Matemática
Contemporânea
Vol. 57, 3–22
http://doi.org/10.21711/231766362023/rmc572

A survey about a Do Carmo's question on complete stable constant mean curvature hypersurfaces

Barbara Nelli

Universitá degli Studi dell'Aquila, L'Aquila, Italy

Abstract. We give an overview of results about the following question by M. P. Do Carmo: Is a noncompact, complete, stable, constant mean curvature hypersurface M of $\mathbb{R}^{n+1}$ necessarily minimal?

Keywords: constant mean curvature, stable.

2020 Mathematics Subject Classification: 53C42, 53A10.

1 Introduction

In a Lecture Notes of 1989 [16], M. P. Do Carmo asked for the following question:

Is a noncompact, complete, stable, constant mean curvature hypersurface M of $\mathbb{R}^{n+1}$ necessarily minimal?

The question is very natural because in 1989, in $\mathbb{R}^3$, it was proved by F. Lopez and A. Ros [31] and independently by A. da Silveira [15] that the question has an affirmative answer (see Theorem 3.3).

For a long time, there was no answer to do Carmo question in higher dimension (except with further assumptions). Then, in 2007, F. Elbert,

The author is partially supported by INdAM-GNSAGA.

E-mail: barbara.nelli@univaq.it

the author and H. Rosenberg [21], and independently X. Cheng [11], gave an affirmative answer in $\mathbb{R}^4$ and $\mathbb{R}^5$.

In dimension larger than 5, only partial results are known.

Notice that in $\mathbb{R}^3$, once one proves that a complete, stable, constant mean curvature surface is minimal, then it is a plane as it was proven independently by M. Do Carmo, C.K. Peng [17], D. Fischer-Colbrie, R. Schoen, [23] and A. Pogorelov [37] (see Theorem 3.4).

Recently O. Chodosh, C. Li [12] have been able to prove that a minimal stable complete hypersurface in $\mathbb{R}^4$ is a hyperplane. This crucial result, joint with the affirmative answer to do Carmo's question for M in $\mathbb{R}^4$, yields that M is a hyperplane.

In this article we give an overview of results related to Do Carmo's question along the years. It is worthwhile to mention that Do Carmo's question has been explored in non Euclidean ambient spaces, too. Nevertheless, we will mainly deal with the results in $\mathbb{R}^{n+1}$.

2 Preliminaries

Let $x : M \longrightarrow \mathcal{N}^{n+1}$ be an immersion of an orientable n-manifold in a Riemannian $n+1$-manifold. By abuse of notation we will use loosely M to indicate $x(M)$. Let A be the second fundamental form of M and denote by $\kappa_1, \ldots, \kappa_n$ its eigenvalues, that is, the principal curvatures of M. The mean curvature of M is

$$H = \frac{\kappa_1 + \cdots + \kappa_n}{n}.$$

We assume that H is constant and orient M by a unit normal vector field N such that H is nonnegative. When $H \neq 0$, the vector HN is known as the mean curvature vector of M.

It is well known that one can characterize constant mean curvature hypersurfaces in $\mathcal{N}^{n+1}$ in a variational way. More precisely, a hypersurface M in $\mathcal{N}^{n+1}$ has constant mean curvature H if and only if it is a critical point for a functional related to area, with respect to variations defined as follows.

Let D be a relatively compact domain in M with smooth boundary. By a variation of $x_{|D}$ we mean a differentiable map $X : (-\varepsilon, \varepsilon) \times D \longrightarrow \mathcal{N}^{n+1}$ such that $X_t = X(t, \cdot)$ is an immersion for each $t \in (-\varepsilon, \varepsilon)$, $X_0 = x$, and $X_{t|\partial D} = x_{|\partial D}$.

We define the area and the volume functions $A,\ V : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}$ by

$$A(t) = \int_M dM_t \quad \text{and} \quad V(t) = \int_{[0,t]\times M} X^* dv,$$

where dM_t is the volume form on M_t, dv is the volume element of $\mathcal{N}^{n+1}$, X^* is the standard linear map on forms induced by X and X^*dv is the induced (algebraic) volume form.

It is proved in [3] (see also [2]) that, writing $f = \langle \frac{\partial X}{\partial t}|_{t=0}, N\rangle$, one has

$$\frac{dA}{dt}(0) = -n\int_D HfdM \quad \text{and} \quad \frac{dV}{dt}(0) = \int_D fdM.$$

Notice that, without loss of generality we can integrate over M, extending f to be zero outside the domain D.

Now, we consider the function $G : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}$ defined by $G(t) = A(t) + nHV(t)$. Clearly, $\frac{dG}{dt}(0) = 0$. Furthermore, one computes (see [3])

$$\frac{d^2G}{dt^2}(0) = -\int_M (f\Delta f + (Ric(N) + |A|^2)f^2)dM,$$

where $|A|$ is the norm of the second fundamental form of M, while $Ric(N)$ is the Ricci curvature of the ambient space in the direction of N.

