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The extremal problem for Sobolev inequalities with upper order remainder terms

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Abstract. Given a smooth compact Riemannian n-manifold (M, g), we prove existence and compactness results of extremal functions for sharp Sobolev inequalities which are closely related to the embedding of $H^{1,q}(M)$ into $L^{qn/(n-q)}(M)$ where the L^q remainder term is replaced by upper order terms.

Keywords: Sharp Sobolev type inequalities, extremal functions, compactness.

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1 Introduction

A lot of attention has been dedicated to the so-called sharp Sobolev type inequalities (Aubin [2], Beckner [5], Brezis and Nirenberg [7], Brouttelande [8], Ceccon and Montenegro [12], Druet [16, 17], Escobar [19], Hebey and Vaugon [25], Lieb [28], Moser [31], Talenti [34], Trudinger [35], among others). Frequently, these inequalities are in connection with concrete problems from geometry and physics (Aubin [3], Carlen and Loss [10], Lieb and Thirring [29], Schoen [32]).

Considerable work has been devoted to the study of extremal functions to sharp Sobolev inequalities in recent decades (see Aubin [2], Aubin and Li [4], Brouttelande [9], Carleson and Chang [11], Collion, Hebey and Vaugon [13], Demyanov and Nazarov [14], Djadli and Druet [15], Druet, Hebey and Vaugon [18], Hebey [21, 23], Humbert [26], Li [27], Struwe [33] and Zhu [36]). Such functions are connected, for instance, with the computation of ground state energy in some physical models.

The goal of the present paper is to discuss the existence of extremal functions of Sobolev type inequality modeled on smooth compact Riemannian manifolds, precisely sharp Riemannian Sobolev-Poincaré inequalities involving also upper order remainder terms. Before we go further and exhibit our target problems, a little bit of notation and overview should be presented.

For $n \geq 2$, it was shown by Aubin [1] and Talenti [34] that, for $1 \leq q < n$ and $q* = qn/(n-q)$,

$$K(n,q) = \sup \left\{ \frac{\|\nabla u\|_{L^q(\mathbb{R}^n)}}{\|u\|_{L^{q*}(\mathbb{R}^n)}} : u \not\equiv 0, u \in L^{q*}(\mathbb{R}^n)\ , \nabla u \in L^q(\mathbb{R}^n) \right\}$$

is achieved and the extremal functions are found. In particular,

$$K(n,q) = \frac{q-1}{n-q} \left[\frac{n-q}{n(q-1)} \right]^{\frac{1}{q}} \left[\frac{\Gamma(n+1)}{\Gamma(\frac{n}{q})\Gamma(n+1-\frac{n}{q})\omega_{n-1}} \right]^{\frac{1}{n}}$$

for $1 < q < n$ and

$$K(n,1) = \frac{1}{n} \left[\frac{n}{\omega_{n-1}} \right]^{\frac{1}{n}}$$

where Γ is the gamma function and ω_{n-1} denotes the volume of the standard $(n-1)$-sphere. All the extremal functions for $1 \leq q < n$ are given by

$$u(x) = c \left(\frac{1}{\mu + |x - x_0|^{q/q-1}} \right)^{\frac{n-q}{q}}$$

where c, μ, $x_0 \in \mathbb{R}^n$. It is easy to see that for some $\tilde{c}$, $\tilde{\mu} > 0$ the corresponding

$$v(x) = \tilde{c}\left(\frac{1}{\tilde{\mu} + |x|^{q/q-1}}\right)^{\frac{n-q}{q}}$$

is the unique minimizer which satisfies:

$$v(0) = 1,\ \nabla v(0) = 0,\quad \int_{\mathbb{R}^n} v^{q*}dx = 1 \text{ and } -\nabla v = K(n,q)^{-q}v^{q*-1}.$$

On a compact Riemannian manifold n-dimensional (M, g), the Sobolev embedding theorem holds: the inclusion $H^{1,q}(M) \subset L^{q*}(M)$ is continuous for $1 \le q < n$. Thus, there exists a real constant C_0 such that any $u \in H^{1,q}(M)$ satisfies $\|u\|_{L^{q*}(M)} \le C_0\|\nabla_g u\|_{L^q(M)}$. Moreover, on a compact manifold, the inclusion $H^{1,q}(M) \subset L^{q*}(M)$ is continuous but not compact and $H^{1,q}(M) \subset L^q(M)$ is compact by the Kondrakov theorem. When we are in this situation, there are constants C and A such that

$$\|u\|_{L^{q*}(M)} \le C\|\nabla_g u\|_{L^q(M)} + A\|u\|_{L^q(M)}. \tag{1.1}$$

Define $K = \inf C$ such that some A exists. Then $K > 0$. Aubin [1] proved that K only depends on n and p. So $K = K(n,p)$ is a norm of the inclusion $H^{1,q}(\mathbb{R}^n) \subset L^{q*}(\mathbb{R}^n)$.

