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Diameter and displacement of sphere involutions

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Abstract. We show that spheres in all dimensions ≥ 3 can be deformed to have diameter larger than the distance between any pair of antipodal points. This answers a question of Yurii Nikonorov.

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1 Introduction

The diameter $\operatorname{diam}(M, \mathrm{d})$ of a compact length space is the maximal distance between pairs of points in (M, d); if M is a manifold and $\mathrm{d} = \mathrm{d}_{\mathrm{g}}$ is induced by a Riemannian metric g, we write $\operatorname{diam}(M, \mathrm{g}) = \operatorname{diam}(M, \mathrm{d}_{\mathrm{g}})$. For example, the round n-sphere of radius r has $\operatorname{diam}(\mathbb{S}^n(r)) = \pi\, r$. Nikonorov [Ni01] proved the following:

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Theorem 1.1 (Nikonorov). *If* (M, d) *is a length space homeomorphic to the sphere* $\mathbb{S}^2$ *and* $I\colon M \to M$ *is an isometric involution without fixed points, then there exists* $x \in M$ *such that* $\operatorname{diam}(M, \mathrm{d}) = \mathrm{d}(x, I(x))$.

The above naturally leads to the following question [Ni01]:

Question 1 (Nikonorov). *Is there an analogue of Theorem 1.1 for length spaces homeomorphic to the sphere* $\mathbb{S}^n$ *for some* $n \geq 3$*?*

Podobryaev [Po18b] observed that sufficiently collapsed Berger spheres provide a negative answer in dimension $n = 3$. In fact, this observation can be easily extended to all *odd* dimensions $n \geq 3$, considering the (homogeneous) spheres $(\mathbb{S}^{2q+1}, \mathrm{g}(t))$ obtained scaling the unit round sphere by $t > 0$ in the vertical direction of the Hopf bundle $\mathbb{S}^1 \to \mathbb{S}^{2q+1} \to \mathbb{C}P^q$. For all $t > 0$, the projection onto $\mathbb{C}P^q$ remains a Riemannian submersion, hence $\operatorname{diam}(\mathbb{S}^{2q+1}, \mathrm{g}(t)) \geq \operatorname{diam}(\mathbb{C}P^q) = \frac{\pi}{2}$. Meanwhile, pairs of antipodal points x and $I(x) = -x$ on $(\mathbb{S}^{2q+1}, \mathrm{g}(t))$ are also antipodal points on the totally geodesic fiber $\mathbb{S}^1(t)$, and thus $\mathrm{d}_{\mathrm{g}(t)}(x, I(x)) \leq \pi\, t$. Therefore, $\mathrm{d}_{\mathrm{g}(t)}(x, I(x)) < \operatorname{diam}(\mathbb{S}^{2q+1}, \mathrm{g}(t))$ for all $t < \frac{1}{2}$. The latter actually holds for all $t < \frac{1}{\sqrt{2}}$ due to the explicit computation (3.1) of $\operatorname{diam}(\mathbb{S}^{2q+1}, \mathrm{g}(t))$ by Rakotoniaina [Ra85], recently rediscovered (in dimension 3) by Podobryaev [Po18a].

In this short note, we provide negative answers in *all dimensions* $n \geq 3$.

Our first construction involves the *spherical join* $\mathbb{S}^k(r) * \mathbb{S}^{n-k-1}(r)$, $1 \leq k \leq n-2$, of spheres of radius $0 < r < \frac{1}{2}$, which is a length space (in fact, an Alexandrov space) with diameter $\frac{\pi}{2}$ and which is homeomorphic to $\mathbb{S}^n$, see [GP93, p. 582] or [BH99, p. 63] for details and definitions. Every point in $\mathbb{S}^k(r) * \mathbb{S}^{n-k-1}(r) \setminus (\mathbb{S}^k(r) \cup \mathbb{S}^{n-k-1}(r))$ can be identified via coordinates (x, ρ, y), where $x \in \mathbb{S}^k(r)$, $y \in \mathbb{S}^{n-k-1}(r)$, and $\rho \in \left(0, \frac{\pi}{2}\right)$. There is a natural isometric action of $\mathsf{SO}(k+1) \times \mathsf{SO}(n-k)$ given by $(A, B) \cdot (x, \rho, y) = (Ax, \rho, By)$, whose orbits have diameter $\pi\, r < \frac{\pi}{2}$, since (see, e.g., [BH99, p. 63]),