The linear operator $L := \Delta + Ric(N) + |A|^2$ is called the *stability operator* of M and M is said to be *stable* if L is nonpositive, that is

$$Q(f,f) := -\int_M fLf \geq 0, \quad \forall f \in C_0^\infty(M).$$

For completeness, we define the index of M. Let D be a relatively compact domain of M. Define $i_{L|D}$ the number of negative eigenvalues of $-L = -(\Delta + Ric(\nu,\nu) + |A|^2)$, for the Dirichlet problem on D :

$$-Lf = \lambda f, \quad f_{|\partial D} = 0.$$

The Index(M) is defined as follows

$$\text{Index}(M) := \sup\{i_{L|D} \mid D \subset M \text{ rel. comp.}\}$$

In order to understand the index geometrically, we observe that the index measures the number of linearly independent normal deformations with compact support of M, that decrease the functional Q. Clearly, M is stable if and only if M has index zero.

We conclude this section by introducing a different notion of stability.

One says that M is *weakly stable* if $Q(f,f) := -\int_M fLf \geq 0$, for all $f \in C_0^\infty(M)$ such that $\int_M f = 0$.

Analogously one can define the *weak index*, adding the nullity condition $\int_M f = 0$ to the definition of the index.

The relation between weak stability and stability has been studied by J.L. Barbosa and P. Berard [4]. What is relevant for our discussion is that in the case of a complete non-compact manifold with infinite volume, the two indices are equal, provided some mild assumption on the volume growth is satisfied: it is enough that the volume of the intrinsic geodesic ball of radius $R+1$ is less or equal to a constant times the volume of the intrinsic geodesic ball of radius R. Moreover, the assumption *infinite volume* is not restrictive because of a result by K. Frensel [24, Theorem 1]: a complete, non-compact, constant mean curvature hypersurface in a manifold with bounded geometry has infinite volume.

Furthermore, the following result is well known.

Theorem 2.1. *If M is weakly stable, then, there exists a compact $K \subset M$ such that $M \setminus K$ is stable.*

Proof. [27, Proposition 2.1] If M is stable, we choose $K = \emptyset$ and the result is proved. Assume M is not stable, then there exists $f \in C_0^\infty(M)$ such that $Q(f,f) < 0$. Let $K = supp(f)$, we will prove that $M \setminus K$ is stable, i.e. for any $g \in C_0^\infty(M \setminus K)$, one has $Q(g,g) \geq 0$. Denote by $\alpha = \int_M g$ and $\beta = \int_M f$ and define $h := \alpha f - \beta g$. By a straightforward computation, one has that $\int_M h = 0$. As M is weakly stable, one has $Q(h,h) \geq 0$. As

$supp(f) \cap supp(g) = \emptyset$, using the bi-linearity of Q one has

$$a0 \leq Q(h,h) = \alpha^2 Q(f,f) + \beta^2 Q(g,g). \tag{2.1}$$

As $Q(f,f) < 0$, inequality (2.1) implies that $\beta \neq 0$ and $Q(g,g) \geq 0$. Hence $M \setminus K$ is stable. □

Let us finish this introductory section by explaining why we consider M noncompact since the beginning. The following result, due to L. Barbosa, M. Do Carmo and J. Eschenburg holds for stable compact constant mean curvature hypersurfaces in space forms (the analogous result in $\mathbb{R}^{n+1}$ is [2, Theorem 1.3]).

Theorem 2.2. *[3, Theorem 1.2] Let M be a compact hypersurface of a space form $\mathcal{N}$, with constant mean curvature. Then M is weakly stable if and only if M is a geodesic sphere.*

Moreover, it is shown in [2] that geodesic spheres in space forms are not stable.

3 When the answer to do Carmo's question is known.

3.1 A Toy Theorem.

Let us start by proving a simple case where the answer to Do Carmo question is affirmative.

Theorem 3.1. *There is no entire graph in $\mathbb{R}^n$ with constant mean curvature $H \neq 0$.*

Theorem 3.1 is related with Do Carmo question, because a constant mean curvature graph is stable. In fact, assume that M is a graph with respect to the vertical direction e_{n+1} in $\mathbb{R}^{n+1}$. Without loss of generality, we can assume that the mean curvature vector $\vec{H} = HN$ of M points upwards. As M is a graph, then $f = \langle N, e_{n+1} \rangle$ is a positive function on M. By a straightforward computation one gets that f satisfies

$$\Delta f + (Ric(N) + |A|^2) f = 0$$

that is f is a positive Jacobi field on M. By [23, Theorem 1], this yields that M is stable.

