R}(f),$$

where if $\gamma' \leq 0$ this should be read as "$\mathcal{R}(f)$ is one reconstruction of $f^{\leq \gamma'}$".

4.3.2 The convolution operator $\mathcal{K}$

We already know how to perform the convolution with K at the level of coherent and homogeneous germs in a way that is compatible with the reconstruction, recall the operator $\mathcal{K}$ defined in 4.12. Given a model $M \in \mathcal{M}$ and a modelled distribution $f \in \mathcal{D}_M^\gamma$, this means that the germ $\mathcal{K}(\langle f, \Pi \rangle)$ is a natural candidate for defining the convolution on f. From the explicit formula (4.3), it is clear that $\mathcal{K}(\langle f, \Pi \rangle)$ can still be written as a finite linear combination of basis germs. But does it correspond to an actual modelled distribution? The answer is affirmative, see [16, Theorem 5.12], and takes the following form in general.

Theorem 4.25 (Multi-level Schauder estimates, see [16, 3])**.** *Let K be a β-regularising kernel, $M = (\Pi, \Gamma) \in \mathcal{M}$ be a model and $f \in \mathcal{D}_M^\gamma$ be a modelled distribution, where $\gamma \in \mathbb{R}$. Assume that $\gamma + \beta \notin \mathbb{N}_0$, then there exists an explicit new model $\hat{M} = (\hat{\Pi}, \hat{\Gamma})$ on a possibly larger set of symbols, and an explicit new modelled distribution $\mathcal{K}f \in \mathcal{D}_{\hat{M}}^{\gamma+\beta}$ relatively to this new model such that*

$$\langle \mathcal{K}f, \hat{\Pi} \rangle = \mathcal{K}(\langle f, \Pi \rangle), \qquad \textit{and thus (see (4.4)):} \qquad \mathcal{R}(\mathcal{K}f) = \mathsf{K} * \mathcal{R}(f).$$

In this theorem, the convention for $\gamma < 0$ is the same as in Theorem 4.12 above, see Remark 4.13.

Furthermore, this operation is linear and continuous with the following bound: there exists $C > 0$ and $K \subset \mathbb{R}^d$ compact such that

$$\|\mathcal{K}(f)\|_{\mathcal{D}^{\gamma+\beta}_{\hat{M}}} \leq C\|\Pi\|_{\mathcal{M}^\alpha(K)}\|f\|_{\mathcal{D}^\gamma_M}.$$

In the context of the model $M^\epsilon = (\Pi^\epsilon, \Gamma^\epsilon)$ defined by Table 4.1 and (4.7) above, and for the kernel K defined at the beginning of Section 4.2, the theorem simplifies slightly: since M^ϵ is admissible, recall Remark 4.21, then it follows from the proof of Theorem 4.25 that the new model and the new set of symbols remain M^ϵ and I respectively, as long as we accept to truncate the resulting modelled distribution $\mathcal{K}f$ just above homogeneity 1. More precisely:

Corollary 4.26 (Multi-level Schauder in our context)**.** *Let K be the 2-regularising kernel defined at the beginning of Section 4.2. Let $\epsilon > 0$ and $M^\epsilon = (\Pi^\epsilon, \Gamma^\epsilon)$ be the model defined by Table 4.1 and (4.7). Assume that $0 < \kappa < 1/3$ and fix $\gamma_0 \in (1+\kappa, 2-2\kappa)$. Let $0 < \gamma \leq \gamma_0$, then there exists a continuous linear map*

$$\mathcal{K} \colon \mathcal{D}^\gamma_{M^\epsilon} \to \mathcal{D}^{\min(\gamma+2,\gamma_0)}_{M^\epsilon},$$

such that

$$\langle \mathcal{K}f, \Pi^\epsilon \rangle = \mathcal{K}(\langle f, \Pi^\epsilon \rangle), \qquad \mathcal{R}(\mathcal{K}f) = \mathsf{K} * \mathcal{R}(f),$$

along with the continuity estimate

$$\|\mathcal{K}(f)\|_{\mathcal{D}^{\min(\gamma+2,\gamma_0)}_{M^\epsilon}} \leq C\|\Pi^\epsilon\|_{\mathcal{M}^\alpha(K)}\|f\|_{\mathcal{D}^\gamma_{M^\epsilon}}.$$

Remark 4.27 (Parsimony)**.** Let us illustrate a non-trivial use of Theorems 4.7 and 4.25 for negative exponents $\gamma \leq 0$ (this was communicated to us by Hendrik Weber). In concrete situations such as Corollary 4.26, there are two ways of constructing the convolution operator $\mathcal{K}$ on modelled distributions when $\gamma + 2 \geq \gamma_0$.

The first approach is to apply Theorem 4.25 to f and then truncate the resulting modelled distribution at order γ_0. One downside is that applying Theorem 4.25 automatically creates new symbols of homogeneity larger than γ_0, only to instantly discard them by truncation, thus artificially enriching the symbol set with unused new symbols.

The second approach is to truncate f at order $\gamma_0 - 2$ *before* applying Theorem 4.25. This might sound problematic, because the definition of the map $\mathcal{K}$ contains the reconstruction map (recall Theorem 4.12) which is not uniquely defined when $\gamma_0 - 2 < 0$. That turns out not to be a problem: it suffices to construct $\mathcal{K}$ with the canonical data of $\mathcal{R}(f)$ as input, rather than the non-canonical $\mathcal{R}(f^{\leq \gamma_0 - 2})$. This way, no artificial symbol has to be added.

Let us also mention that a similar idea of using multi-level Schauder estimates for negative exponents was recently implemented in [18], although in a slightly different context.

Note that in order to fully perform the convolution with the fundamental solution G of (2.1), we need to take into account the remainder R, recall the decomposition $\mathsf{G} = \mathsf{K} + \mathsf{R}$. But because R, being in the Schwartz class, is regularising of *any* order, we can simply define its lift by considering the corresponding Taylor germ: the following proposition is a straightforward consequence of the second item in Theorem A.20, along with the Definition 4.22 of a modelled distribution.

Proposition 4.28 (Lifting of R)**.** *Let $\mathsf{R} \in \mathcal{S}$ be any function in the Schwartz class and $\gamma, \gamma' > 0$ be any exponents, then the map*

$$\begin{aligned} \mathcal{K}^{\mathsf{R}} : \quad \mathcal{D}^{\gamma}_{M^\epsilon} \quad &\longrightarrow \quad \mathcal{D}^{\gamma'}_{M^\epsilon} \\ f \quad &\longmapsto \quad \sum_{|k| \leq \gamma'} \frac{\partial^k (\mathsf{R} * (\mathcal{R} f))(\cdot)}{k!} X^k, \end{aligned}$$

is well-defined, linear, and satisfies

$$\mathcal{R}(\mathcal{K}^{\mathsf{R}} f) = \mathsf{R} * \mathcal{R}(f),$$

along with the following continuity estimate: there exist $C > 0, K \subset \mathbb{R}^d$ *such that*

$$\|\mathcal{K}^{\mathsf{R}}(f)\|_{\mathcal{D}^{\gamma'}_{M^\epsilon}} \leq C\|\Pi^\epsilon\|_{\mathcal{M}^\alpha(K)}\|f\|_{\mathcal{D}^{\gamma}_{M^\epsilon}}.$$

4.3.3 Multiplication

Since there exists a formal multiplication at the level of symbols (recall e.g. that $\uparrow\bullet = \updownarrow$), it turns out that it is possible to multiply modelled distributions. For instance, in the particular case where one wants to multiply a modelled distribution f by $\bullet$, it is straightforward to check that the following proposition is a simple consequence of the definition of a modelled distribution.

Proposition 4.29 (Multiplication by the noise)**.** *Let* $M \in \mathcal{M}$ *be a model and* $f = f_1\mathbf{1} + f_2\uparrow + f_3X_1 + f_4X_2 \in \mathcal{D}^\gamma_{M;\mathrm{fp}}$ *for some* $\gamma > 1+\kappa$*. Set*

$$\bullet f \coloneqq f_1\bullet + f_2\updownarrow + f_3\bullet X_1 + f_4\bullet X_2.$$

Then $\bullet f \in \mathcal{D}^{\gamma-1-\kappa}_M$ *with the continuity bound*

$$\|\bullet f\|_{\mathcal{D}^{\gamma-1-\kappa}_M} = \|f\|_{\mathcal{D}^\gamma_M}.$$

Note that when $M = M^\epsilon$ is the model defined by Table 4.1 and (4.7), then by multiplicativity, recall Remark 4.15, one has

$$\langle \bullet f, \Pi^\epsilon\rangle = \langle f, \Pi^\epsilon\rangle\xi_\epsilon,$$

as a product of smooth functions, and thus from Example 4.10:

$$\mathcal{R}(\bullet f) = \mathcal{R}(f)\xi_\epsilon,$$

as a product of smooth functions. Note that the reconstruction operators above are uniquely defined because $\gamma - 1 - \kappa > 0$ (recall Theorem 4.7).

Remark 4.30. It is in fact possible to multiply modelled distributions, say f and g, in a general way and under mild assumptions, see [16, Theorem 4.7]. Proposition 4.29 corresponds to the particular case where $g = \bullet$.

4.3.4 Composition with smooth functions

We finally remark that one can also define the composition $F(f)$ for modelled distributions $f \in \mathcal{D}^{\gamma}_{M;\mathrm{fp}}$ and sufficiently smooth functions $F\colon \mathbb{R} \to \mathbb{R}$.

Proposition 4.31. *Let $M \in \mathcal{M}$ be a model and $f = f_1\mathbf{1} + f_2\upharpoonleft + f_3 X_1 + f_4 X_2 \in \mathcal{D}^{\gamma}_{M;\mathrm{fp}}$ for some $\gamma \in (1+\kappa, 2(1-\kappa))$. Let $F \in \mathcal{C}^{\frac{\gamma}{1-\kappa}}$ and set*

$$F(f) \coloneqq x \mapsto F(f_1(x))\mathbf{1} + F'(f_1(x))\big(f_2(x)\upharpoonleft + f_3(x)X_1 + f_4(x)X_2\big).$$

Then $F(f) \in \mathcal{D}^{\gamma}_{M}$ and if furthermore $F \in \mathcal{C}^{\frac{\gamma}{1-\kappa}+1}$, then the map $f \mapsto F(f)$ is locally Lipschitz continuous. Note also that if $\Pi_x(\tau)$ for $\tau \in \{\mathbf{1}, \upharpoonleft, X_1, X_2\}$ are all continuous functions, then by Example 4.10, $\mathcal{R}(F(f)) = F(\mathcal{R}(f))$.

The proof is a straighforward calculation from the assumptions and the Definition 4.22 of a modelled distribution. We refer to [16, Theorem 4.16] for a more general discussion.