$$\mathrm{d}^{\mathrm{sph}}_{\mathrm{join}}\big((x_1,\rho,y_1),(x_2,\rho,y_2)\big) = \arccos\big(\cos^2\rho\,\cos(\mathrm{d}(x_1,x_2)) + \sin^2\rho\,\cos(\mathrm{d}(y_1,y_2))\big),$$

which is bounded from above by $\max\{\mathrm{d}(x_1,x_2),\mathrm{d}(y_1,y_2)\} \leq \pi r$, where d is used for distances in $\mathbb{S}^k(r)$ and $\mathbb{S}^{n-k-1}(r)$. The involution $I(x,\rho,y) = (-x,\rho,-y)$ induced by the antipodal maps of each sphere is an isometry without fixed points, and corresponds to the antipodal map of $\mathbb{S}^n$ under the above homeomorphism. Since I commutes with the $\mathsf{SO}(k+1)\times\mathsf{SO}(n-k)$-action, it leaves invariant each orbit, and thus its maximal displacement is $\pi\, r < \frac{\pi}{2}$. Therefore, $\mathbb{S}^k(r) * \mathbb{S}^{n-k-1}(r)$, with $1 \leq k \leq n-2$ and $0 < r < \frac{1}{2}$, yields a negative answer to Question 1 for all $n \geq 3$.

The spherical join $\mathbb{S}^k(r)*\mathbb{S}^{n-k-1}(r)$ is a smooth Riemannian manifold if and only if $r=1$, in which case it is isometric to $\mathbb{S}^n(1)$. However, inspired by this construction, we can also produce *smooth* counter-examples to Question 1, as follows:

THEOREM. For all $n \geq 3$, there is a family of smooth Riemannian metrics $(\mathrm{g}_s)_{s\geq 0}$ on $\mathbb{S}^n$, such that g_0 is the unit round metric, $\operatorname{diam}(\mathbb{S}^n,\mathrm{g}_s) \geq \frac{\pi}{2}$, and the antipodal map $I(x) = -x$ is an isometry of $(\mathbb{S}^n,\mathrm{g}_s)$ satisfying $\mathrm{d}_{\mathrm{g}_s}(x,I(x)) \leq \frac{\pi}{\sqrt{1+\frac{s}{2}}}$ for all $x \in \mathbb{S}^n$.

Clearly, for $s > 6$, the spheres $(\mathbb{S}^n,\mathrm{g}_s)$ provide a negative answer to Question 1 in all dimensions $n \geq 3$. These spheres are Cheeger deformations of $\mathbb{S}^n(1) \subset \mathbb{R}^{n+1}$ with respect to the block diagonal subgroup of isometries $\mathsf{SO}(k+1)\times\mathsf{SO}(n-k)$ in $\mathsf{SO}(n+1)$, with $1 \leq k \leq n-2$. In particular, they are cohomogeneity one manifolds with geometric features similar to $\mathbb{S}^k(r)*\mathbb{S}^{n-k-1}(r)$; e.g., both are positively curved and converge in Gromov–Hausdorff sense to $\left[0,\frac{\pi}{2}\right]$ as $s \nearrow +\infty$, respectively $r \searrow 0$. In fact, the unifying feature of all constructions in this note is that they are spheres with a distance-nonincreasing map onto $\left[0,\frac{\pi}{2}\right]$ whose fibers are invariant under the antipodal map and can be deformed to have arbitrarily small intrinsic diameter.