The proof of the Theorem 3.1 is well known and, to the best of our knowledge, handed down orally. Hence we do it here.

Proof. Assume that M is a graph with respect to the vertical direction and that the mean curvature vector of M points upwards. Consider a geodesic ball B of mean curvature H. Up to vertical translation, B and M are disjoint and B lies above M. Translate down B towards M and consider the first contact point p between M and B. As B is above M around p and they have the same mean curvature vector, they coincide, by the maximum principle. This is a contradiction because B is compact and M is not. □

Remark 3.2. *We notice that, as soon as one consider different ambient spaces, in order to get Theorem 3.1, one has to add some natural assumptions. For example, in hyperbolic space, for any $H \in (0, 1)$ there is a rotational non minimal graph over the entire hyperbolic plane, with constant mean curvature H. The result analogous to Theorem 3.1 in hyperbolic space holds for hypersurfaces with constant mean curvature larger than one, after choosing a suitable notion of graph. Another interesting example is given by constant mean curvature graphs in $\mathbb{H}^n \times \mathbb{R}$ with respect to the vertical direction. For any $H \in (0, \frac{n-1}{n}]$, there is an entire rotational graph over $\mathbb{H}^n$ with constant mean curvature H. The result analogous to Theorem 3.1 in $\mathbb{H}^n \times \mathbb{R}$, holds for hypersurfaces with constant mean curvature larger than $\frac{n-1}{n}$.*

3.2 The answer in $\mathbb{R}^3$

Do Carmo question in $\mathbb{R}^3$ has been completely answered by F. Lopez and A. Ros [31] and independently by A. da Silveira [15]. We state Lopez-Ros Theorem.

Theorem 3.3. *The only weakly stable constant mean curvature surfaces in $\mathbb{R}^3$ are the plane and the sphere.*

In the same article, F. Lopez and A. Ros show that a complete surface with constant mean curvature has finite index if and only if it is either compact or a minimal surface with finite total curvature. Notice that examples of compact constant mean curvature surfaces other than the sphere are the famous Wente tori, constructed by H. Wente [41].

The strategy of Lopez-Ros proof is as follows. They prove that, if the mean curvature is different from zero, then the surface must be compact, then it is the sphere by Theorem 2.2. If the mean curvature is zero, then one apply the following result, that we already cited in the introduction.

Theorem 3.4. *A minimal stable surface in $\mathbb{R}^3$ is a plane.*

Theorem 3.4 has been proven independently by M. Do Carmo, C.K. Peng [17], D. Fischer-Colbrie, R. Schoen, [23] and A. Pogorelov [37].

The fact that so many high level mathematicians were working on the subject almost at the same time is just a further evidence that the end of the seventies was the beginning of *a new golden age of minimal surface theory* [36].

Notice that, the result of Theorem 3.4 can be viewed as an generalization of the celebrated Bernstein's theorem.

Theorem 3.5. *[5] A minimal entire graph in $\mathbb{R}^3$ is a plane.*

3.3 The answer in $\mathbb{R}^4$ and $\mathbb{R}^5$

For almost 20 years, there was no answer to Do Carmo's question in dimension larger than three, without some further assumption. Then, in

2007, M. F. Elbert, the author and H. Rosenberg answered to Do Carmo's question in $\mathbb{R}^4$ and $\mathbb{R}^5$ [21, Corollary 1]. The analogous result was obtained by X. Cheng independently [11, Theorem 1.2] where the author observed that the result holds for finite index hypersurfaces, as well.

Theorem 3.6. *Any complete, stable, constant mean curvature, hypersurface in $\mathbb{R}^{n+1}$ for $n = 3, 4$ is minimal.*

The result by F. Elbert, the author and H. Rosenberg and X. Cheng, holds in a general Riemannian manifold, provided the mean curvature of the hypersurface is larger than a constant depending on the sectional curvature of the ambient space. In fact the result follows immediately from the following distance estimate.

Theorem 3.7. *[21, Theorem 1] Let M^n be any stable, constant mean curvature H, hypersurface of a riemannian manifold $\mathcal{N}^{n+1}$ for $n = 3, 4$. There exists a constant c depending on n, H, $sec(\mathcal{N}^{n+1})$, such that*

$$dist_{M^n}(p, \partial M^n) \leq c$$

whenever $H > 2\sqrt{|\min(0, sec(\mathcal{N}^{n+1})|}$.

The proof is based on a Bonnet-Myers method and inspires in [38].

Proof. (sketch) Stability implies that there exists $u > 0$ on M such that $Lu = 0$ [22]. Consider the metric on M given by $d\tilde{s}^2 = u^{2k}ds^2$, where $\frac{5(n-1)}{4n} \leq k \leq \frac{4}{n-1}$. Notice that in order to have a k satisfying the previous bound, n must be smaller than 5.