Let (M, g) be a smooth compact Riemannian n-manifold and $q \le p < qn/(n-q)$, such that there exists a constant $B_0(p,n,g) > 0$ where, for any $C_0^\infty(M)$, we have the following sharp inequality

$$||u||^q_{L^{q*}(M)} \le K(n,q)^q||\nabla_g u||^q_{L^q(M)} + B_0(p,n,g)||u||^q_{L^p(M)} \tag{1.2}$$

for all $u \in H^{1,q}(M)$.

The constant $B_0(p,n,g) = \inf A$ such that (1.1) occurs with $C = K(n,q)$, $B_0(p,n,g)$ depends only on p and (M, g). By summarising the works of Aubin [2], Druet [16], Hebey and Vaugon [24] , the inequality above is valid in the cases:

1. On any smooth, compact Riemannian n-manifold, $n \ge 3$ and $q = 2$.

2. For all q on any 2-dimensional smooth, compact Riemannian manifold.

3. For all q on compact flat spaces, compact hyperbolic spaces and smooth, compact n-manifolds of nonpositive sectional curvature as long as the Cartan-Hadamard n-manifold conjecture is true (see Hebey [22] section 8.2). In particular, $n = 3$ or $n = 4$.

Special attention has also been paid to the existence problem of extremal functions to (1.2). A non-zero function $u_0 \in C^\infty(M)$ is said to be an extremal to (1.2), if

$$||u_0||^q_{L^{q^*}(M)} = K(n,q)^q ||\nabla_g u_0||^q_{L^q(M)} + B_0(p,n,g)||u_0||^q_{L^p(M)} \,.$$

Denote by $E_p(g)$ the set of the extremal functions to (1.2) with unit L^{q^*}-norm.

Our main result in this paper is summarized in the next theorem.

Theorem 1.1. *Let (M,g) be a smooth compact Riemannian n-manifold without boundary of dimension $n \geq 4$ such that the inequality (1.2) is true. Then the set $E_p(g)$ is non-empty for any $1 < q < p < q^*$.*

The general idea of the proof and its nature are well-known and were developed in various works (cf. [2], [4], [15], [24], among others). The tools are based on blow-up techniques, concentration analysis and PDE estimates. What happens is that each proof has its specific technical difficulties inherent to the problem addressed, for instance, by the range of values of p in our inequalities. The ideas of the proofs are mainly inspired in the works of Aubin and Li [4]. The key points are the so-called L^p concentration estimates.

In Section 2, we define the PDE framework and formulate functions u_α, aimed at minimizing the Euler-Lagrange functional J_α. Section 3 is dedicated to construct an extremal function $u \in C^\infty(M)$, the weak limit of u_α. We study in detail the case $u = 0$. We then perform a comprehensive study of blow-up, concentration and priori estimates on the generated family of minimizers. Moving to Section 4, we show that u is non-zero and conclude the proof of Theorem 1.1, by contraction.

2 The PDE setting

Let $1 < q < p < q^*$ and $(\alpha) \subset \mathbb{R}$ be a sequence of positive real numbers converging to $B_0(p, n, g)$ with $\alpha < B_0(p, n, g)$. For each α, we consider the functional

$$J_\alpha(u) = \int_M |\nabla_g u|^q \, dv_g + \alpha K(n, q)^{-q} \left(\int_M |u|^p \, dv_g \right)^{\frac{q}{p}}$$

defined on

$$\Lambda_\alpha = \left\{ u \in H^{1,q}(M) : ||u||_{L^{q^*}(M)} = 1 \right\}$$

where dv_g is the Riemannian volume element of g and $H^{1,q}(M)$ denotes the completion of $C^\infty(M)$ under the norm

$$||u||_{H^{1,q}(M)} = \left(||\nabla_g u||^q_{L^q(M)} + ||u||^q_{L^q(M)} \right)^{1/q} .$$