2 Main construction

Let $\mathsf{G} = \mathsf{SO}(k+1) \times \mathsf{SO}(n-k) \subset \mathsf{SO}(n+1)$ be the subgroup of block diagonal matrices that act on $\mathbb{R}^{n+1} = \mathbb{R}^{k+1} \oplus \mathbb{R}^{n-k}$ preserving this orthogonal splitting. Clearly, G acts on the unit sphere $\mathbb{S}^n(1) \subset \mathbb{R}^{n+1}$, and the unit speed geodesic $\gamma\colon \left[0, \frac{\pi}{2}\right] \to \mathbb{S}^n(1)$, given by $\gamma(\rho) = \cos\rho\, e_1 + \sin\rho\, e_{n+1}$, where $\{e_j\}$ is the canonical basis of $\mathbb{R}^{n+1}$, meets all G-orbits in $\mathbb{S}^n(1)$ orthogonally. The orbits $\mathsf{G}(\gamma(0)) \cong \mathbb{S}^k(1) \times \{0\}$ and $\mathsf{G}(\gamma(\frac{\pi}{2})) \cong \{0\} \times \mathbb{S}^{n-k-1}(1)$ are singular orbits; all the other orbits $\mathsf{G}(\gamma(\rho)) \cong \mathbb{S}^k(\cos\rho) \times \mathbb{S}^{n-k-1}(\sin\rho)$, $0 < \rho < \frac{\pi}{2}$, are principal orbits. Using this framework, we may define G-invariant metrics on $\mathbb{S}^n$ by specifying their values on the (open and dense) subset of principal points as the doubly warped product

$$\mathrm{g} = \mathrm{d}\rho^2 + \varphi(\rho)^2\, \mathrm{g}_{\mathbb{S}^k} + \psi(\rho)^2\, \mathrm{g}_{\mathbb{S}^{n-k-1}}, \quad 0 < \rho < \tfrac{\pi}{2}, \tag{2.1}$$

where φ and ψ are positive functions satisfying appropriate smoothness conditions at $\rho = 0$ and $\rho = \frac{\pi}{2}$, and $\mathrm{g}_{\mathbb{S}^d}$ is the unit round metric on $\mathbb{S}^d$. Cohomogeneity one metrics of the form (2.1) are called *diagonal*. For example, the unit round metric $\mathrm{g}_0 = \mathrm{g}_{\mathbb{S}^n}$ is of the above form, with functions $\varphi_0(\rho) = \cos\rho$ and $\psi_0(\rho) = \sin\rho$.

The *Cheeger deformation* of g_0 is the 1-parameter family g_s, $s \geq 0$, of diagonal cohomogeneity one metrics (2.1) determined by the functions

$$\varphi_s(\rho) = \frac{\cos\rho}{\sqrt{1 + s\cos^2\rho}} \quad \text{and} \quad \psi_s(\rho) = \frac{\sin\rho}{\sqrt{1 + s\sin^2\rho}}, \tag{2.2}$$

see [AB15, Ex 6.46]. For all $s \geq 0$, the metric g_s is C^∞ smooth and G-invariant, the orbit space of the G-action on $(\mathbb{S}^n, \mathrm{g}_s)$ is $\mathbb{S}^n/\mathsf{G} = \left[0, \frac{\pi}{2}\right]$, and γ remains a unit speed geodesic orthogonal to all G-orbits. As the projection $\mathbb{S}^n \to \mathbb{S}^n/\mathsf{G}$ is distance-nonincreasing, we have

$$\operatorname{diam}(\mathbb{S}^n, \mathrm{g}_s) \geq \tfrac{\pi}{2}, \qquad \text{for all } s \geq 0. \tag{2.3}$$

Moreover, $(\mathbb{S}^n, \mathrm{g}_s)$ has $\sec \geq 0$ for all $s \geq 0$, and it converges in Gromov–Hausdorff sense to $\mathbb{S}^n/\mathsf{G} = \left[0, \frac{\pi}{2}\right]$ as $s \nearrow +\infty$.