Let $p \in M$ and r such that the ball B_r of radius r centered at p is contained in M. Let γ be a geodesic, minimizing for $d\tilde{s}$, joining p to ∂B_r. Let a be the ds-length of γ. One has $a \geq r$, then it is enough to bound a.

From the formula of the second variation along a minimizing geodesic one has

$$\int_0^{\tilde{r}} \left((n-1) \left(\frac{d\varphi}{d\tilde{s}} \right)^2 - \tilde{R}_{11}\varphi^2 \right) d\tilde{s} \geq 0$$

where $\tilde{r}$ is the $d\tilde{s}$-length of γ, $\varphi : [0, \tilde{r}] \longrightarrow M$, $\varphi(0) = \varphi(\tilde{r}) = 0$ and $\tilde{R}_{11}$ is the $d\tilde{s}$ Ricci curvature in the direction tangent to γ. After a long manipulation, using the expression of $\tilde{R}_{11}$ in terms of the ds Ricci curvature R_{11}, in the direction tangent to γ, the fact that $Lu = 0$ and the bounds on k, one gets that there exist two constant A, B depending on k, n, H such that

$$\int_0^a (\varphi_{ss} A + B\varphi)\varphi ds \leq 0.$$

Now, choosing $\varphi(s) = \sin(\pi s a^{-1})$, $s \in [0, a]$ one gets

$$a^2 \leq \frac{A\pi^2}{B}$$

that gives the desired result. □

Remark 3.8. *For completeness, we point out that the distance estimates in* $\mathbb{R}^3$ *given in [38, Theorem 2] is* $\frac{\pi}{2H}$. *This estimate was improved to* $\frac{\pi}{H}$ *by L. Mazet [32, Theorem 1]. Notice that Mazet's result is sharp, as it is realized in half of the round sphere of radius* $\frac{1}{H}$, *when one considers the distance between the north pole and the equator.*

4 Partial results in $\mathbb{R}^{n+1}$ for $n \geq 5$

In dimension larger than 5, there are only partial results concerning Do Carmo's question. In the following we will describe the principal results and issues.

4.1 Assumptions on the growth of the total curvature

Let M be a hypersurface in $\mathbb{R}^{n+1}$, denote by A the shape operator of M, by H its mean curvature and by $\Phi = -A + HI$, the traceless second fundamental form. H. Alencar and M. Do Carmo, in 1997, proved the following result.

Theorem 4.1. *[1, Theorem 01] Let* M *be a complete, non compact hypersurface of* $\mathbb{R}^{n+1}$, $n \leq 5$, *with constant mean curvature* H. *Assume that* M *is stable and that*

$$\lim_{R\longrightarrow\infty} \frac{\int_{B_R} |\Phi|^2 dM}{R^{2+2q}} = 0,\ \ q \leq \frac{2}{6n+1}.$$

Then M is a hyperplane.

The proof of Theorem 4.1 relies on a Simons' type inequality for the traceless second fundamental form and on classical computations similar to those by Do Carmo and Peng in [18]. One gets that $|\Phi| = 0$, that implies that M is totally umbilical and hence an hyperplane.

In 2000, M. Do Carmo and D. Zhou were able to improve by one the dimension in Theorem 4.1 but need to assume the following more restrictive hypothesis on the growth of the total curvature.

Theorem 4.2. *[19, 20] Let M be a complete, non compact hypersurface of $\mathbb{R}^{n+1}$, $n \leq 6$, with constant mean curvature H. Assume that M is stable and that*

$$\lim_{R\longrightarrow\infty} \frac{\int_{B_R} |\Phi|^2 dM}{R^{2-2/n}} = 0.$$

Then M is a hyperplane.

More recently, in 2012, Theorem 4.1 was generalized as follows by S. Ilias, the author and M. Soret.

Theorem 4.3. *[27, Corollary 6.3] Let M be a complete, non compact hypersurface of $\mathbb{R}^{n+1}$, $n \leq 5$, with constant mean curvature H. Assume that M is stable and there exists $s \geq 1$ such that*

$$\lim_{R\longrightarrow\infty} \frac{\int_{B_{2R}\setminus B_R} |\Phi|^2 dM}{R^{2s}} = 0.$$

Then M is a hyperplane.

In Theorem 4.3, the assumption on the growth of the total curvature is weaker than in Theorem 4.1. In fact, any polynomial growth is allowed for the total curvature. The strategy of the proof of Theorem 4.3 is very similar to the strategy of the proofs of the previous theorems by do Carmo, but an improved Caccioppoli's type inequality [27, Theorem 5.6] allows the authors to get a more general result.

We finally remark that Corollary 6.3 in [27] is stated in Space Forms.