4.4 Back to the PDE

Let us show how the above results allow to study the elliptic singular PDE (E). Given the model $M^{\epsilon} = (\Pi^{\epsilon}, \Gamma^{\epsilon})$ constructed in sections 4.2.1 and 4.2.2, if we set (recall (3.12))

$$U_{\epsilon}(x) \coloneqq u_{\epsilon}(x)\mathbf{1} + (a + b u_{\epsilon}(x))\upharpoonleft + \sum_{i=1}^{2} \partial_i f_x(x)\, X_i$$

then $U_{\epsilon} \in \mathcal{D}^{1+2\kappa}_{M^{\epsilon}}$ for a $\kappa > 0$ small. This U_{ϵ} is such that $\langle \Pi^{\epsilon}, U_{\epsilon}\rangle_x(x) = u_{\epsilon}(x)$, namely $\mathcal{R}\langle \Pi^{\epsilon}, U_{\epsilon}\rangle = u_{\epsilon}$. Now we have

$$\mathcal{R}\langle \Pi^{\epsilon}, U_{\epsilon}\rangle = u_{\epsilon} = \mathsf{K} * ((a + b u_{\epsilon})\xi_{\epsilon}).$$

Can we find a modelled distribution whose reconstruction gives the right-hand side as well? Let us recall that

$$\Pi^{\epsilon}_x(\bullet) = \xi_{\epsilon}, \quad \Pi^{\epsilon}_x(\updownarrow) = (\mathsf{K} * \xi_{\epsilon} - \mathsf{K} * \xi_{\epsilon}(x))\,\xi_{\epsilon}, \quad \Pi^{\epsilon}_x(\bullet X_i) = (\cdot_i - x_i)\xi_{\epsilon}.$$

Then by Proposition 4.29 (note the product rule $\bullet\uparrow = \updownarrow$), recalling (3.9),

$$\begin{aligned} V_\epsilon &:= \bullet\,(a+bU_\epsilon)(x) \\ &:= (a+bu_\epsilon(x))\,\bullet + b(a+bu_\epsilon(x))\updownarrow + b\,\partial_i \mathsf{K} * ((u_\epsilon - u_\epsilon(x))\xi_\epsilon)(x)\,\bullet X_i \end{aligned}$$

defines a modelled distribution in $\mathcal{D}^\kappa_{M^\epsilon}$ with $\kappa > 0$ small and

$$\mathcal{R}\langle \Pi^\epsilon, V_\epsilon\rangle = (a+bu_\epsilon)\xi_\epsilon.$$

Then we have by Corollary 4.26

$$\mathsf{K} * ((a+bu_\epsilon)\xi_\epsilon) = \mathsf{K} * \mathcal{R}\langle \Pi^\epsilon, \bullet\,(a+bU_\epsilon)\rangle = \mathcal{R}\circ\mathcal{K}\langle \Pi^\epsilon, \bullet\,(a+bU_\epsilon)\rangle$$

where $\mathcal{K} : \mathcal{D}^\kappa_{M^\epsilon} \to \mathcal{D}^{1+2\kappa}_{M^\epsilon}$ is the convolution operator on coherent germs defined in Theorem 4.12. It turns out that in this case we have (see below)

$$\begin{aligned} \mathcal{K}V_\epsilon(x) = \mathsf{K} * ((a+bu_\epsilon)\xi_\epsilon)(x)\,\mathbf{1} &+ (a+bu_\epsilon(x))\uparrow \\ &+ \sum_{i=1}^{2} b\,\partial_i \mathsf{K} * ((u_\epsilon - u_\epsilon(x))\xi_\epsilon)(x)\, X_i. \end{aligned}$$

In a classical setting we would like to find u_ϵ as the fixed point of a map

$$z \mapsto \mathsf{K} * ((a+bz)\xi_\epsilon)\,, \tag{4.12}$$

in such a way moreover that the (deterministic) map $\xi_\epsilon \mapsto u_\epsilon$ is *continuous* in a (distributional) $\mathcal{C}^{-1-\kappa}$ topology. This is however *impossible*, as we have seen at the beginning of Section 3.

Regularity structures give an alternative approach, by lifting the equation to a space of modelled distributions. Given any model $M = (\Pi, \Gamma)$, for a $Z \in \mathcal{D}^{1+2\kappa}_{(\Pi,\Gamma)}$ of the form

$$Z(x) = z(x)\mathbf{1} + Z^{\uparrow}(x)\uparrow + \sum_{i=1}^{2} Z^{X_i}(x)\, X_i, \qquad z = \mathcal{R}Z, \tag{4.13}$$

where $z(x), Z^{\uparrow}(x)$ and $Z^{X_i}(x)$ are real numbers, we write

$$\bullet\, Z(x) = z(x)\,\bullet + Z^{\uparrow}(x)\updownarrow + \sum_{i=1}^{2} Z^{X_i}(x)\,\bullet X_i.$$

Then by Proposition 4.29 we have $\bullet\, Z \in \mathcal{D}^{\kappa}_{(\Pi,\Gamma)}$ and therefore $\bullet\,(a+bZ) \in \mathcal{D}^{\kappa}_{(\Pi,\Gamma)}$. By Corollary 4.26 we can apply the integration operator $\mathcal{K}$ defined in (4.3) and obtain a modelled distribution $\mathcal{K}(\bullet\,(a+bZ)) \in \mathcal{D}^{1+2\kappa}_{(\Pi,\Gamma)}$, which has the form

$$\mathcal{K}(\bullet\,(a+bZ))(x) = \mathsf{K} * T(x)\,\mathbf{1} + (a+bz(x))\,\overset{\bullet}{|} + \sum_{i=1}^{2} b\,\partial_i \mathsf{K} * ((z-z(x))\xi)(x)\, X_i,$$

where the distribution T is defined as $\mathcal{R}(\bullet\,(a+bZ))$. Therefore we have lifted the map (4.12) to a map $Z \mapsto \mathcal{K}(\bullet\,(a+bZ))$ in the space of modelled distributions $\mathcal{D}^{1+2\kappa}_{(\Pi,\Gamma)}$.

The main analytical idea is that, while the product $(u,\xi) \mapsto u\xi$ is ill-defined in this context, the map $\mathcal{D}^{1+2\kappa}_{(\Pi,\Gamma)} \ni Z \mapsto \bullet\, Z \in \mathcal{D}^{\kappa}_{(\Pi,\Gamma)}$ is perfectly well-defined and continuous, *once the model* (Π,Γ) *is constructed*. Moreover suitable estimates give continuity with respect of the model of all the above objects. We recall that the crucial term $\Pi_x(\overset{\bullet}{\underset{\bullet}{|}})$ in the model fixes a value for the ill-defined product $(\mathsf{K} * \xi)\xi$, which is the missing information to give a sense to the map (4.12) for $\xi \in \mathcal{C}^{-1-\kappa}$.

Now the fixed point equation $U = \mathcal{K}(\bullet\,(a+bU))$ is equivalent to the system of equations

$$\begin{aligned} u &= \mathsf{K} * (\mathcal{R}(\bullet\,(a+bU))), \\ U^{\overset{\bullet}{|}}(x) &= a + bu(x), \\ U^{X_i}(x) &= b\,\partial_i \mathsf{K} * ((u-u(x))\xi)(x). \end{aligned}$$

It is interesting to see that in this system the first equality is still the equation (now for a general model), while the next two equalities merely fix the remaining coefficients of U in (4.13). We recall that such coefficients of U are the constitutive elements of the semi-norms $\|U\|_{\mathcal{D}^{1+2\kappa}_{(\Pi,\Gamma)}}$ defined in (4.11). Therefore, lifting the PDE to a space of modelled distributions does not modify the equation but imposes a stronger topology on the solutions. This is the key element of the crucial continuity property $(\Pi,\Gamma) \mapsto U$, which associates to a model the solution to the fixed point $U = \mathcal{K}(\bullet\,(a+bU))$.

In the next subsection we explore this fixed point equation more closely.

4.4.1 Fixed point

Fix $\gamma \in (1+\kappa, 2-2\kappa)$ and consider the following iteration map, which is the equivalent of (2.3) in the space of modelled distributions:

$$\begin{array}{rccc} \Phi^\epsilon : & \mathcal{D}^\gamma_{M^\epsilon;\mathrm{fp}} & \longrightarrow & \mathcal{D}^\gamma_{M^\epsilon;\mathrm{fp}} \\ & f & \longmapsto & (\mathcal{K}+\mathcal{K}^{\mathsf{R}})(a\bullet + b\bullet f). \end{array}$$

By Corollary 4.26 and Proposition 4.29, Φ^ϵ is well defined, and combining all the corresponding continuity bounds, there exist constants $C, k_0 > 0$ and a compact set $K \subset \mathbb{R}^d$ such that

$$\|\Phi^\epsilon(f) - \Phi^\epsilon(g)\|_{\mathcal{D}^\gamma_{M^\epsilon}} \le b\, C \|M^\epsilon\|^{k_0}_{\mathcal{M}(K)} \|f-g\|_{\mathcal{D}^\gamma_{M^\epsilon}},$$

whence the existence of a unique fixed-point $f_\epsilon \in \mathcal{D}^\gamma_{M^\epsilon;\mathrm{fp}}$ after choosing b small enough in function of M^ϵ.

Note that, because ξ_ϵ is smooth, and because the operations constructed above are compatible with the reconstruction, applying the reconstruction operator to the fixed point equation shows that $u_\epsilon := \mathcal{R}(f_\epsilon)$ is a solution to

$$u_\epsilon = \mathsf{G} * ((a+bu_\epsilon)\xi_\epsilon) \quad \text{i.e. by Theorem A.17:} \quad (1-\Delta)u_\epsilon = (a+bu_\epsilon)\xi_\epsilon,$$

where all concerned modelled distributions can be uniquely reconstructed because $\gamma > 1+\kappa$ (recall that $\bullet f_\epsilon \in \mathcal{D}^{\gamma-1-\kappa}$).

Now the reader may wonder why we spent so much effort solving (E) for a *smooth* ξ^ϵ: surely we could have done this by more elementary methods. The advantage of the approach presented above is that we have in fact exhibited a factorisation

$$\begin{array}{ccccccc} \mathcal{C}^{-1-\kappa} & \longrightarrow & \mathcal{M}_{\mathrm{adm}} & \longrightarrow & \bigsqcup_{M\in\mathcal{M}_{\mathrm{adm}}} \mathcal{D}^\gamma_M & \xrightarrow{\mathcal{R}} & \mathcal{C}^{1-\kappa} \\ \xi_\epsilon & \longmapsto & M^\epsilon & \longmapsto & f_\epsilon & \longmapsto & u_\epsilon, \end{array} \tag{4.14}$$

of the solution map, where the last two arrows are *continuous* with respect to the natural topologies involved. Thus, studying the convergence of $(u_\epsilon)_{\epsilon>0}$ as $\epsilon \to 0$ amounts to studying the convergence of $(M^\epsilon)_{\epsilon>0}$ in the space $\mathcal{M}$ of models: this is particularly nice, because M^ϵ consists in a finite number of explicit objects, recall Table 4.1.

The purpose of the following Section 5 is to study the family $(M^\epsilon)_{\epsilon>0}$. As it turns out, recall Remark 4.16, this is a divergent family as $\epsilon \to 0$: however, there will be a natural way of "curing" the divergence via a suitable renormalisation procedure, so that the family of corresponding renormalised solutions $\hat{u}_\epsilon$ will converge, to what we will call the solution to (E).

Remark 4.32. Note that the continuity of the second arrow in (4.14) does not immediately follow from Theorem 4.25 and Proposition 4.29, because the continuity estimates in those theorems compare modelled distributions on the same model.

However, it is possible to provide "enhanced" continuity estimates in Theorem 4.25 and Proposition 4.29, i.e. comparing modelled distributions on different models. We refer the reader to [16] for more details.

5 The equation in dimension $d = 2$: probabilistic aspects

We now consider the (lack of) convergence as $\epsilon \to 0$ of the random family $\big(M^\epsilon = (\Pi^\epsilon, \Gamma^\epsilon)\big)_{\epsilon>0}$ defined by Table 4.1 and (4.7), in the set of admissible models $\mathcal{M}_{\mathrm{adm}}$. We proceed in three steps:

1. we consider a "Kolmogorov-type criterion" on the convergence of a family of models (Section 5.1),

2. we modify in a natural way the family of models defined by Table 4.1 and (4.7), in order to make it convergent (Sections 5.2 and 5.3),

3. we study the effect of this renormalisation on the equation and its solution (Sections 5.4 and 5.5).

5.1 A Kolmogorov criterion for the convergence of random models

A sufficient condition for the convergence of a family of random models, on the structure considered here, is provided by the following result, see [16, Theorem 10.7].