By the definition of $B_0(p, n, g)$,

$$\lambda_\alpha := \inf_{\Lambda_\alpha} J_\alpha(u) < K(n, q)^{-q}. \tag{2.1}$$

For α close enough to $B_0(p, n, g)$, we claim that (2.1) leads to the existence of a positive smooth minimizer u_α for λ_α. The Euler-Lagrange equation satisfied by such a minimizer is

$$- \Delta_q u_\alpha + \alpha K(n, q)^{-q} ||u_\alpha||^{q-p}_{L^p(M)} u_\alpha^{p-1} = \lambda_\alpha u_\alpha^{q^*-1}, \tag{2.2}$$

where $\Delta_q u = \mathrm{div}_g(|\nabla_g u|^{q-2} \nabla_g u)$ is the q-Laplacian operator associated to the metric g. Since J_α is of C^1 class on Λ_α, by the Ekeland's variational principle [20], there exists a minimizing sequence $(u_m) \subset \Lambda_\alpha$ such that $||DJ_\alpha(u_m)||_{(T_{u_m}\Lambda_\alpha)^*} \to 0$, where DJ_α denotes the Fréchet derivative of J_α on Λ_α and T represents the tangent space. Since the sequence (u_m) is bounded in $H^{1,q}(M)$, there exists $u_\alpha \in H^{1,q}(M)$ such that, up to a subsequence, (u_m) converges weakly to u_α in $H^{1,q}(M)$, strongly in $L^q(M)$ and in $L^p(M)$, and almost everywhere as $m \to +\infty$. Moreover, there exist bounded nonnegative measures μ and ν such that

$$|\nabla_g u_m|^q dv_g \rightharpoonup \mu, \quad |u_m|^{q^*} dv_g \rightharpoonup \nu. \tag{2.3}$$

By a standard concentration-compactness principle of Lions [30], there exists at most a countable set $\mathcal{T}$, $\{x_j\}_{j\in\mathcal{T}} \subset M$ and positive numbers $\{\mu_j\}_{j\in\mathcal{T}}$ and $\{\nu_j\}_{j\in\mathcal{T}}$ such that

$$\mu \geq |\nabla_g u_\alpha|^q dv_g + \sum_{j\in\mathcal{T}} \mu_j \delta_{x_j}, \quad \nu = |u_\alpha|^{q^*} dv_g + \sum_{j\in\mathcal{T}} \nu_j \delta_{x_j} \tag{2.4}$$

with $K(n,q)^q \mu_j \geq \nu_j^{q/q^*}$ for all $j \in \mathcal{T}$, where δ_{x_j} represents the Dirac mass centered at x_j.

Fix $k \in \mathcal{T}$ and choose a cutoff function $\varphi_\varepsilon \in C_0^\infty\left(B(x_k, 2\varepsilon)\right)$ satisfying $0 \leq \varphi_\varepsilon \leq 1$, $\varphi_\varepsilon = 1$ in $B(x_k,\varepsilon)$ and $|\nabla_g \varphi_\varepsilon| \leq \frac{c}{\varepsilon}$ for some constant $c > 0$ independent of ε, where $B(x_k,\varepsilon)$ denotes the geodesic ball, with respect to g, of radius ε centered at x_k. Write

$$\varphi_\varepsilon u_m = \tau_m + \left(\int_M |u_m|^{q^*} \varphi_\varepsilon \, dv_g\right) u_m,$$

where

$$\tau_m := \left[\varphi_\varepsilon - \left(\int_M |u_m|^{q^*} \varphi_\varepsilon \, dv_g\right)\right] u_m \in T_{u_m}\Lambda_\alpha.$$

The boundness of (u_m) in $H^{1,q}(M)$ implies

$$\begin{aligned}&\int_M |\nabla_g u_m|^{q-2} \langle \nabla_g u_m, \nabla_g \tau_m\rangle \, dv_g + \\ &\qquad + \alpha K(n,q)^{-q} ||u_m||_p^{q-p} \int_M |u_m|^{p-2} u_m \tau_m \, dv_g \to 0.\end{aligned}$$