The G-orbits in $(\mathbb{S}^n, \mathrm{g})$, where g is the cohomogeneity one diagonal metric (2.1), are isometric to the product $\mathsf{G}(\gamma(\rho)) = \mathbb{S}^k(\varphi(\rho)) \times \mathbb{S}^{n-k-1}(\psi(\rho))$ of round spheres of radii $\varphi(\rho)$ and $\psi(\rho)$. Thus, the distance between any $x, y \in \mathsf{G}(\gamma(\rho))$ is

$$\begin{aligned} \mathrm{d}_{\mathrm{g}}(x,y) &\leq \operatorname{diam}(\mathsf{G}(\gamma(\rho)), \mathrm{g}) \\ &= \sqrt{\operatorname{diam}(\mathbb{S}^k(\varphi(\rho)))^2 + \operatorname{diam}(\mathbb{S}^{n-k-1}(\psi(\rho)))^2} = \pi\sqrt{\varphi(\rho)^2 + \psi(\rho)^2}. \end{aligned}$$

Setting φ and ψ to be the functions in (2.2), one easily checks that the maximum value of the above is achieved at $\rho = \frac{\pi}{4}$ for all $s \geq 0$, and is equal to $\frac{\pi}{\sqrt{1+\frac{s}{2}}}$.

The antipodal map $I\colon \mathbb{S}^n \to \mathbb{S}^n$, which acts as $I = -\mathrm{Id} \in \mathsf{O}(n+1)$, commutes with the G-action on $(\mathbb{S}^n, \mathrm{g}_s)$, thus I leaves invariant all G-orbits. In fact, I restricts to the antipodal map on each sphere factor in $\mathsf{G}(\gamma(\rho))$, $\rho \in \left[0, \frac{\pi}{2}\right]$. Thus, the displacement of I on $(\mathbb{S}^n, \mathrm{g}_s)$ satisfies

$$\mathrm{d}_{\mathrm{g}_s}(x, I(x)) \leq \max_{\rho \in \left[0, \frac{\pi}{2}\right]} \operatorname{diam}(\mathsf{G}(\gamma(\rho)), \mathrm{g}_s) = \frac{\pi}{\sqrt{1+\frac{s}{2}}}.$$

Together with (2.3), this proves the Theorem in the Introduction. $\square$

Remark 2.1. Not all G-invariant metrics on $\mathbb{S}^n$ are diagonal, i.e., of the form (2.1), if n is odd. For instance, let $n = 3$ and $k = 1$. For all $t \neq 1$, the isometry group of the Berger sphere $(\mathbb{S}^3, \mathrm{g}(t))$ is $\mathsf{U}(2) \subset \mathsf{SO}(4)$, which contains $\mathsf{G} = \mathsf{SO}(2) \times \mathsf{SO}(2)$, so $\mathrm{g}(t)$ is G-invariant. However, $\mathrm{g}(t)$ is not of the form (2.1) if $t \neq 1$. Indeed, principal G-orbits in $(\mathbb{S}^3, \mathrm{g}(t))$ are isometric to flat 2-tori $(\mathsf{G}(\gamma(\rho)), \mathrm{g}(t)) \cong \mathbb{R}^2/\Gamma_{(\rho,t)}$ and none of the lattices $\Gamma_{(\rho,t)}$ are rectangular if $t \neq 1$. Meanwhile, principal G-orbits in $(\mathbb{S}^3, \mathrm{g})$, with g as in (2.1), are rectangular flat tori $(\mathsf{G}(\gamma(\rho)), \mathrm{g}) \cong \mathbb{R}^2/2\pi\varphi(\rho)\mathbb{Z} \oplus 2\pi\psi(\rho)\mathbb{Z}$.

3 Final remarks

3.1 Berger spheres

Let us expand on our discussion of the spheres $(\mathbb{S}^{2q+1}, \mathrm{g}(t))$, whose Hopf circles are closed geodesics of length $2\pi\, t$. According to [Ra85, Po18a],

$$\operatorname{diam}(\mathbb{S}^{2q+1}, \mathrm{g}(t)) = \begin{cases} \dfrac{\pi}{2\sqrt{1-t^2}}, & \text{if } 0 < t \leq \frac{1}{\sqrt{2}}, \\ \pi\, t, & \text{if } \frac{1}{\sqrt{2}} < t \leq 1, \\ \pi, & \text{if } 1 < t. \end{cases} \tag{3.1}$$

As pairs of antipodal points x and $I(x)$ are joined by half of the Hopf circle to which they belong, $\mathrm{d}_{\mathrm{g}(t)}(x, I(x)) \leq \pi\, t < \operatorname{diam}(\mathbb{S}^{2q+1}, \mathrm{g}(t))$ for all $t < \frac{1}{\sqrt{2}}$, see Figure 3.1.