Theorem 5.1 (Kolmogorov continuity for models)**.** *Let* $(M^\epsilon = (\Pi^\epsilon, \Gamma^\epsilon))_{\epsilon>0}$ *be a family of stationary admissible models in* $\mathcal{M}_{\mathrm{adm}}$ *(recall Section 4.2 for the corresponding definitions), defined on the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*. We assume that there exist* $\theta, \theta' > 0$ *such that for all symbols* $\tau \in I$ *of negative homogeneity* $\alpha_\tau < 0$*, all test-functions* $\varphi \in C_c^\infty$*, and all* $p \geq 1$*,* $\epsilon, \epsilon_1, \epsilon_2, \lambda \in (0,1)$*,*

$$\mathbb{E}\big[|\Pi_0^\epsilon(\tau)(\varphi^\lambda)|^p\big] \leq C_{\tau,\varphi,p}\lambda^{p(\alpha_\tau+\theta)},$$
$$\mathbb{E}\big[|(\Pi_0^{\epsilon_1}(\tau) - \Pi_0^{\epsilon_2}(\tau))(\varphi^\lambda)|^p\big] \leq C_{\tau,\varphi,p}(\epsilon_1+\epsilon_2)^{p\theta'}\lambda^{p(\alpha_\tau+\theta)}.$$

Then for all $p \geq 1$*, the family* $(M^\epsilon)_{\epsilon>0}$ *converges in* $L^p\big((\Omega, \mathcal{F}, \mathbb{P}); \mathcal{M}_{\mathrm{adm}}\big)$*.*

Remark 5.2. Since we work here with Gaussian variables, we can generally exploit results of the type "equivalence of moments". Indeed, all the random variables we will consider in fact belong to a Wiener chaos of some finite order, where such a result is known, see e.g. [16, Lemma 10.5]. Thus, for convenience, in what follows we will only check the bounds for $p = 2$, i.e. that (renaming $2\theta, 2\theta'$ as θ, θ')

$$\mathbb{E}\big[|\Pi_0^\epsilon(\tau)(\varphi^\lambda)|^2\big] \leq C_{\tau,\varphi}\lambda^{2\alpha_\tau+\theta}, \tag{5.1}$$
$$\mathbb{E}\big[|(\Pi_0^{\epsilon_1}(\tau) - \Pi_0^{\epsilon_2}(\tau))(\varphi^\lambda)|^2\big] \leq C_{\tau,\varphi}(\epsilon_1+\epsilon_2)^{\theta'}\lambda^{2\alpha_\tau+\theta}. \tag{5.2}$$

Recall from Table 4.1 that the symbols of negative homogeneity are here:

$$\{\bullet, \overset{\bullet}{\bullet}, \bullet X_1, \bullet X_2\}.$$

In the remainder of this section, we check the bounds (5.1)-(5.2) for $\tau \in \{\bullet, \bullet X_1, \bullet X_2\}$. The last symbol $\tau = \overset{\bullet}{\bullet}$ is more delicate to treat – this is due to the noise appearing twice – and will require more advanced techniques such as Wick calculus: we delay the corresponding discussion to Section 5.2.

Example 5.3. Let us establish (5.1)-(5.2) for the symbol $\tau = \bullet$. Since $\bullet$ has homogeneity $\alpha_\bullet = -1-\kappa$ it suffices to establish for some $\theta' > 0$,

$$\mathbb{E}\big[|\Pi_0^\epsilon(\bullet)(\varphi^\lambda)|^2\big] \leq C_{\tau,\varphi}\lambda^{-2-\kappa},$$
$$\mathbb{E}\big[|(\Pi_0^{\epsilon_1}(\bullet) - \Pi_0^{\epsilon_2}(\bullet))(\varphi^\lambda)|^2\big] \leq C_{\tau,\varphi}(\epsilon_1+\epsilon_2)^{\theta'}\lambda^{-2-\kappa},$$

which corresponds to taking $\theta = \kappa$ in (5.1)-(5.2). We will use Young's convolution inequality in $\mathbb{R}^d$:

$$\|f * g\|_{L^r} \leq \|f\|_{L^p}\|g\|_{L^q}, \quad \text{if} \quad \frac{1}{p}+\frac{1}{q}=\frac{1}{r}+1, \quad 1 \leq p,q,r \leq +\infty.$$

Recall that by definition,

$$\Pi_0^\epsilon(\bullet)(\varphi^\lambda) = \xi_\epsilon(\varphi^\lambda) = \xi(\rho^\epsilon * \varphi^\lambda),$$

so that by the isometry property of the Gaussian white noise ξ, recalling that $f_k(\cdot) \coloneqq f(\cdot - k)$,

$$\begin{aligned}\mathbb{E}^{\frac{1}{2}}\big[|\Pi_0^\epsilon(\bullet)(\varphi^\lambda)|^2\big] &= \|(\rho^\epsilon * \varphi^\lambda)^{\mathrm{per}}\|_{L^2([0,1]^2)} \\ &= \|\sum_{k\in\mathbb{Z}^d}(\rho^\epsilon * \varphi^\lambda)_k\|_{L^2([0,1]^2)} \\ &\leq \sum_{k\in\mathbb{Z}^d}\|(\rho^\epsilon * \varphi^\lambda)_k\|_{L^2([0,1]^2)}.\end{aligned}$$

Since ρ and φ have compact support, the function $\rho^\epsilon * \varphi^\lambda$ is supported in a region of $\mathbb{R}^d$ which is bounded uniformly in $\epsilon, \lambda \in (0,1]$. Thus, after taking into account the recentering by k, the sum above admits a bounded number of non-zero terms, which depends only on the support of φ (and of ρ). Applying Young's convolution inequality to each of those terms leads to

$$\mathbb{E}\big[|\Pi_0^\epsilon(\bullet)(\varphi^\lambda)|^2\big] \lesssim_\varphi \|\rho^\epsilon\|_{L^1}^2\|\varphi^\lambda\|_{L^2}^2 = \|\rho\|_{L^1}^2\|\varphi\|_{L^2}^2\lambda^{-2},$$

as wanted. For the second bound, we have similarly after periodization and for reasons of support

$$\mathbb{E}^{\frac{1}{2}}\left[|(\Pi_0^{\epsilon_1}(\bullet)-\Pi_0^{\epsilon_2}(\bullet))(\varphi^\lambda)|^2\right]\le\sum_k\|(\rho^{\epsilon_1}-\rho^{\epsilon_2})*\varphi_k^\lambda\|_{L^2([0,1]^2)}.$$

over a finite sum whose size depends only on the support of φ. By definition of the convolution, and using the fact that $\rho^{\epsilon_1,\epsilon_2}:=\rho^{\epsilon_1}-\rho^{\epsilon_2}$ has vanishing integral, we express

$$(\rho^{\epsilon_1}-\rho^{\epsilon_2})*\varphi_k^\lambda(x)=\int\rho^{\epsilon_1,\epsilon_2}(z)(\varphi_k^\lambda(x-z)-\varphi_k^\lambda(x))\,\mathrm{d}z.$$

Thus,

$$\begin{aligned}&\big((\rho^{\epsilon_1}-\rho^{\epsilon_2})*\varphi_k^\lambda(x)\big)^2\\&=\int\rho^{\epsilon_1,\epsilon_2}(z_1)\rho^{\epsilon_1,\epsilon_2}(z_2)(\varphi_k^\lambda(x-z_1)-\varphi_k^\lambda(x))(\varphi_k^\lambda(x-z_2)-\varphi_k^\lambda(x))\,\mathrm{d}z_1\,\mathrm{d}z_2,\end{aligned}$$

and integrating against x,

$$\|\rho^{\epsilon_1,\epsilon_2}*\varphi_k^\lambda\|_{L^2}^2=\iint\rho^{\epsilon_1,\epsilon_2}(z_1)\rho^{\epsilon_1,\epsilon_2}(z_2)F_\lambda(z_1,z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2,\tag{5.3}$$

where (recall that $d=2$)

$$\begin{aligned}F_\lambda(z_1,z_2)&:=\int(\varphi_k^\lambda(x-z_1)-\varphi_k^\lambda(x))(\varphi_k^\lambda(x-z_2)-\varphi_k^\lambda(x))\,\mathrm{d}x\\&=\lambda^{-2}\int\Big(\varphi_k(u-\frac{z_1}{\lambda})-\varphi_k(u)\Big)\Big(\varphi_k(u-\frac{z_2}{\lambda})-\varphi_k(u)\Big)\,\mathrm{d}u.\end{aligned}$$

Since φ has compact support, the domain of the integral above is of bounded size, and using the fact that φ is θ'-Hölder for any $\theta'\in(0,1)$:

$$|F_\lambda(z_1,z_2)|\le C_{\varphi,\theta'}\lambda^{-2}\Big|\frac{z_1}{\lambda}\Big|^{\theta'}\Big|\frac{z_2}{\lambda}\Big|^{\theta'}=C_{\varphi,\theta'}\lambda^{-2-2\theta'}|z_1|^{\theta'}|z_2|^{\theta'},$$

so that plugging this estimate in (5.3) gives

$$\|(\rho^{\epsilon_1}-\rho^{\epsilon_2})*\varphi_k^\lambda\|_{L^2}^2\le C_{\varphi,\theta'}\lambda^{-2-2\theta'}\Big(\int|(\rho^{\epsilon_1}-\rho^{\epsilon_2})(z)||z|^{\theta'}\,\mathrm{d}z\Big)^2,$$

whence the announced bound after taking θ' sufficiently small, because

$$\int|(\rho^{\epsilon_1}-\rho^{\epsilon_2})(z)||z|^{\theta'}\,\mathrm{d}z\le\int(|\rho^{\epsilon_1}(z)|+|\rho^{\epsilon_2}(z)|)|z|^{\theta'}\,\mathrm{d}z\le(\epsilon_1+\epsilon_2)^{\theta'}.$$

Example 5.4. Let us establish (5.1)-(5.2) for the symbol $\tau = \bullet X_1$ (a slight variant of the same argument also treats $\tau = \bullet X_2$). Since $\tau = \bullet X_1$ has homogeneity $\alpha_{\bullet X_1} = -\kappa$, it suffices to establish

$$\mathbb{E}\big[|\Pi_0^\epsilon(\bullet X_1)(\varphi^\lambda)|^2\big] \leq C_{\tau,\varphi}\lambda^{-\kappa},$$
$$\mathbb{E}\big[|(\Pi_0^{\epsilon_1}(\bullet X_1) - \Pi_0^{\epsilon_2}(\bullet X_1))(\varphi^\lambda)|^2\big] \leq C_{\tau,\varphi}(\epsilon_1+\epsilon_2)^{\theta'}\lambda^{-\kappa}.$$

We appeal to the calculations of Example 5.3: define $\eta(\cdot) := \cdot_1\varphi(\cdot)$ so that $\cdot_1\varphi^\lambda(\cdot) = \lambda\eta^\lambda(\cdot)$. Then by definition,

$$\Pi_0^\epsilon(\bullet X_1)(\varphi^\lambda) = \xi_\epsilon(\cdot_1\varphi^\lambda(\cdot)) = \xi(\rho^\epsilon * (\cdot_1\varphi^\lambda(\cdot))) = \lambda\xi(\rho^\epsilon * \eta^\lambda),$$

so that the announced estimates follow directly from those of Example 5.3.