4.2 Assumptions on the volume entropy

For a long time, the only concrete answers to Do Carmo's question were obtained by taking assumptions on total curvature. In 2016 [28], S. Ilias, the author and M. Soret considered a different point of view and study Do Carmo's question with respect to assumptions on the volume growth.

In order to study this problem, we introduce the volume entropy of a Riemannian manifold, that is, roughly speaking, the exponential volume growth of the balls. We follow the notation by R. Brooks [6, 7], who first introduced this invariant of the metric.

Let M be a n-manifold, B_R be a geodesic ball in M, of radius R, centered at a fixed point $\sigma \in M$ and denote by $|B_R|$ its volume. The *volume entropy* of M is

$$\mu_M := \limsup_{R \longrightarrow \infty} \left(\frac{\ln |B_R|}{R} \right).$$

The notion of volume entropy does not depend on the center σ of the balls. Moreover, having volume entropy equal to zero is equivalent to the following limit being satisfied.

$$\limsup_{R \longrightarrow \infty} \frac{|B_R|}{e^{\alpha R}} = 0, \ \forall \alpha > 0.$$

Then, it is natural to say that M has *subexponential growth* if $\mu_M = 0$ and *exponential growth* if $\mu_M > 0$.

Notice that, having subexponential growth is a weaker assumption than being bounded by a polynomial of any degree.

4.2.1 The volume entropy and the spectrum by R. Brooks

In order to relate the volume entropy with stability, we need to give some further definitions. Let M be a n-manifold, Δ be the Laplacian on M and $\sigma(M)$ be the spectrum of the opposite of the Laplacian on M, $-\Delta$.

The bottom of the spectrum $\sigma(M)$ is denoted by $\lambda_0(M)$ and is characterized as follows (see for example [9, Chapter 1]).

$$\lambda_0(M) = \inf\{\sigma(M)\} = \inf_{\substack{f\in C_0^\infty(M)\\ f\neq 0}} \left(\frac{\int_M |\nabla f|^2}{\int_M f^2}\right).$$

One also defines the essential spectrum $\sigma_{ess}(M)$ of $-\Delta$ and its bottom, as follows

$$\lambda_0^{ess}(M) = \inf\{\sigma_{ess}(M)\} = \sup_K \lambda_0(M \setminus K) \tag{4.1}$$

where K runs through all compact subsets of M.

Another invariant related to the spectrum is the *Cheeger constant* [10]. Recall that the Cheeger constant h_M of a Riemannian manifold M is defined as

$$h_M = \inf_\Omega \frac{|\partial\Omega|}{|\Omega|}$$

where Ω runs over all compact domains of M, with piecewise smooth boundary, and $|\cdot|$ indicates the volume.

We summarize in the following Theorem two results by Brooks and Cheeger.

Theorem 4.4. *[6, 10] If M has infinite volume, then*

$$\frac{h_M^2}{4} \leq \lambda_0(M) \leq \lambda_0(M \setminus K) \leq \lambda_0^{ess}(M) \leq \frac{\mu_M^2}{4}$$

where K is any compact subset of M.

We observe that the infiniteness of the volume follows, for example, from the existence of a Sobolev inequality [8, 25] or in the case where M is a complete noncompact submanifold with bounded mean curvature of a manifold with bounded geometry, by the result by K. Frensel [24] recalled in Section 2.

4.2.2 The case of zero volume entropy

The next theorem by S. Ilias, the author and M. Soret answers positively to Do Carmo's question provided the entropy of the hypersurface is zero. Notice that, there is neither dimensional nor curvature assumptions on the manifold M

Theorem 4.5. *[28, Corollary 8] There is no complete, non-compact, finite index hypersurface M in $\mathbb{R}^{n+1}$ with constant mean curvature $H \neq 0$ and $\mu_M = 0$.*

Theorem 4.5 follows easily from the following result.

Theorem 4.6. *[28, Theorem 9] There is no complete, noncompact, finite index hypersurface M immersed in a manifold $\mathcal{N}$, provided the mean curvature H of M satisfies $nH^2 + Ric(N) \geq \delta$, where N is a unit normal vector field to M, δ is a constant such that $\delta > \frac{\mu_M^2}{4}$.*

Proof. Assume that such M exists. By [22, Proposition 1], the finite index implies that there exists a compact K in M such that $M \setminus K$ is stable. Then, for any $f \in C_0^\infty(M \setminus K)$, one has

$$0 \leq Q(f, f) = \int_{M \setminus K} |\nabla f|^2 - (|A|^2 + Ric(N))f^2.$$

Then $|A|^2 + Ric(N) \geq nH^2 + Ric(N) \geq \delta$, yields $0 \leq \int_{M\setminus K} |\nabla f|^2 - \delta \int_{M\setminus K} f^2$, By definition of λ_0, one has $\lambda_0(M \setminus K) \geq \delta > \frac{\mu_M^2}{4}$. This contradicts Theorem 4.4. $\square$

We finish this section by stating a recent result, answering to Do Carmo's question in the case of constant mean curvature foliations of a manifold with zero volume entropy.