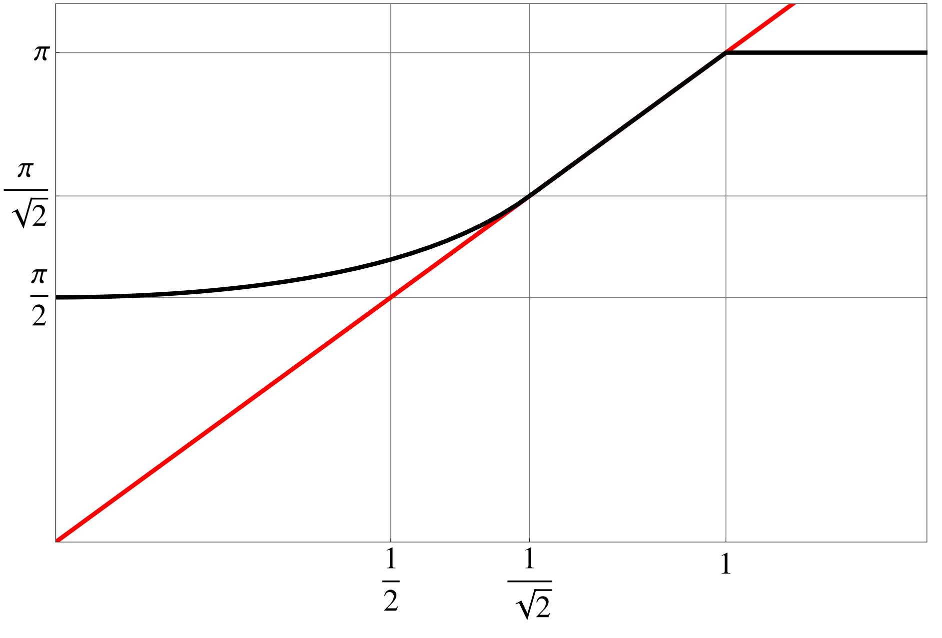

Figure 3.1: Diameter (black) and half length of Hopf circle (red) in $(\mathbb{S}^{2q+1}, \mathrm{g}(t))$.

A similar situation occurs on the Berger spheres $(\mathbb{S}^{4q+3}, \mathrm{h}(t))$ and $(\mathbb{S}^{15}, \mathrm{k}(t))$ obtained by scaling the unit round sphere by $t > 0$ in the vertical direction of the Hopf bundles $\mathbb{S}^3 \to \mathbb{S}^{4q+3} \to \mathbb{H}P^q$ and $\mathbb{S}^7 \to \mathbb{S}^{15} \to \mathbb{S}^8\big(\frac{1}{2}\big)$, respectively. Namely, for all $t > 0$, the projection map of these bundles remains a Riemannian submersion, and thus $\operatorname{diam}(\mathbb{S}^{4q+3}, \mathrm{h}(t)) \geq \operatorname{diam}(\mathbb{H}P^q) = \frac{\pi}{2}$ and $\operatorname{diam}(\mathbb{S}^{15}, \mathrm{k}(t)) \geq \operatorname{diam}(\mathbb{S}^8\big(\frac{1}{2}\big)) = \frac{\pi}{2}$. Pairs of antipodal points belong to the same Hopf circle, hence to the same (totally geodesic) fiber, which is isometric to $\mathbb{S}^3(t)$ or $\mathbb{S}^7(t)$, so $\mathrm{d}_{\mathrm{g}(t)}(x, I(x)) \leq \pi\, t$. Thus, for sufficiently small $t > 0$, these spheres also provide a negative answer to Question 1.