5.2 Elements of Wick calculus

The last remaining symbol to tackle is $\mathbf{I}$. Recall that its homogeneity is $\alpha_{\mathbf{I}} = -2\kappa$ so that the goal is to establish

$$\mathbb{E}\big[|\Pi_0^\epsilon(\mathbf{I})(\varphi^\lambda)|^2\big] \leq C_{\tau,\varphi}\lambda^{-\theta}, \tag{5.4}$$
$$\mathbb{E}\big[|\Pi_0^{\epsilon_1}(\mathbf{I})(\varphi^\lambda) - \Pi_0^{\epsilon_2}(\mathbf{I})(\varphi^\lambda)|^2\big] \leq C_{\tau,\varphi}(\epsilon_1+\epsilon_2)^{\theta'}\lambda^{-\theta}, \tag{5.5}$$

where

$$\begin{aligned}\Pi_0^\epsilon(\mathbf{I})(\varphi^\lambda) &= \int (\mathsf{K} * \xi_\epsilon(x) - \mathsf{K} * \xi_\epsilon(0))\xi_\epsilon(x)\varphi^\lambda(x)\,\mathrm{d}x \\ &= \int \xi\big(\mathsf{K}_\epsilon(\cdot - x) - \mathsf{K}_\epsilon(\cdot)\big)\xi\big(\rho^\epsilon(\cdot - x)\big)\varphi^\lambda(x)\,\mathrm{d}x,\end{aligned} \tag{5.6}$$

and where we denote for simplicity $\mathsf{K}_\epsilon := \mathsf{K} * \rho^\epsilon$. The difficulty is that this is not a Gaussian variable anymore, and in particular we can not apply the same strategy as in the section above. Still, (5.6) retains some stochastic structure: crucially, the expression (5.6) is bilinear in ξ, which makes it natural to express it in tensor product notation, as

$$\begin{aligned}\Pi_0^\epsilon(\mathbf{I})(\varphi^\lambda) &= \langle \xi \otimes \xi, \int \big(\mathsf{K}_\epsilon(\cdot - x) - \mathsf{K}_\epsilon(\cdot)\big) \otimes \big(\rho^\epsilon(\cdot - x)\big)\varphi^\lambda(x)dx\rangle \\ &= \langle \xi \otimes \xi, \int \big(\mathsf{K}_\epsilon^{\mathrm{per}}(\cdot - x) - \mathsf{K}_\epsilon^{\mathrm{per}}(\cdot)\big) \otimes \big(\rho^{\epsilon,\mathrm{per}}(\cdot - x)\big)\varphi^\lambda(x)dx\rangle,\end{aligned}$$

where the right-hand term in the bracket is deterministic, explicit, and 1-periodic. This motivates that we should study the behaviour of $\xi \otimes \xi$ on the tensor product $L^2(\mathbb{T}^d)^{\otimes 2}$. On simple tensors, i.e. of the form $f \otimes g$ for $f, g \in L^2(\mathbb{T}^d)$, observe that

$$\langle \xi \otimes \xi, f \otimes g \rangle = \langle \xi, f \rangle \langle \xi, g \rangle,$$

is the product of two Gaussian variables, with expectation

$$\mathbb{E}[\langle \xi \otimes \xi, f \otimes g \rangle] = \langle f, g \rangle,$$

by definition of Gaussian white noise. Now we may also estimate its variance with Isserlis' theorem [19]-[2, Theorem 4.2.2] on the product of the entries of a centered Gaussian vector $(X_1, \cdots X_n)$, stating that

$$\mathbb{E}[X_1 \cdots X_n] = \mathbb{1}_{\{n \text{ is even}\}} \sum_{\text{Pairings } \mathscr{P} \text{ of } \{1,\cdots,n\}} \prod_{\{i,j\} \in \mathscr{P}} \mathbb{E}[X_i X_j],$$

yielding in our case

$$\mathrm{Var}[\langle \xi \otimes \xi, f \otimes g \rangle] = \langle f, f \rangle \langle g, g \rangle + \langle f, g \rangle^2 \leq 2 \|f\|_{L^2}^2 \|g\|_{L^2}^2 = 2 \|f \otimes g\|_{L^2}^2,$$

where we used the Cauchy–Schwarz inequality. More generally, setting

$$\hat{I}_2(f \otimes g) \coloneqq \langle \xi \otimes \xi, f \otimes g \rangle - \langle f, g \rangle, \tag{5.7}$$

and extending $\hat{I}_2$ by linearity on $L^2(\mathbb{T}^d)^{\otimes 2}$ one finds for $h \in L^2(\mathbb{T}^d)^{\otimes 2}$,

$$\mathbb{E}[\hat{I}_2(h)] = 0, \qquad \mathbb{E}[\hat{I}_2(h)^2] \leq 2 \|h\|_{L^2}^2. \tag{5.8}$$

We wil exploit the properties (5.7)-(5.8) in the following way: we rewrite by definition of $\hat{I}_2$,

$$\Pi_0^\epsilon(:)(\varphi^\lambda) = \int \langle \mathsf{K}_\epsilon^{\mathrm{per}}(\cdot - x) - \mathsf{K}_\epsilon^{\mathrm{per}}(\cdot), \rho^{\epsilon,\mathrm{per}}(\cdot - x) \rangle \varphi^\lambda(x) \, \mathrm{d}x \tag{5.9}$$

$$+ \hat{I}_2\Big(\int \big(\mathsf{K}_\epsilon^{\mathrm{per}}(\cdot - x) - \mathsf{K}_\epsilon^{\mathrm{per}}(\cdot)\big) \otimes \big(\rho^{\epsilon,\mathrm{per}}(\cdot - x)\big) \varphi^\lambda(x) dx \Big), \tag{5.10}$$

where the first term is deterministic and explicit, while the second one is random but controlled by (5.8). Note that the operation we have performed corresponds to a "renormalisation" since we have isolated the expectation in (5.7).

Remark 5.5. Let us mention that the approach sketched above can be widely generalised. In particular, it is well-known that there exists an isometric isomorphism

$$I\colon \bigoplus_{n\geq 0} \big(L^2(\mathbb{T}^d)\big)^{\otimes_s n} \to L^2(\Omega, \mathcal{F}, \mathbb{P}),$$

called *Wiener isometry*, which is explicit (in terms of Hermite polynomials): denoting I_n the restriction of I to the symmetrized tensor product $\big(L^2(\mathbb{T}^d)\big)^{\otimes_s n}$, then

$$I_1(f) = \xi(f), \qquad I_2(f\otimes f) = \frac{1}{\sqrt{2}}\hat{I}_2(f\otimes f) = \frac{1}{\sqrt{2}}(\xi(f)^2 - \|f\|^2), \qquad \text{etc.}$$

Furthermore, products of the form $I_n(f)I_m(g)$ can be expressed via an explicit multiplication formula, see [16, Lemma 10.3]-[24, Proposition 1.1.3].

The decomposition of a random variable along $\oplus_{n\geq 0}\big(L^2(\mathbb{T}^d)\big)^{\otimes_s n}$ via I, such as in (5.9)-(5.10), is called its *Wiener chaos decomposition*: one advantage is that each of these terms can be bounded separately by the isometry property of I.

5.3 Renormalisation of M^ϵ

Recalling (5.9)-(5.10), we rewrite (assuming without loss of generality that $\int \varphi = 1$):

$$\Pi_0^\epsilon(\updownarrow)(\varphi^\lambda) = \hat{I}_2\bigg(\int_{\mathbb{R}^2} W_\epsilon(x;\cdot_1,\cdot_2)\varphi^\lambda(x)\,\mathrm{d}x\bigg) + \int_{\mathbb{R}^d} \tilde{W}_\epsilon(x)\varphi^\lambda(x)\,\mathrm{d}x + c_\epsilon,$$

where

$$\begin{aligned} W_\epsilon(x;z_1,z_2) &:= \big(\mathsf{K}_\epsilon^{\mathrm{per}}(z_1 - x) - \mathsf{K}_\epsilon^{\mathrm{per}}(z_1)\big)\rho^{\epsilon,\mathrm{per}}(z_2 - x), \\ \tilde{W}_\epsilon(x) &:= -\langle \mathsf{K}_\epsilon^{\mathrm{per}}(\cdot), \rho^{\epsilon,\mathrm{per}}(\cdot - x)\rangle = -\mathsf{K}_\epsilon^{\mathrm{per}} * \rho^{\epsilon,\mathrm{per}}(x) \\ c_\epsilon &:= \langle \mathsf{K}_\epsilon^{\mathrm{per}}(\cdot - x), \rho^{\epsilon,\mathrm{per}}(\cdot - x)\rangle = \langle \mathsf{K}_\epsilon^{\mathrm{per}}, \rho^{\epsilon,\mathrm{per}}\rangle, \end{aligned} \tag{5.11}$$

are deterministic and explicit in function of K. Now, recalling (5.9)-(5.10), the following proposition implies that the desired estimate (5.4) fails, but in a "renormalisable" way:

$$\mathbb{E}\big[|\Pi_0^\epsilon(\updownarrow)(\varphi^\lambda)|^2\big] \to_{\epsilon\to 0} \infty \qquad \text{but} \qquad \mathbb{E}\big[|\Pi_0^\epsilon(\updownarrow)(\varphi^\lambda) - c_\epsilon|^2\big] \leq C_{\tau,\varphi}\lambda^{-\theta}.$$

Proposition 5.6 (See [16, Theorem 10.19]). *In the context described just above,*

1. *For $\epsilon > 0$ small enough,*
$$c_\epsilon = -\frac{1}{2\pi}\log|\epsilon| + O_{\epsilon\to 0}(1).$$
2. *For any $\theta > 0$ sufficiently small,*
$$\Big|\int_{\mathbb{R}^2} \tilde{W}_\epsilon(x)\varphi^\lambda(x)\,\mathrm{d}x\Big|^2 \le C_{\varphi,\theta}\lambda^{-\theta}.$$
3. *For any $\theta > 0$ sufficiently small,*
$$\Big\|\int_{\mathbb{R}^2} W_\epsilon(x;\cdot_1,\cdot_2)\varphi^\lambda(x)\,\mathrm{d}x\Big\|^2_{L^2([0,1]^2)} \le C_{\varphi,\theta}\lambda^{-\theta}.$$

For convenience, we will prove only the first two points, as the third one can be obtained with similar techniques but the corresponding calculations are more tedious. We refer to [16, Theorem 10.19] for more details.

Proof (of Items 1 and 2). Recall that by construction,
$$\mathsf{K}(x) = -\frac{1}{2\pi}\log|x| + O_{x\to 0}(1). \tag{5.12}$$
Proof of Item 1. For ϵ small enough in function of ρ, the periodization $\rho^{\epsilon,per}$ coincides with ρ^ϵ on $[-\frac{1}{2},\frac{1}{2}]^2$ which is our region of integration, thus we may write
$$\begin{aligned} c_\epsilon &= \langle \mathsf{K}^{\mathrm{per}}_\epsilon, \rho^{\epsilon,\mathrm{per}}\rangle_{\mathbb{L}^2(\mathbb{T}^2)} \\ &= \iint_{[-\frac{1}{2},\frac{1}{2}]^2} \mathsf{K}(z)\rho^\epsilon(x-z)\rho^\epsilon(x)\,\mathrm{d}x\,\mathrm{d}z \\ &= \iint_{[-\frac{1}{2\epsilon},\frac{1}{2\epsilon}]^2} K(\epsilon z)\rho(x-z)\rho(x)\,\mathrm{d}x\,\mathrm{d}z, \end{aligned}$$
so using (5.12), the fact that ρ integrates to 1, and the integrability of log at the origin,
$$\begin{aligned} c_\epsilon &= -\iint_{[-\frac{1}{2\epsilon},\frac{1}{2\epsilon}]^2} \frac{1}{2\pi}\log|\epsilon z|\rho(x-z)\rho(x)\,\mathrm{d}x\,\mathrm{d}z + O_{\epsilon\to 0}(1) \\ &= -\frac{1}{2\pi}\log|\epsilon| + O_{\epsilon\to 0}(1), \end{aligned}$$

which establishes the claim.

Proof of Item 2. By definition,

$$\begin{aligned}\int_{\mathbb{R}^2} \tilde{W}_\epsilon(x)\varphi^\lambda(x)\,\mathrm{d}x &= -\int_{\mathbb{R}^2} \mathsf{K} * \rho^{\epsilon,\mathrm{per}} * \rho^{\epsilon,\mathrm{per}}(x)\varphi^\lambda(x)\,\mathrm{d}x \\ &= -\sum_{k,l\in\mathbb{Z}^2} \int_{\mathbb{R}^2} (\mathsf{K} * (\rho_k * \rho_l)^\epsilon)(\lambda u)\varphi(u)\,\mathrm{d}u, \qquad (5.13)\end{aligned}$$

over a finite sum whose number of terms depends only on φ and K (for reasons of support).