Since

$$\begin{aligned}&\int_M |\nabla_g u_m|^{q-2} \langle \nabla_g u_m, \nabla_g \tau_m\rangle \, dv_g + \\ &\qquad + \alpha K(n,q)^{-q} ||u_m||_p^{q-p} \int_M |u_m|^{p-2} u_m \tau_m \, dv_g\end{aligned}$$

$$
\begin{aligned}
&= \int_M |\nabla_g u_m|^{q-2} \nabla_g u_m \nabla_g(\varphi_\varepsilon u_m)\, dv_g \\
&\quad - \left(\int_M |u_m|^{q^*} \varphi_\varepsilon\, dv_g\right)\left(\int_M |\nabla_g u_m|^{q-2} \langle \nabla_g u_m, \nabla_g u_m\rangle\, dv_g\right) \\
&\quad + \alpha K(n,q)^{-q} ||u_m||_p^{q-p} \int_M |u_m|^{p-2} u_m(\varphi_\varepsilon u_m)\, dv_g \\
&- \left(\int_M |u_m|^{q^*} \varphi_\varepsilon\, dv_g\right) \alpha K(n,q)^{-q} ||u_m||_p^{q-p} \int_M |u_m|^{p-2} u_m^2\, dv_g,
\end{aligned}
$$

so that, by (2.3),

$$
\begin{aligned}
&\lim_{m\to+\infty} \left(\int_M |\nabla_g u_m|^{q-2} \langle \nabla_g u_m, \nabla_g(\varphi_\varepsilon u_m)\rangle\, dv_g + \right. \\
&\qquad \left. + \alpha K(n,q)^{-q} ||u_m||_p^{q-p} \int_M |u_m|^{p-2} u_m(\varphi_\varepsilon u_m)\, dv_g\right) \\
&= \lim_{m\to+\infty} \left(\int_M |u_m|^{q^*} \varphi_\varepsilon\, dv_g\right) \times \\
&\qquad \times \left(\int_M |\nabla_g u_m|^q\, dv_g + \alpha K(n,q)^{-q} \left(\int_M |u_m|^p\, dv_g\right)^{\frac{q}{p}}\right) \\
&= \lambda_\alpha \left(\int_M \varphi_\varepsilon\, d\nu\right). \qquad (2.5)
\end{aligned}
$$

On the other hand, from (2.3), we also get

$$
\begin{aligned}
&\lim_{m\to+\infty} \left(\int_M |\nabla_g u_m|^{q-2} \langle \nabla_g u_m\ , \nabla_g(\varphi_\varepsilon u_m)\rangle\, dv_g + \right. \\
&\qquad \left. + \alpha K(n,q)^{-q} ||u_m||_p^{q-p} \int_M |u_m|^{p-2} u_m(\varphi_\varepsilon u_m)\, dv_g\right) \\
&= \lim_{m\to+\infty} \left(\int_M u_m |\nabla_g u_m|^{q-2} \langle \nabla_g u_m\ , \nabla_g \varphi_\varepsilon\rangle + \varphi_\varepsilon |\nabla_g u_m|^q\, dv_g + \right. \\
&\qquad \left. + \alpha K(n,q)^{-q} ||u_m||_p^{q-p} \int_M \varphi_\varepsilon |u_m|^p\, dv_g\right) \\
&= \lim_{m\to+\infty} \left(\int_M u_m |\nabla_g u_m|^{q-2} \langle \nabla_g u_m\ , \nabla_g \varphi_\varepsilon\rangle\, dv_g + \right. \\
&\qquad \left. + \alpha K(n,q)^{-q} ||u_m||_p^{q-p} \int_M \varphi_\varepsilon |u_m|^p\, dv_g\right) + \int_M \varphi_\varepsilon\, d\mu. \qquad (2.6)
\end{aligned}
$$