3.2 First Laplace eigenvalue

Spectral geometry provides an alternative path to show that Berger spheres yield a negative answer to Question 1, by considering

$$\mathrm{g} \longmapsto \lambda_1(M, \mathrm{g}) \operatorname{diam}(M, \mathrm{g})^2,$$

where $\lambda_1(M, \mathrm{g})$ is the smallest positive eigenvalue of the Laplace–Beltrami operator. This scale-invariant functional is bounded from below by $\frac{\pi^2}{4}$ on compact connected homogeneous spaces [Li80]. Moreover, one has that $\lambda_1(\mathbb{S}^{2q+1}, \mathrm{g}(t)) \leq 4(q+1)$ for all $t > 0$, since

$$\lambda_1(\mathbb{S}^{2q+1}, \mathrm{g}(t)) = \min\left\{4(q+1),\, 2q + \tfrac{1}{t^2}\right\} = \begin{cases} 4(q+1), & \text{if } t \leq \frac{1}{\sqrt{2q+4}}, \\ 2q + \frac{1}{t^2}, & \text{if } t \geq \frac{1}{\sqrt{2q+4}}, \end{cases}$$

see [BP13, Prop. 5.3]. Similar upper bounds on λ_1 for $(\mathbb{S}^{4q+3}, \mathrm{h}(t))$ and $(\mathbb{S}^{15}, \mathrm{k}(t))$ can be obtained from [BLP22]. This yields a positive diameter lower bound, independent of $t > 0$, that could be used in lieu of the exact value (3.1) for $(\mathbb{S}^{2q+1}, \mathrm{g}(t))$ or of the submersion lower bound $\frac{\pi}{2}$ in general. However, this spectral lower bound on the diameter is weaker than the latter, and becomes arbitrarily small as $q \nearrow +\infty$.

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References

[AB15] M. M. Alexandrino and R. G. Bettiol. *Lie groups and geometric aspects of isometric actions.* Springer, Cham, 2015. DOI: 10.1007/978-3-319-16613-1.

[BP13] R. G. Bettiol, P. Piccione. *Bifurcation and local rigidity of homogeneous solutions to the Yamabe problem on spheres.* Calc. Var. Partial Differential Equations **47**:3–4 (2013), 789–807. DOI: 10.1007/s00526-012-0535-y.

[BLP22] R. G. Bettiol, E. A. Lauret, P. Piccione. *The first eigenvalue of a homogeneous CROSS.* J. Geom. Anal. **32** (2022), 76. DOI: 10.1007/s12220-021-00826-7.

[BH99] M. Bridson, A. Haefliger. *Metric spaces of non-positive curvature.* Grundlehren Math. Wiss., 319. Springer-Verlag (1999). DOI: 10.1007/978-3-662-12494-9.

[GP93] K. Grove, P. Petersen. *A radius sphere theorem.* Invent. Math. **112**:3 (1993), 577–583. DOI: 10.1007/BF01232447.

[Li80] P. Li. *Eigenvalue estimates on homogeneous manifolds.* Comment. Math. Helvetici **55** (1980), 347–363. DOI: 10.1007/BF02566692.

[Ni01] Yu. G. Nikonorov. *On the geodesic diameter of surfaces possessing an involutory isometry* (Russian). Tr. Rubtsovsk. Ind. Inst. **9** (2001), 62–65. English translation: arXiv:1811.01173.

[Po18a] A. V. Podobryaev. *Diameter of the Berger Sphere*. Math. Notes **103**:5–6 (2018), 846–851. DOI: 10.1134/S0001434618050188.

[Po18b] A. V. Podobryaev. *Antipodal points and diameter of a sphere*. Russ. J. Nonlinear Dyn. **14**:4 (2018), 579–581. DOI: 10.20537/nd180410.

[Ra85] C. Rakotoniaina. *Cut locus of the B-spheres*. Ann. Global Anal. Geom. **3**:3 (1985), 313–327. DOI: 10.1007/BF00130483.