Fix $\theta > 0$, then from the logarithmic estimate (5.12) and the fact that K is compactly supported (by construction), one can estimate

$$|\mathsf{K}(x)| \leq C|x|^{-\theta},$$

and thus it suffices to prove that for any given mollifier $\eta \in C_c^\infty$ (we will choose $\eta = \rho_k * \rho_l$), the convolution $\mathsf{K} * \eta^\epsilon$ still satisfies such a bound uniformly in ϵ: this is done in [16, Lemma 10.17], but let us replicate the argument here. Without loss of generality, assume $\mathrm{supp}(\eta) \subset B(0,1)$. Let $x \in \mathbb{R}^d$. If $|x| \leq 2\epsilon$, we write

$$\mathsf{K} * \eta^\epsilon(x) = \int \mathsf{K}(z)\eta^\epsilon(x-z)\,\mathrm{d}z,$$

so that for reasons of support, $|z| \leq 3\epsilon$ in this integral. We bound $|\eta^\epsilon(x-z)| \leq C\epsilon^{-2}$, hence,

$$\begin{aligned}|\mathsf{K} * \eta^\epsilon(x)| &\leq C\epsilon^{-2}\int_{B(0,3\epsilon)} |\mathsf{K}(z)|\,\mathrm{d}z \\ &\leq C\epsilon^{-2}\int_{B(0,3\epsilon)} |z|^{-\theta}\,\mathrm{d}z \\ &\leq C\epsilon^{-\theta} \leq C|x|^{-\theta}.\end{aligned}$$

On the other hand, if $|x| \geq 2\epsilon$ we write

$$\mathsf{K} * \eta^\epsilon(x) = \int \mathsf{K}(x-z)\eta^\epsilon(z)\,\mathrm{d}z,$$

so that for reasons of support, $|z| \leq \epsilon$ in this integral and $|x-z| \geq |x| - |z| \geq |x| - \epsilon \geq |x| - |x|/2 = |x|/2$, thus

$$|\mathsf{K} * \eta^\epsilon(x)| \leq C\int |x-z|^{-\theta}|\rho^\epsilon(z)|\,\mathrm{d}z \leq C|x|^{-\theta},$$

and combining this estimate with (5.13) yields the announced bound. □

Remark 5.7. One can tackle in the same way the second bound (5.5). We admit this fact without proof because the corresponding calculations become technical. We refer to [16, Theorem 10.19] where the calculations are fully carried out.

5.4 The renormalised model

The discussion of the previous section showed that the model M^ϵ diverges, because of the logarithmically divergent term c_ϵ. However, convergence holds for the *renormalised model* $\hat{M}^\epsilon$ defined for $\tau \in I$ by

$$\hat{\Pi}_x^\epsilon(\tau) := \begin{cases} \Pi_x^\epsilon(\updownarrow - c_\epsilon \mathbf{1}) & \text{if } \tau = \updownarrow, \\ \Pi_x^\epsilon(\tau) & \text{else,} \end{cases} \qquad \text{and} \qquad \hat{\Gamma}^\epsilon := \Gamma^\epsilon. \tag{5.14}$$

where c_ϵ defined in (5.11) diverges as

$$c_\epsilon = -\frac{1}{2\pi} \log|\epsilon| + O_{\epsilon\to 0}(1).$$

Remark 5.8. The renormalised model $\hat{\Pi}^\epsilon$ is *not* multiplicative, contrary to Π^ϵ, recall Remark 4.15. Indeed, we have $\hat{\Pi}^\epsilon(\updownarrow) \neq \hat{\Pi}^\epsilon(\bullet)\hat{\Pi}^\epsilon(\uparrow)$.

Remark 5.9. The definition (5.14) shows that for a model (Π, Γ) the collection of basis germs Π is not uniquely determined by Γ in general, even for admissible models.

Remark 5.10. The fact that the renormalised expansion operator $\hat{\Gamma}^\epsilon$ coincides with Γ^ϵ is not true for general equations, as the renormalisation terms may be propagated into $\hat{\Gamma}^\epsilon$ by convolution.

5.5 The renormalised equation

The situation at this point is the following: we have a model Π^ϵ and a renormalised model $\hat{\Pi}^\epsilon$, but only the latter converges (in the space of admissible models) as $\epsilon \to 0$. Let us recall that the reconstruction and convolution operator on modelled distributions depend on the underlying model, so that we note $\mathcal{R}^\epsilon, \mathcal{K}^\epsilon$ resp. $\hat{\mathcal{R}}^\epsilon$, $\hat{\mathcal{K}}^\epsilon$ those operators for the model Π^ϵ resp. $\hat{\Pi}^\epsilon$ (where we include both the singular part K and the remainder R of the kernel G in the convolution operators).

By the discussions above, if we fix a (random) $b > 0$ small enough, there exist modelled distributions $f_\epsilon \in \mathcal{D}^\gamma_{M^\epsilon;\mathrm{fp}}$, $\hat{f}_\epsilon \in \mathcal{D}^\gamma_{\hat{M}^\epsilon;\mathrm{fp}}$ solving the fixed point problem

$$f_\epsilon = \mathcal{K}^\epsilon(a\,\bullet + b\, f_\epsilon \bullet), \qquad \hat{f}_\epsilon = \hat{\mathcal{K}}^\epsilon(a\,\bullet + b\, \hat{f}_\epsilon \bullet), \tag{5.15}$$

and we denote their reconstruction

$$u_\epsilon := \mathcal{R}^\epsilon(f_\epsilon), \qquad \hat{u}_\epsilon := \hat{\mathcal{R}}^\epsilon(\hat{f}_\epsilon),$$

so that by continuity of (4.14), the family $(\hat{u}_\epsilon)_{\epsilon>0}$ converges as $\epsilon \to 0$. We now show that $\hat{u}^\epsilon$ is also the solution to a *renormalised* equation, see also [16, Section 9.3].

By construction, recall Section 4.4.1, the modelled distribution $\hat{f}_\epsilon$ is of the form

$$\hat{f}_\epsilon(x) = f_{1,\epsilon}(x)\mathbf{1} + f_{2,\epsilon}(x)\overset{\bullet}{|} + f_{3,\epsilon}(x)X_1 + f_{4,\epsilon}(x)X_2, \tag{5.16}$$

for some functions $f_{1,\epsilon}, f_{2,\epsilon}, f_{3,\epsilon}, f_{4,\epsilon}$. In fact, expressing the fixed-point identity (5.15), one observes that

$$f_{2,\epsilon}(x) = a + b f_{1,\epsilon}(x), \tag{5.17}$$

Now, on the one hand, since the reconstruction commutes with the convolution operator,

$$\hat{u}_\epsilon = \hat{\mathcal{R}}^\epsilon(\hat{f}_\epsilon) = \hat{\mathcal{R}}^\epsilon(\hat{\mathcal{K}}^\epsilon(a\,\bullet + b\hat{f}_\epsilon \bullet)) = \mathsf{G} * \big(\hat{\mathcal{R}}^\epsilon(a\,\bullet + b\hat{f}_\epsilon \bullet)\big). \tag{5.18}$$

On the other hand, since $\Pi^\epsilon, \hat{\Pi}^\epsilon$ are constituted of (smooth) functions which in fact coincide except on the symbol $\overset{\bullet}{\underset{\bullet}{|}}$ which does not appear in the decomposition (5.16), one deduces (recall Example 4.10) that

$$\hat{u}_\epsilon(x) = \hat{\mathcal{R}}^\epsilon(\hat{f}_\epsilon)(x) = \hat{\Pi}^\epsilon_x(\hat{f}_\epsilon(x))(x) = \Pi^\epsilon_x(\hat{f}_\epsilon(x))(x) = f_{1,\epsilon}(x). \tag{5.19}$$

Similarly, the right-hand side of (5.18) can be expressed as

$$
\begin{aligned}
&\hat{\mathcal{R}}^\epsilon(a\bullet + b\hat{f}_\epsilon\bullet)(x)\\
&= \hat{\Pi}^\epsilon_x(a\bullet + b\hat{f}_\epsilon(x)\bullet)(x)\\
&= \hat{\Pi}^\epsilon_x\big((a + bf_{1,\epsilon}(x))\bullet + bf_{2,\epsilon}(x)\ddagger + bf_{3,\epsilon}(x)\bullet X_1 + bf_{4,\epsilon}(x)\bullet X_2\big)(x)\\
&= \Pi^\epsilon_x\big((a + bf_{1,\epsilon}(x))\bullet + bf_{2,\epsilon}(x)(\ddagger - c_\epsilon\mathbf{1}) + bf_{3,\epsilon}(x)\bullet X_1 + bf_{4,\epsilon}(x)\bullet X_2\big)(x)\\
&= \Pi^\epsilon_x\big(a\bullet - c_\epsilon bf_{2,\epsilon}(x)\mathbf{1} + b\hat{f}_\epsilon(x)\bullet\big)(x).
\end{aligned}
$$

Now we plug in (5.17)-(5.19), and use the multiplicativity property of Π^ϵ,

$$
\begin{aligned}
&\hat{\mathcal{R}}^\epsilon(a\bullet + b\hat{f}_\epsilon\bullet)(x)\\
&= a\Pi^\epsilon_x(\bullet)(x) + b\Pi^\epsilon_x(\hat{f}_\epsilon(x))(x)\Pi^\epsilon_x(\bullet)(x) - c_\epsilon bf_{2,\epsilon}(x)\Pi^\epsilon_x(\mathbf{1})(x)\\
&= a\xi_\epsilon(x) + b\hat{u}_\epsilon(x)\xi_\epsilon(x) - c_\epsilon b(a + bf_{1,\epsilon}(x))\\
&= a\xi_\epsilon(x) + b\hat{u}_\epsilon(x)\xi_\epsilon(x) - c_\epsilon b(a + b\hat{u}_\epsilon(x))\\
&= (a + b\hat{u}_\epsilon(x))(\xi_\epsilon(x) - bc_\epsilon),
\end{aligned}
$$

whence replacing in (5.18) gives

$$
\hat{u}_\epsilon = \mathsf{G} * \big((a + b\hat{u}_\epsilon)(\xi_\epsilon - bc_\epsilon)\big),
$$

i.e. recalling Theorem A.17, $\hat{u}_\epsilon$ is the unique (smooth) 1-periodic solution to

$$
(1-\Delta)\hat{u}_\epsilon = \big(a + b\hat{u}_\epsilon\big)\big(\xi_\epsilon - bc_\epsilon\big).
$$

This concludes the proof of Theorem 1.2.

Remark 5.11 (Family of solutions)**.** For any constant $c \in \mathbb{R}$, we could replace c_ϵ by $c_\epsilon + c$ in the calculations above and still obtain a limiting solution $\hat{u} = \hat{u}(c) \in \mathcal{C}^{1-\kappa}$. Thus, we have in fact obtained a whole family of solutions $(\hat{u}(c))_{c\in\mathbb{R}}$ to the equation (E), indexed by $\mathbb{R}$.

5.6 Adding a non-linearity

In this section we discuss the more general equation

$$
(1-\Delta)u = (a + bF(u))\xi \qquad \text{on } \mathbb{T}^{d=2}, \tag{5.20}
$$

for some non-linarity $F\colon \mathbb{R} \to \mathbb{R}$ of suitable regularity.

More precisely, fix $\kappa > 0$ sufficiently small, $\gamma \in (1+\kappa, 2-2\kappa)$ and $F \in \mathcal{C}^{\frac{\gamma}{1-\kappa}+1}$. Then from the discussion of Section 4.3.4, it is still possible to lift (5.20) to the following fixed-point problems in spaces of modelled distributions $\mathcal{D}^\gamma$, for the *same* models Π^ϵ and $\hat{\Pi}^\epsilon$ which we introduced above:

$$f_\epsilon = \mathcal{K}^\epsilon(a \bullet + bF(f_\epsilon)\bullet), \qquad \hat{f}_\epsilon = \hat{\mathcal{K}}^\epsilon(a \bullet + bF(\hat{f}_\epsilon)\bullet),$$

where $F\colon \mathcal{D}^\gamma \to \mathcal{D}^\gamma$ is the operation defined in Proposition 4.31.