Theorem 4.7. *[29, Theorem 2]. Let $\mathcal{N}$ be a manifold such that $\mu_{\mathcal{N}} = 0$. Let $\mathcal{F}$ be a codimension one C^3 foliation of $\mathcal{N}$ by non-compact leaves of constant mean curvature H. Then $H = 0$.*

5 An interesting application: the Maximum Principle at Infinity.

We would like to complete this survey by recalling a theorem that can be proved once one has an affirmative answer to Do Carmo's question.

Let us start by a result for minimal surfaces, the so called *Strong Half-space Theorem* proved by D. Hoffman and W. Meeks in 1990.

Theorem 5.1. *[26, Theorem 2] Two connected minimal surfaces, properly immersed in* $\mathbb{R}^3$, *must intersect, unless they are parallel planes.*

Let us give an outline of the proof of Theorem 5.1. Assume by contradiction that two such surfaces M_1 and M_2 exist. Being minimal, M_1 and M_2 are not compact. Then, there is connected non-compact domain W such that $\partial W = M_1 \cup M_2$.

Now, translate M_1 towards M_2. If there is a first interior contact point, one gets a contradiction by the maximum principle, then, the first contact point is at infinity and we can assume that M_1 and M_2 are asymptotic one to the other. We consider a relatively compact domain S in M_1 with boundary $\Gamma = \partial S$ a Jordan curve. One can solve the Plateau problem with boundary Γ in W (using M_1 and M_2 as barriers). By letting S being larger and larger, solving Plateau problems for the sequence of boundary of S and getting the limit, one finds a complete stable minimal surface Σ in W. By Theorem 3.4, Σ must be a plane, hence M_1 lies on one side of a plane. This is a contradiction by the following *Halfspace Theorem.*

Theorem 5.2. *[26, Theorem 1] A connected, nonplanar minimal surface M properly immersed in* $\mathbb{R}^3$ *is not contained in a halfspace.*

We notice that, the proof that we sketched is not the proof contained in [26], but has the advantage that can be generalized to many other situations and can be related to the positive answer to Do Carmo's question.

In the litteraure, a result analogous to the strong half-space theorem, for surfaces with constant mean curvature different from zero is usually called *Maximum Principle at Infinity.*

As an example, we state the pioneer theorem proved by A. Ros and H. Rosenberg in 2010.

Theorem 5.3. *[38, Theorem 1] Let M_1 and M_2 be connected disjoint surfaces properly embedded in* $\mathbb{R}^3$, *with constant mean curvature H. Then M_2 is not on the mean convex side of M_1.*

We give a sketch of the strategy of the proof of Theorem 5.3. Assume, by contradiction, that M_2 is contained in the mean convex side of M_1. If either M_1 or M_2 is compact, one uses the classical maximum principle to get a contradiction. Otherwise, one is able to prove that there exists a complete stable surface with constant mean curvature H between M_1 and M_2, as in the proof of the Strong Halfspace Theorem. This is a contradiction because such surface does not exist, by Theorem 3.3.

Let us do some remarks about (strong) halfspace theorem and maximum principle at infinity.

In $\mathbb{R}^{n+1}$, $n > 2$, a theorem analogous to the Halfspace Theorem (Theorem 5.2) does not hold because catenoids (minimal hypersurfaces, invariant by rotations) are contained in a slab (see for example [40]). As a consequence, the analogous to the Strong Halfspace Theorem (Theorem 5.1) does not hold in higher dimension.

As for the higher dimensional version of Theorem 5.3, it clearly it holds in $\mathbb{R}^{n+1}$, $n = 3, 4$, because of Theorem 3.6. We notice that, as Theorem 3.6 holds in a more general ambient manifold with uniformly bounded sectional curvature, the Maximum Principle at Infinity holds as well ([21, Theorem 2]).

The Maximum Principle at Infinity has been studied in different ambient spaces of dimension three. Let us describe some examples.

In the hyperbolic space, a Maximum Principle at Infinity for surfaces with constant mean curvature larger than one, with bounded Gaussian curvature is proved by R. F. de Lima and W. Meeks III [30], while in $\mathbb{H}^2 \times \mathbb{R}$, provided the mean curvature is larger than $\frac{1}{\sqrt{3}}$, it is proved by the author and H. Rosenberg in [34, Theorem B]. H. Rosenberg proved a Maximum Principle at Infinity in any homogeneously regular three manifold, provided the mean curvature is large enough.