Note, however, that it is not clear whether such fixed points now uniquely exist: this is because the corresponding map

$$\begin{array}{rccc} \Phi^\epsilon: & \mathcal{D}^\gamma_{M^\epsilon;\mathrm{fp}} & \longrightarrow & \mathcal{D}^\gamma_{M^\epsilon;\mathrm{fp}} \\ & f & \longmapsto & \mathcal{K}(a\bullet + bF(f)\bullet), \end{array}$$

might not be a contraction anymore for some choice of b sufficiently small: indeed, the map $f \mapsto F(f)$ is only *locally* Lipschitz in general, recall Proposition 4.31, and thus the continuity constants may depend on the norm of f.

Arguing exactly as in Section 5.5, one may write the corresponding renormalised equation satisfied by $\hat{u}_\epsilon := \mathcal{R}(\hat{f}_\epsilon)$ (assuming such a fixed-point $\hat{f}_\epsilon$ has been found):

$$(1-\Delta)\hat{u}_\epsilon = (a + bF(\hat{u}_\epsilon))(\xi_\epsilon - bc_\epsilon F'(\hat{u}_\epsilon)).$$

A A useful toolbox: distributions, Hölder regularity, Gaussian white noise

A.1 Distributions

One of the central ingredients in equation (E) is the Gaussian white noise ξ, which we define in Appendix A.5 below. Its sample paths turn out to be so irregular that they can not be understood as functions; they rather live in the world of *distributions*.

We fix some $d \in \mathbb{N}$ and recall some notations and definitions. For $r \in \mathbb{N}_0$, we will denote $C^r = C^r(\mathbb{R}^d)$ the space of r-times continuously differentiable functions, and note correspondingly

$$\|f\|_{C^r} := \sup_{|k| \leq r} \|\partial^k f\|_\infty,$$

where the supremum is taken over multi-indices $k = (k_1, \cdots, k_d)$ and $|k| = \sum_{i=1}^d k_i$.

Definition A.1 (Test-functions and distributions)**.** A *test-function* is a compactly supported smooth function $\varphi \in C_c^\infty(\mathbb{R}^d)$. A *distribution* is a linear functional

$$f : C_c^\infty(\mathbb{R}^d) \to \mathbb{R},$$

which is continuous in the following sense: for every compact set $K \subset \mathbb{R}^d$, there exists $r \in \mathbb{N}_0$ and $C > 0$ such that for all test-functions φ supported in K,

$$|f(\varphi)| \leq C\|\varphi\|_{C^r}.$$

We will denote $\mathcal{D}'(\mathbb{R}^d)$ the space of distributions.

Note that any locally integrable function $f : \mathbb{R}^d \to \mathbb{R}$ defines a distribution through

$$f(\varphi) := \int_{\mathbb{R}^d} f(x)\varphi(x)\,\mathrm{d}x, \tag{A.1}$$

which corresponds to the prototypical way of understanding the pairing between a distribution f and a test-function φ. Another well-known distribution is the Dirac δ defined by

$$\delta(\varphi) := \varphi(0).$$

One advantage of the "distributional point of view" is that *linear* transformations on test-functions can usually be automatically transferred to distributions by duality (on the other hand, nonlinear operations such as products can typically *not* be extended to distributions). Let us list some examples which will be used throughout this article:

Definition A.2 (Some operations on distributions)**.** Let $f \in \mathcal{D}'(\mathbb{R}^d)$.

1. (Derivation) Let $k \in \mathbb{N}_0^d$ be a multi-indice, then we define the derivative $\partial^k f \in \mathcal{D}'(\mathbb{R}^d)$ by
$$\partial^k f(\varphi) \coloneqq (-1)^{|k|} f(\partial^k \varphi),$$
which extends the usual integration by parts formula, recall (A.1).
2. (Multiplication with C^∞) Let $\chi \in C^\infty$, then we define the product $\chi f \in \mathcal{D}'(\mathbb{R}^d)$ by $(\chi f)(\varphi) \coloneqq f(\chi \varphi)$.
3. (Translation) For $a \in \mathbb{R}^d$, let $\tau_a \colon \mathbb{R}^d \to \mathbb{R}^d$ be the translation by a i.e. $\tau_a(x) = x + a$. We define $(f \circ \tau_a)(\varphi) \coloneqq f(\varphi \circ \tau_{-a})$.

In what follows it will be convenient to consider periodic distributions.

Definition A.3 (Periodic distributions)**.** We say that a distribution $f \in \mathcal{D}'(\mathbb{R}^d)$ is *periodic of period* $a \in \mathbb{R}^d$ if $f \circ \tau_a = f$.

Notation A.4. We will denote $\mathcal{D}'(\mathbb{T}^d)$ the vector space of periodic distributions of period e_i for all $i \in \{1, \ldots, d\}$, where e_i denote the i-th vector of the canonical basis of $\mathbb{R}^d$. We will also just write $\mathcal{D}'$ when it is clear whether we mean $\mathcal{D}'(\mathbb{R}^d)$ or $\mathcal{D}'(\mathbb{T}^d)$.

Remark A.5. This may seem like an abuse of notation, since it would be more natural to define $\mathcal{D}'(\mathbb{T}^d)$ as the space of continuous linear functionals $f \colon C^\infty(\mathbb{T}^d) \to \mathbb{R}$. However, it is possible to see that those two constructions actually result in isomorphic vector spaces, hence we will stick with Notation A.4 in the remainder of this article.

A.2 Hölder regularity

We have already mentioned above that Gaussian white noise ξ is particularly irregular. Of course, it is now natural to look for a quantification of this irregularity. Recall the classical notion of Hölder continuity for functions:

Definition A.6 (α-Hölder regularity, $0 < \alpha < 1$)**.** Let $\alpha \in (0,1)$. A function $f\colon \mathbb{R}^d \to \mathbb{R}$ is said to be *locally α-Hölder* if for every compact $K \subset \mathbb{R}^d$ there exists $C > 0$ such that for $x, y \in K$,

$$|f(y) - f(x)| \leq C|y - x|^\alpha. \tag{A.2}$$

Hölder regularity is widely used in stochastic analysis: for instance, it is well-known that the sample paths of brownian motion are α-Hölder for any $\alpha \in (0, 1/2)$.

When $\alpha > 1$, it can be checked that only constant functions satisfy condition (A.2), which indicates the need to adapt the definition. A natural generalisation consists in substracting further terms of the Taylor expansion of f in (A.2):

Definition A.7 (α-Hölder regularity, $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}_0$)**.** Let $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}_0$. A function $f \in C^{\lfloor \alpha \rfloor}(\mathbb{R}^d; \mathbb{R})$ is said to be *locally α-Hölder* if for every compact $K \subset \mathbb{R}^d$ there exists $C > 0$ such that for $x, y \in K$,

$$\Big| f(y) - \sum_{|k| \leq \alpha} \frac{f^{(k)}(x)}{k!} (y - x)^k \Big| \leq C|y - x|^\alpha. \tag{A.3}$$

Of course, Definitions A.6 and A.7 are suitable only for functions, while white noise turns out to be a distribution: we now discuss how to generalise the notion of Hölder regularity to negative exponents and distributions. For this purpose, let us understand how (A.3) transforms under the pairing (A.1). We will need to control the position and scale of test-functions, which can be modulated as follows:

Definition A.8 (Recentering and rescaling)**.** Let φ be a test-function, $x \in \mathbb{R}$ and $\lambda > 0$. We define a new test-function φ_x^λ by setting, for $z \in \mathbb{R}^d$,

$$\varphi_x^\lambda(z) \coloneqq \lambda^{-d} \varphi\Big(\frac{z - x}{\lambda}\Big).$$

We say that φ_x^λ is *recentered* at the point x and *rescaled* with the scale λ.

Observe that φ_x^λ is concentrated around x with a scale λ. Thus, $f(\varphi_x^\lambda)$ naturally describes the behaviour of f in the ball centered around x with radius λ. More precisely, let f be an α-Hölder function for some $\alpha \in \mathbb{R}_+ \backslash \mathbb{N}_0$, φ be a test-function, and $x \in \mathbb{R}^d, \lambda > 0$. Applying (A.1) and the natural change of variable in the integral, observe that

$$f(\varphi_x^\lambda) = \int_{\mathbb{R}^d} f(z)\varphi_x^\lambda(z)\,\mathrm{d}z = \int_{\mathbb{R}^d} f(x+\lambda u)\varphi(u)\,\mathrm{d}u,$$

is a $O(1)$ as the localisation scale λ goes to 0. Crucially, this estimate is improved if we further assume that φ annihilates polynomials, in the sense that $\int_{\mathbb{R}^d} \varphi(x)x^k\,\mathrm{d}x = 0$ for all multi-indices $|k| \le \alpha$. Indeed, in this case we may substract the Taylor expansion in the integral:

$$f(\varphi_x^\lambda) = \int_{\mathbb{R}^d} \Big(f(z) - \sum_{|k| \le \alpha} \frac{f^{(k)}(x)}{k!}(z-x)^k \Big) \varphi_x^\lambda(z)\,\mathrm{d}z,$$

yielding after applying the assumption of Hölder regularity, $f(\varphi_x^\lambda) = O(\lambda^\alpha)$. It turns out that "reversing" those observations yields a satisfactory definition of Hölder regularity for distributions: to give precise statements, let us introduce the following notation.

Definition A.9 (Balls of test-functions)**.** Let $r \in \mathbb{N}_0$, $\delta \in \mathbb{R}$, and denote

$$\begin{aligned}
&\mathscr{B}^r := \{\varphi \in C_c^\infty : \operatorname{supp}\varphi \subset B(0,1), \|\varphi\|_{C^r} \le 1\}, \\
&\mathscr{B}_\delta := \{\varphi \in C_c^\infty : \operatorname{supp}\varphi \subset B(0,1), \int_{\mathbb{R}^d} \varphi(x)x^k\,\mathrm{d}x = 0 \text{ for } 0 \le |k| \le \delta\}, \\
&\mathscr{B}_\delta^r := \mathscr{B}^r \cap \mathscr{B}_\delta.
\end{aligned}$$

We can now define the notion of Hölder regularity for distributions.

Definition A.10 (α-Hölder regularity, $\alpha \in \mathbb{R}$)**.** Let $\alpha \in \mathbb{R}$, and let $r = r_\alpha \in \mathbb{N}_0$ be the smallest integer such that $\alpha + r > 0$. A distribution $f \in \mathcal{D}'(\mathbb{R}^d)$ is said to be *locally α-Hölder* if for every compact $K \subset \mathbb{R}^d$,

$$\|f\|_{\mathcal{C}^\alpha(K)} := \sup_{\substack{x \in K, \\ \varphi \in \mathscr{B}^r}} |f(\varphi_x)| + \sup_{\substack{x \in K, \\ \lambda \in (0,1] \\ \varphi \in \mathscr{B}_\alpha^r}} \frac{|f(\varphi_x^\lambda)|}{\lambda^\alpha} < +\infty. \tag{A.4}$$

We will note:

1. $\mathcal{C}^{\alpha}_{\mathrm{loc}}(\mathbb{R}^d)$ the Fréchet space topologized by the collection of seminorms (A.4) over any (countable) exhaustion of $\mathbb{R}^d$ by compacts.

2. $\mathcal{C}^{\alpha}(\mathbb{T}^d)$ the Banach space of α-Hölder 1-periodic distributions, normed by (A.4) for $K = [0,1]^d$.

It is a good exercise in mollification to prove that when $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}_0$, Definition A.10 actually coincides with Definition A.7.

Remark A.11. Note that Definition A.10 also covers integer exponents $n \in \mathbb{N}_0$. Perhaps surprisingly, the corresponding space $\mathcal{C}^{n}_{\mathrm{loc}}(\mathbb{R}^d)$ is *strictly* larger than the space C^n of n-times continuously differentiable functions (whence our use of the calligraphic letter $\mathcal{C}$ in the notation for Hölder spaces). As an example, note that the distribution $x \mapsto \log|x|$ belongs to $\mathcal{C}^{0}_{\mathrm{loc}}(\mathbb{R}^d)$ but is not continuous at the origin.

A.3 Multiplication in Hölder spaces

One advantage of the Hölder spaces is their general stability with respect to usual operations. For instance, while it is known [26] that it is impossible to multiply two distributions in general, it is in fact possible to multiply two Hölder distributions as long as the regularity of one compensates the irregularity of the other.

Theorem A.12 ("Young" multiplication)**.** *Let $\alpha, \beta \in \mathbb{R}$ with $\alpha + \beta > 0$. Then there exists a canonical continuous bilinear "multiplication" map*

$$\begin{array}{rcl} \mathcal{C}^{\alpha}(\mathbb{T}^d) \times \mathcal{C}^{\beta}(\mathbb{T}^d) & \longrightarrow & \mathcal{C}^{\min(\alpha,\beta)}(\mathbb{T}^d) \\ (f,g) & \longmapsto & f \cdot g, \end{array}$$

which coincides with the usual pointwise multiplication on periodic smooth functions.

The assumption $\alpha + \beta > 0$ in this theorem is crucial: when $\alpha + \beta \leq 0$, such a canonical map does not exist anymore.

A.4 Integration in Hölder spaces

Another important operation is that of integrating against singular kernels. To motivate this, recall the notion of a fundamental solution to a differential operator.

Definition A.13 (Fundamental solution)**.** Let L be a differential operator on $\mathbb{R}^d$. A *fundamental solution* to L is any distribution $\mathsf{G} \in \mathcal{D}'(\mathbb{R}^d)$ such that $L\mathsf{G} = \delta$.

This notion is not void, as fundamental solutions can be explicitly provided for many differential operators of interest.

Example A.14. This paper is motivated by the example $L = 1 - \Delta$ in $\mathbb{R}^d$ for $d = 1, 2$, which admits a fundamental solution given by:

$$\mathsf{G}(x) = \frac{1}{2} e^{-|x|} \quad (\text{for } d = 1), \qquad \mathsf{G}(x) = \frac{1}{2\pi} K_0(|x|) \quad (\text{for } d = 2),$$

where $K_0 \colon \mathbb{R}_+^* \to \mathbb{R}_+$ denotes the modified Bessel function of the second kind with index 0, which is a smooth function with known asymptotic behaviour:

$$\text{At } 0\text{: } K_0(x) = -\log(x) + O_{x\to 0}(1), \text{ and } \partial^k K_0(x) = O_{x\to 0}(x^{-k}), \; k \geq 1,$$

$$\text{At } \infty\text{: for } k \geq 0, \; \partial^k K_0(x) \sim_{x\to +\infty} (-1)^k \sqrt{\frac{\pi}{2x}} e^{-x}.$$

A fundamental solution allows to "invert" the differential operator L by convolution: this is because δ is the neutral element for convolution. We now recall and clarify the properties of convolution of distributions. We will use the following notations:

Notation A.15. We will denote:

1. $\mathcal{E}'(\mathbb{R}^d)$ the *space of compactly supported distributions* i.e. of distributions $f \in \mathcal{D}'(\mathbb{R}^d)$ such that there exists a test-function $\chi \in C_c^\infty$ such that $\chi f = f$.

2. $\mathcal{S}(\mathbb{R}^d)$ the *Schwartz class of rapidly decreasing smooth functions* i.e. the space of smooth functions $f\colon \mathbb{R}^d \to \mathbb{R}$ such that for all multi-indices $\alpha, \beta \in \mathbb{N}_0^d$, $\sup_{x\in\mathbb{R}^d} |x^\alpha \partial^\beta f(x)| < \infty$.

Definition A.16 (Convolution)**.** Convolution products may be defined on various spaces of functions and distributions.

1. The convolution $*\colon C_c^\infty \times C^\infty \to C^\infty$ is defined for $\varphi \in C_c^\infty, \psi \in C^\infty$, $x \in \mathbb{R}^d$, by
$$\varphi * \psi(x) \coloneqq \int_{\mathbb{R}^d} \varphi(y)\psi(x-y)dy.$$

2. The convolution $*\colon C_c^\infty \times \mathcal{D}'(\mathbb{R}^d) \to C^\infty$ is defined by duality: for $\varphi \in C_c^\infty$, $f \in \mathcal{D}'(\mathbb{R}^d)$, and $x \in \mathbb{R}^d$, set
$$\varphi * f(x) \coloneqq f(\varphi(x-\cdot)).$$

It turns out that the operation defined in item 2 can further be extended:

3. to a map $*\colon \mathcal{E}'(\mathbb{R}^d) \times \mathcal{D}'(\mathbb{T}^d) \to \mathcal{D}'(\mathbb{T}^d)$,

4. to a map $*\colon \mathcal{S}(\mathbb{R}^d) \times \mathcal{D}'(\mathbb{T}^d) \to C^\infty(\mathbb{T}^d)$,

and all definitions above are compatible and guarantee the relations: $\partial^k(f * g) = (\partial^k f) * g = f * (\partial^k g)$.

The importance of the notion of fundamental solution is reflected by the following "well-posedness" result.

Theorem A.17. *For $d \in \{1, 2\}$ and $f \in \mathcal{D}'(\mathbb{T}^d)$, then the equation of unknown $u \in \mathcal{D}'(\mathbb{T}^d)$,*
$$(1 - \Delta)u = f,$$
admits a unique solution given by $u = \mathsf{G} * f$, where G is given by (2.1) and the convolution is well-defined because G can be decomposed as the sum of a compactly supported distribution and a Schwartz function.

If L is a differential operator of order $r \in \mathbb{N}_0$, we expect that applying L to a function or to a distribution will reduce its regularity by r. Looking at Theorem A.17, we conversely expect that the operation of convolution against the corresponding kernels should have a regularising effect of degree r: such statements are known as Schauder estimates.

The smoothing effect of convolution is well-known: as recalled in Definition A.16, the convolution of any distribution with any compactly supported smooth function always results in a smooth function. Note however that we can not directly apply this fact here because the kernels in (2.1) are not smooth at the origin. It turns out that they still display a regularising effect, to a degree which depends on the order of their singularity at the origin.

Definition A.18 (Regularising kernel)**.** Let $\beta > 0$. We say that a function $\mathsf{K}\colon \mathbb{R}^d \to \mathbb{R} \cup \{\pm\infty\}$ is a *β-regularising kernel* if it is C^∞ on $\mathbb{R}^d \setminus \{0\}$ and if there exists a constant $\rho > 0$ such that for all multi-indices $k \in \mathbb{N}_0^d$, there exists a constant $C_k > 0$ such that

$$|\partial^k \mathsf{K}(x)| \leq \begin{cases} C_k |x|^{\beta - d - |k|} \mathbb{1}_{\{|x| \leq \rho\}} & \text{if } \beta \neq d, \\ C_k |x|^{-|k|} \log\left(1 + |x|^{-1}\right) \mathbb{1}_{\{|x| \leq \rho\}} & \text{if } \beta = d. \end{cases}$$

Remark A.19 (Truncation)**.** Note that the functions in (2.1) do not satisfy Definition A.18 because they are not compactly supported.

However, it is straightforward to see that one can decompose them as $\mathsf{G} = \mathsf{K} + \mathsf{R}$, where K is 2-regularising, and $\mathsf{R} \in \mathcal{S}(\mathbb{R}^d)$ is a Schwartz function.

The regularisation properties of convolution are made precise in the following theorem, see e.g. [11, Section 14.3] for a discussion.

Theorem A.20 (Regularisation by convolution)**.** *The following properties hold:*

1. *Let* $\mathsf{R} \in \mathcal{S}(\mathbb{R}^d)$ *be a Schwartz function. Then for any* $\alpha, \alpha' \in \mathbb{R}$ *the*

convolution map

$$\begin{array}{rcl} \mathcal{C}^{\alpha}(\mathbb{T}^d) & \longrightarrow & \mathcal{C}^{\alpha'}(\mathbb{T}^d) \\ f & \longmapsto & \mathsf{R} * f, \end{array}$$

is well-defined, linear, and continuous.

2. *("Classical Schauder estimates") Let* K *be a* β*-regularising kernel for some* $\beta > 0$*. Then for any* $\alpha \in \mathbb{R}$*, the convolution map*

$$\begin{array}{rcl} \mathcal{C}^{\alpha}(\mathbb{T}^d) & \longrightarrow & \mathcal{C}^{\alpha+\beta}(\mathbb{T}^d) \\ f & \longmapsto & \mathsf{K} * f, \end{array}$$

is well-defined, linear, and continuous.

A.5 Gaussian white noise

We now turn to the definition of our main probabilistic object. As for the usual definition of brownian motion (and Gaussian processes in general), we define Gaussian white noise in two steps:

1. we first prescribe a Gaussian covariance structure;

2. then we show that – up to modification – we can impose suitable regularity assumptions on the sample paths.

The prescription of covariance is contained in the following definition.

Definition A.21 (Gaussian white noise)**.** We call (periodic) *Gaussian white noise* any linear isometry

$$\begin{array}{rrcl} \xi : & L^2(\mathbb{T}^d) & \longrightarrow & L^2(\Omega, \mathcal{F}, \mathbb{P}) \\ & h & \longmapsto & \xi(h), \end{array}$$

on some complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that for all $h \in L^2(\mathbb{T}^d)$, $\xi(h)$ is a real-valued centered Gaussian variable.

Remark A.22 (Existence)**.** These conditions characterize the marginals of ξ in a consistent way, so that Kolmogorov's extension theorem guarantees the existence of such a ξ.

Remark A.23 (Covariance)**.** The assumption of isometry means that for $h_1, h_2 \in L^2(\mathbb{T}^d)$, $\mathbb{E}[\xi(h_1)\xi(h_2)] = \langle h_1, h_2\rangle_{L^2(\mathbb{T}^d)}$. In particular, for $h \in L^2(\mathbb{T}^d)$ this gives the variance of the centered Gaussian variable $\xi(h)$, and thus its law: $\xi(h) \sim \mathcal{N}(0, \|h\|^2)$. Thus, Definition A.21 provides a rigorous formulation of the (nonrigorous) intuition that the covariance function of ξ is the Dirac: "$\mathbb{E}[\xi(x_1)\xi(x_2)] = \delta(x_2 - x_1)$".

Remark A.24 (Periodization)**.** In Definition A.21, the map ξ is defined against periodic functions. In the remainder of this article, we will see such a map as a random periodic distribution, where the pairing against an arbitrary test-function $\varphi \in C_c^\infty$ is defined by $\xi(\varphi) := \xi(\varphi^{\mathrm{per}})$ where $\varphi^{\mathrm{per}}(x) := \sum_{k\in\mathbb{Z}^d} \varphi(x+k)$ defines the *periodization* of φ. In this context, the covariance of ξ is thus given for $\varphi, \psi \in C_c^\infty$ by $\mathbb{E}[\xi(\varphi)\xi(\psi)] = \langle \varphi^{\mathrm{per}}, \psi^{\mathrm{per}}\rangle_{L^2([0,1]^d)}$.

We are now interested in the regularity properties of ξ, which we study up to modification, as usual in stochastic analysis:

Definition A.25 (Modification)**.** Let $f\colon L^2(\mathbb{T}^d) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$. A *modification* of f is a map $\tilde{f}\colon L^2(\mathbb{T}^d) \times \Omega \to \mathbb{R}$ such that for each $h \in L^2(\mathbb{T}^d)$, $\tilde{f}(h)\colon \Omega \to \mathbb{R}$ is a random variable almost surely equal to $f(h)$.