The careful reader could remark that in the Maximum Principle at Infinity in $\mathbb{H}^2 \times \mathbb{R}$, the natural bound on the mean curvature should be H larger than larger than $\frac{1}{2}$. We believe that the bound $\frac{1}{\sqrt{3}}$ in [34, Theorem B] is due to technical reasons, but this is an open question: does the

Maximum Principle at Infinity in $\mathbb{H}^2 \times \mathbb{R}$ holds for surfaces with constant mean curvature between $\frac{1}{\sqrt{3}}$ and $\frac{1}{2}$?

Different maximum principles at infinity in $\mathbb{H}^2 \times \mathbb{R}$ are contained for example in [33, 35, 39].

We finish this section with a three-dimensional case where the Strong Half Space Theorem is still open: the Heisenberg space, Nil_3.

We recall that the simply connected homogeneous 3-manifolds Nil_3 is one of the eight Thurston geometries. Moreover, minimal surfaces in such Thurston geometries has been deeply studied in the last twenty years.

The more general partial result about Halfspace Theorem in Nil_3 is the following theorem, due to B. Daniel, W. Meeks III and H. Rosenberg.

Theorem 5.4. *[14, Theorem 1.4] Let S be a properly immersed minimal surface in Nil_3. If S lies on one side of some entire minimal graph G, then S is the image of G by a vertical translation.*

A weaker result was proved by B. Daniel and L. Hauswirth, that showed that if S is contained on one side of a vertical plane P, then S is a vertical plane parallel to P [13, Theorem 6.3]. In order to get the Strong Halfspace Theorem, one should replace the graph G with a general minimal surface in Nil_3.

Acknowledgements

The author would like to thank the organizers of TYAN, Thematic Workshops in Mathematics: Differential Geometry, held online on February 21, 2022, for the kind invitation to write this survey.

References

[1] H. Alencar, M. Do Carmo: *Remarks on the growth of functions and the weak stability of hypersurfaces with constant mean curvature,* An. Acad. Brasil. Ciênc. 69, 2 (1997) 163-166.

[2] J. L. Barbosa, M. Do Carmo: *Stability of hypersurfaces with constant mean curvature,* Math. Z. 185, 3 (1984) 339-353.

[3] J. L. Barbosa, M. Do Carmo, J. Eschenburg: *Stability of hypersurfaces with constant mean curvature in Riemannian manifolds,* Math. Z. 197, 1 (1988) 123-138.

[4] J. L. Barbosa, P. Berard:*Eigenvalue and twisted eigenvalue problems, applications to CMC surfaces,* J. Math. Pures et Appl. 79, 5 (2000) 427-450.

[5] S. Bernstein: *Sur un théorème de géométrie et ses applications aux équations aux dérivées partielles elliptiques,* Comm. Inst. Sci. Math. Mech. Univ. Kharkov 15 (1915-17) 38-45.

[6] R. Brooks: *Exponential growth and the spectrum of the laplacian,* Proc. Am. Math. Soc. 82(3) (1981) 473-477.

[7] R. Brooks: *On the spectrum of non-compact manifold with finite volume,* Math. Zeit.187 (1984) 425- 432.

[8] G. Carron: *Inégalités isopérimétriques et inégalités de Faber-Krahn,* Séminaire de géométrie spectrale et géométrie, tome 13 (1994) 63-66.

[9] I. Chavel: *Eigenvalues in Riemannian Geometry,* Academic Press (1984).

[10] J. Cheeger: *A lower bound for the smallest eigenvalue of the Laplacian* , In: Gunning, R.C. (ed.) Problems in analysis. A symposium in honor os Salomon Bochner (Princeton 1969) 195-199. Princeton University Press, Princeton (1970).

[11] X. Cheng: *On constant mean curvature surfaces with finite index,* Archiv der Mathematik 86 (2006) 365-374.

[12] O. Chodosh, C. Li: *Stable minimal hypersurfaces in* $\mathbb{R}^4$, arXiv:2108.11462 [math.DG] (2021) to appear in Acta Math.

[13] B. Daniel and L. Hauswirth: *Half-space theorem, embedded minimal annuli and minimal graphs in the Heisenberg group,* Proc. Lond. Math. Soc. (3), 98(2) (2009) 445-470.

[14] B. Daniel, W. Meeks, H. Rosenberg: *Half-space theorems for minimal surfaces in* Nil_3 *and* Sol_3, J. Differential Geom. 88, 1 (2011) 41-59.

[15] A. Da Silveira: *Stability of complete noncompact surfaces with constant mean curvature,* Math. Ann. 277, 4 (1987) 629-638.

[16] Manfredo Perdigão Do Carmo: *Hypersurfaces of Constant Mean Curvature,* LNM n.1410, Differential Geometry, Peñiscola 1989.