For this purpose, we will apply the following version of Kolmogorov's continuity theorem:

Theorem A.26 (Kolmogorov continuity for distributions)**.** *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a complete probability space, and let* $f\colon L^2(\mathbb{T}^d) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$ *be a continuous linear map. Assume that there exist* $\alpha < 0$, $p \geq 1$ *and* $\varphi \in C_c^\infty(\mathbb{R}^d)$ *with* $\int \varphi \neq 0$, *such that for all* $k \in \mathbb{N}_0$, $x \in \mathbb{R}^d$,

$$\mathbb{E}\Big[\big|f(\varphi_x^{2^{-k}})\big|^p\Big] \leq C_{p,\varphi} 2^{-k\alpha p}. \tag{A.5}$$

Then f *admits a modification* $\tilde{f}$ *such that for all* $\omega \in \Omega$, *and* $\alpha' < \alpha - d/p$,

$$\tilde{f}(\omega) \in \mathcal{C}^{\alpha'}(\mathbb{T}^d).$$

Proof. It is known that from only the assumptions of linearity and continuity, the mapping f has a modification (which we still call f for simplicity) which is in $\mathcal{D}'(\mathbb{T}^d)$ for all $\omega \in \Omega$: we refer the reader e.g. to [28, Corollary 4.2] for the (technical) proof.

It remains to obtain the Hölder regularity: for this purpose, we will use an improved characterisation of the Definition A.10 of Hölder regularity for distributions. Namely, in order to show that a distribution $f \in \mathcal{D}'$ is in $\mathcal{C}^{\alpha'}_{\text{loc}}$ for some $\alpha' < 0$, it suffices to exhibit *one* test-function $\rho \in C_c^\infty(\mathbb{R}^d)$ with $\int \rho \neq 0$, such that for all compacts $K \subset \mathbb{R}^d$,

$$\sup_{x \in K \cap \mathbb{Q}^d} \sup_{n \in \mathbb{N}_0} 2^{n\alpha'} |f(\rho_x^{2^{-n}})| < \infty.$$

See [6, Theorem 12.5] for an elementary proof of this fact. We choose $\rho := \varphi * \varphi \in C_c^\infty$, where $\varphi \in C_c^\infty$ is the test-function provided in the statement of the theorem. By the usual properties of convolution in $\mathcal{D}'$, we have for all $\omega \in \Omega$, $x \in K$, $n \in \mathbb{N}_0$,

$$f(\rho_x^{2^{-n}}) = f(\varphi^{2^{-n}} * \varphi_x^{2^{-n}}) = \int_{K'} \varphi_x^{2^{-n}}(y) f(\varphi_y^{2^{-n}}) dy.$$

where K' is a suitable enlargement of K, e.g. $K' := K \oplus \operatorname{supp} \varphi$. Applying Hölder's inequality to the exponent p from the statement of the theorem and its conjugate $q := \frac{p}{p-1}$,

$$|f(\rho_x^{2^{-n}})| \leq \left(\int_{\mathbb{R}^d} |\varphi_x^{2^{-n}}(y)|^q dy \right)^{\frac{1}{q}} \left(\int_{K'} |f(\varphi_y^{2^{-n}})|^p dy \right)^{\frac{1}{p}}.$$

The integral on the left evaluates to $2^{\frac{nd}{p}} \|\varphi\|_{L^q}$. Now we bound

$$\begin{aligned} \left(\sup_{x \in K \cap \mathbb{Q}^d} \sup_{n \in \mathbb{N}_0} 2^{n\alpha'} |f(\rho_x^{2^{-n}})| \right)^p &\leq \sup_{x \in K} \sum_{n \in \mathbb{N}_0} \left(2^{n\alpha'} |f(\rho_x^{2^{-n}})| \right)^p \\ &\leq \sum_{n \in \mathbb{N}_0} 2^{n(\alpha' p + d)} \|\varphi\|_{L^q}^p \int_{K'} |f(\varphi_y^{2^{-n}})|^p dy. \end{aligned}$$

For each $n \in \mathbb{N}_0$, the map $(\omega, y) \mapsto f(\varphi_y^{2^{-n}})$ is jointly measurable, because it is measurable in ω and continuous in y. Thus by Tonelli's theorem the map $\omega \mapsto \int_{K'} |f(\varphi_y^{2^{-n}})|^p dy$ is measurable and

$$\mathbb{E}\left[\int_{K'} |f(\varphi_y^{2^{-n}})|^p dy\right] = \int_{K'} \mathbb{E}\left[|f(\varphi_y^{2^{-n}})|^p\right] dy \lesssim_{p,\varphi,K} 2^{-n\alpha p}.$$

This implies

$$\mathbb{E}\left[\left(\sup_{x\in K\cap\mathbb{Q}^d}\sup_{n\in\mathbb{N}_0} 2^{n\alpha'}|f(\rho_x^{2^{-n}})|\right)^p\right] \lesssim_{p,\varphi,K} \sum_{n\in\mathbb{N}_0} 2^{np(\alpha'+\frac{d}{p}-\alpha)},$$

which is finite as soon as $\alpha' < \alpha - d/p$. Thus, $f \in \mathcal{C}^{\alpha'}$ on an event $E \in \mathcal{F}$ of measure 1: the result follows after setting $\tilde{f} := f$ on E and $\tilde{f} := 0$ outside. □

Other proofs of such Kolmogorov statements can also be found in the literature with techniques from wavelet theory [12] or Littlewood–Paley decompositions [23, Lemma 5.2], [27].

Application A.27 (Regularity of Gaussian white noise)**.** Let ξ be a Gaussian white noise, as per Definition A.21, then ξ satisfies (A.5) for $\alpha = -d/2$ and any $p \geq 1$. Indeed, let $\varphi \in C_c^\infty(\mathbb{R}^d)$ then

$$\begin{aligned}
\mathbb{E}\Big[\big|\xi(\varphi_x^{2^{-k}})\big|^p\Big] &\leq C_p \mathbb{E}\Big[\big|\xi(\varphi_x^{2^{-k}})\big|^2\Big]^{p/2} && \text{(Gaussian moments)}\\
&= C_p \big\|(\varphi_x^{2^{-k}})^{\mathrm{per}}\big\|_{L^2}^p && \text{(isometry and Remark A.24)}\\
&\leq C_{p,\mathrm{supp}(\varphi)} \|\varphi\|_{L^2}^p 2^{kdp/2} && \text{(Definition A.8 of scaling)},
\end{aligned}$$

where in the first inequality we have used the following general property of Gaussian variables: for all $p \geq 1$, there exists $C_p > 0$ such that for all real Gaussian centered random variables X, $\mathbb{E}[|X^p|] \leq C_p \mathbb{E}[X^2]^{\frac{p}{2}}$. Thus applying Theorem A.26, ξ admits a modification with sample paths belonging in $\mathcal{C}^\alpha(\mathbb{T}^d)$ for any $\alpha < -d/2$.

Remark A.28 (Besov regularity)**.** Looking back to Application A.27, we have only used the gaussianity assumption in the form of the equivalence of moments. In fact, by exploiting more precise properties of Gaussian processes, it is possible to prove that any Gaussian white noise admits a modification with sample paths belonging to the Besov space $\mathcal{B}^{-\frac{d}{2}}_{p,\infty}(\mathbb{T}^d)$ for all $p \in [1, +\infty)$, see e.g. [27]. This is a slight improvement because of the classical embedding of function spaces

$$\mathcal{B}^{-\frac{d}{2}}_{p,\infty}(\mathbb{T}^d) \subset \mathcal{C}^{-\frac{d}{2}-\frac{d}{p}}(\mathbb{T}^d).$$

References

[1] I. Bailleul and M. Hoshino. A tourist's guide to regularity structures and singular stochastic pdes. *arXiv preprint*, 2020.

[2] N. Berglund. *An Introduction to Singular Stochastic PDEs*. EMS Series of Lectures in Mathematics. European Mathematical Society (EMS), Zürich, 2022.

[3] L. Broux, F. Caravenna, and L. Zambotti. Hairer's multilevel schauder estimates without regularity structures. *arXiv preprint*, 2023.

[4] Y. Bruned, A. Chandra, I. Chevyrev, and M. Hairer. Renormalising SPDEs in regularity structures. *J. Eur. Math. Soc. (JEMS)*, 23(3):869–947, 2021.

[5] Y. Bruned, M. Hairer, and L. Zambotti. Algebraic renormalisation of regularity structures. *Invent. Math.*, 215(3):1039–1156, 2019.

[6] F. Caravenna and L. Zambotti. Hairer's reconstruction theorem without regularity structures. *EMS Surv. Math. Sci.*, 7(2):207–251, 2020.

[7] A. Chandra and M. Hairer. An analytic bphz theorem for regularity structures. *arXiv preprint*, 2016.

[8] I. Chevyrev. Hopf and pre-lie algebras in regularity structures. *arXiv preprint*, 2022.

[9] G. Da Prato and A. Debussche. Strong solutions to the stochastic quantization equations. *Ann. Probab.*, 31(4):1900–1916, 2003.

[10] A. M. Davie. Differential equations driven by rough paths: an approach via discrete approximation. *Appl. Math. Res. Express. AMRX*, pages Art. ID abm009, 40, 2008. [Issue information previously given as no. 2 (2007)].

[11] P. K. Friz and M. Hairer. *A course on rough paths.* Universitext. Springer, Cham, 2020. With an introduction to regularity structures.

[12] M. Furlan and J.-C. Mourrat. A tightness criterion for random fields, with application to the Ising model. *Electron. J. Probab.*, 22:Paper No. 97, 29, 2017.

[13] M. Gubinelli, P. Imkeller, and N. Perkowski. Paracontrolled distributions and singular PDEs. *Forum Math. Pi*, 3:e6, 75, 2015.

[14] M. Gubinelli and N. Perkowski. *Lectures on singular stochastic PDEs*, volume 29 of *Ensaios Matemáticos [Mathematical Surveys].* Sociedade Brasileira de Matemática, Rio de Janeiro, 2015.

[15] M. Hairer. Singular stochastic PDEs. In *Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. 1*, pages 685–709. Kyung Moon Sa, Seoul, 2014.

[16] M. Hairer. A theory of regularity structures. *Invent. Math.*, 198(2):269–504, 2014.

[17] M. Hairer and C. Labbé. A simple construction of the continuum parabolic Anderson model on $\mathbf{R}^2$. *Electron. Commun. Probab.*, 20:no. 43, 11, 2015.

[18] M. Hairer and R. Steele. The bphz theorem for regularity structures via the spectral gap inequality. *arXiv preprint*, 2023.

[19] L. Isserlis. On a formula for the product-moment coefficient of any order of a normal frequency distribution in any number of variables. *Biometrika*, 12(1-2):134–139, 11 1918.

[20] C. Labbé. The continuous Anderson Hamiltonian in $d \leq 3$. *J. Funct. Anal.*, 277(9):3187–3235, 2019.

[21] P. Linares, F. Otto, and M. Tempelmayr. The structure group for quasi-linear equations via universal enveloping algebras. *Comm. Amer. Math. Soc.*, 3:1–64, 2023.

[22] P. Linares, F. Otto, M. Tempelmayr, and P. Tsatsoulis. A diagram-free approach to the stochastic estimates in regularity structures. *arXiv preprint*, 2021.

[23] J.-C. Mourrat and H. Weber. Global well-posedness of the dynamic Φ^4 model in the plane. *Ann. Probab.*, 45(4):2398–2476, 2017.

[24] D. Nualart. *The Malliavin calculus and related topics.* Probability and its Applications (New York). Springer-Verlag, Berlin, second edition, 2006.

[25] F. Otto, J. Sauer, S. Smith, and H. Weber. A priori bounds for quasi-linear spdes in the full sub-critical regime. *arXiv preprint*, 2021.

[26] L. Schwartz. Sur l'impossibilité de la multiplication des distributions. *C. R. Acad. Sci. Paris*, 239:847–848, 1954.

[27] M. C. Veraar. Regularity of Gaussian white noise on the d-dimensional torus. In *Marcinkiewicz centenary volume*, volume 95 of *Banach Center Publ.*, pages 385–398. Polish Acad. Sci. Inst. Math., Warsaw, 2011.

[28] J. B. Walsh. An introduction to stochastic partial differential equations. In *École d'été de probabilités de Saint-Flour, XIV—1984*, volume 1180 of *Lecture Notes in Math.*, pages 265–439. Springer, Berlin, 1986.

[29] P. Zorin-Kranich. The reconstruction theorem in quasinormed spaces. *Rev. Mat. Iberoam.*, August 2022.