[17] Manfredo Perdigão Do Carmo, C. K. Peng: *Stable complete minimal surfaces in* $\mathbb{R}^3$ *are planes,* Bull. of the Amer. Math. Soc, (NS) 1, 6 (1979) 903-906.

[18] M. Do Carmo, K. Peng: *Stable complete minimal hypersurfaces,* Proceedings of the 1980 Symphosium on differential geometry, Beijing 1,2,3 (1982) 1349-1358.

[19] M. Do Carmo, D. Zhou: *Bernstein-type theorems in hypersurfaces with constant mean curvature,* An. Acad. Brasil. Ciênc. 72, 3 (2000) 301-310.

[20] M. Do Carmo, D. Zhou: *Errata to "Bernstein-type theorems in hypersurfaces with constant mean curvature", An. Acad. Brasil. Cienc. 72, 3 (2000) 301-310*, An. Acad. Brasil. Ciênc. 73, 3 (2001) 333-335.

[21] M. F. Elbert, B. Nelli, H. Rosenberg: *Stable constant mean curvature hypersurfaces,* Proc. Amer. Math. Soc. 135, 10 (2007) 3359-3366.

[22] D. Fischer-Colbrie: *On Complete Minimal Surfaces with finite Morse Index in three Manifolds,* Inv. Math. 82 (1985) 121-132.

[23] D. Fischer-Colbrie, R. Schoen: *The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature,* Comm. Pure Appl. Math. 33, 2 (1980) 199-211.

[24] K. Frensel: *Stable complete surfaces with constant mean curvature,* An. Acad. Brasil. Ciênc. 60, 2 (1988) 115-117 (1989).

[25] E. Hebey.: Sobolev Spaces on Riemannian Manifolds, Lecture Notes in Mathematics, vol. 1635. Springer Verlag, Berlin (1996)

[26] D. Hofmann, W. Meeks: *The strong halfspace theorem for minimal surfaces,* Invent. Math. 101, 2 (1990) 373-377.

[27] S. Ilias, B. Nelli, M. Soret: *Caccioppoli's inequalities on constant mean curvature hypersurfaces in Riemannian manifolds,* Ann. Global Anal. Geom. 42, 4 (2012) 443-471.

[28] S. Ilias, B. Nelli, M. Soret: *On the entropies of hypersurfaces with bounded mean curvature,* Math. Ann. 364, 3-4 (2016) 1095-1120.

[29] S. Ilias, B. Nelli, M. Soret: *A Note on Constant Mean Curvature Foliations of Noncompact Riemannian Manifolds,* Int. Jour. of Math. and Math. Sciences (2022).

[30] R. F. De Lima, W. Meeks III: *Maximum principle at infinity for surfaces of bounded mean curvature in* $\mathbb{R}^3$ *and* $\mathbb{H}^3$, Indiana Univ. Math. J. 53, 5 (2004) 1211-1223.

[31] F. Lopez, A. Ros: *Complete minimal surfaces with index one and stable constant mean curvature surfaces,* Comment. Math. Helv. 64,1 (1989) 34-43.

[32] L. Mazet: *Optimal length estimates for stable CMC surfaces in 3-space forms,* Proc. Amer. Math. Soc. 137, 8 (2009) 2761-2765.

[33] L. Mazet: *A general halfspace theorem for constant mean curvature surfaces,* Amer. J. Math. 135, 3 (2013) 801-834.

[34] B. Nelli, H. Rosenberg: *Global properties of constant mean curvature surfaces in* $\mathbb{H}^2 \times \mathbb{R}$, Pac. Journ. Math. 226, 1 (2006) 137-152.

[35] B. Nelli, R. Sa Earp: *A halfspace theorem for mean curvature* $H = \frac{1}{2}$ *surfaces in* $\mathbb{H}^2 \times \mathbb{R}$, J. Math. Anal. Appl. 365, 1 (2010) 167-170.

[36] Joaquin Perez: *A new golden age of minimal surfaces,* Notices of the AMS, 64, 4 (2017) 347-358.

[37] A. Pogorelov: *On the stability of minimal surfaces, (Russian)* Dokl. Akad. Nauk SSSR 260, 2 (1981) 293-295.

[38] A. Ros, H. Rosenberg: *Properly embedded surfaces with constant mean curvature,* Amer. J. Math. 132, 6 (2010) 1429-1443.

[39] H. Rosenberg, F. Schulze, J. Spruck: *The half-space property and entire positive minimal graphs in* $M \times \mathbb{R}$, J. Diff. Geom. 95, 2 (2013) 321-336.

[40] R. Schoen: *Uniqueness, symmetry, and embeddedness of minimal surfaces,* J. Diff. Geom 18, (1983) 791-809.

[41] H. C. Wente: *A counter-example to a conjecture by H. Hopf,* Pac. Jour. of Math.121 (1986) 193-234.