We now show that the last limit tends to zero as $\varepsilon \to 0$. In fact, using Hölder's inequality, we have

$$
\begin{aligned}
&\left| \int_M u_m |\nabla_g u_m|^{q-2} \langle \nabla_g u_m, \nabla_g \varphi_\varepsilon \rangle \, dv_g \right| \\
&\leq \int_M |u_m| |\nabla_g u_m|^{q-1} |\nabla_g \varphi_\varepsilon| \, dv_g \\
&\leq \left(\int_M |\nabla_g u_m|^q \, dv_g \right)^{\frac{q-1}{q}} \left(\int_{B(x_k,2\varepsilon)\setminus B(x_k,\varepsilon)} |u_m|^q |\nabla_g \varphi_\varepsilon|^q \, dv_g \right)^{\frac{1}{q}} \\
&\leq \left(\int_M |\nabla_g u_m|^q \, dv_g \right)^{\frac{q-1}{q}} \left(\int_{B(x_k,2\varepsilon)\setminus B(x_k,\varepsilon)} |\nabla_g \varphi_\varepsilon|^n \, dv_g \right)^{1/n} \times \\
&\qquad \times \left(\int_{B(x_k,2\varepsilon)\setminus B(x_k,\varepsilon)} |u_m|^{q^*} \, dv_g \right)^{1/q^*}
\end{aligned}
$$

Observe also that, from (2.4),

$$
\begin{aligned}
&\limsup_{m\to+\infty} \left| \int_M u_m |\nabla_g u_m|^{q-2} \langle \nabla_g u_m \ , \nabla_g \varphi_\varepsilon \rangle \, dv_g \right| \\
&\leq c \limsup_{m\to+\infty} \left[\left(\int_M |\nabla_g u_m|^q \, dv_g \right)^{\frac{q-1}{q}} \left(\int_{B(x_k,2\varepsilon)\setminus B(x_k,\varepsilon)} |\nabla_g \varphi_\varepsilon|^n \, dv_g \right)^{\frac{1}{n}} \times \right. \\
&\qquad \left. \times \left(\int_{B(x_k,2\varepsilon)\setminus B(x_k,\varepsilon)} |u_m|^{q^*} \, dv_g \right)^{\frac{1}{q^*}} \right] \\
&\leq c \left[\frac{1}{\varepsilon^n} \mathrm{vol}_g \left(B(x_k, 2\varepsilon) \setminus B(x_k, \varepsilon) \right) \right]^{1/n} \times \\
&\qquad \times \lim_{m\to\infty} \left(\int_{B(x_k,2\varepsilon)\setminus B(x_k,\varepsilon)} |u_m|^{q^*} \, dv_g \right)^{\frac{1}{q^*}} \\
&\leq c \left(\int_{B(x_k,2\varepsilon)\setminus B(x_k,\varepsilon)} |u_\alpha|^{q^*} \, dv_g + \sum_{j\in\mathcal{T}} \nu_j \delta_{x_j} vol(B(x_k, 2\varepsilon) \setminus B(x_k, \varepsilon)) \right)^{\frac{1}{q^*}} \\
&\to 0, \text{ as } \varepsilon \to 0.
\end{aligned}
$$

The estimate of the remaining integral is decomposed into two cases.

If $u_\alpha = 0$ on M, then

$$\limsup_{m\to+\infty}\left(||u_m||_p^{q-p}\int_M \varphi_\varepsilon |u_m|^p\, dv_g\right) \le \limsup_{m\to+\infty} ||u_m||_p^q = 0.$$

Otherwise,

$$\limsup_{m\to+\infty}\left(||u_m||_p^{q-p}\int_M \varphi_\varepsilon |u_m|^p\, dv_g\right) = ||u_\alpha||_p^{q-p}\int_M \varphi_\varepsilon |u_\alpha|^p\, dv_g \to 0$$

as $\varepsilon \to 0$.

Therefore, letting $\varepsilon \to 0$ in (2.5) and (2.6), one arrives at

$$\mu_k = \lambda_\alpha \nu_k. \tag{2.7}$$

We claim that $\lambda_\alpha > 0$ for α close enough to $B_0(p,n,g)$. Let $B_0(p,n,g)-\varepsilon < \alpha < B_0(p,n,g)$ with $\varepsilon > 0$ small enough. Evaluating the sharp inequality (1.2) at u_m and applying Hölder's inequality, we obtain

$$||u_m||_{L^p}^q \le ||u_m||_{L^{q^*}}^q v_g(M)^{\frac{q}{p}-\frac{q}{q^*}},$$

so that

$$\begin{aligned} ||u_m||_{L^{q^*}(M)}^q &\le K(n,q)^q\Big(\int_M |\nabla_g u_m|^q\, dv_g\Big) + B_0(p,n,g)||u_m||_{L^p(M)}^2 \\ &= K(n,q)^q J_\alpha(u_m) + (B_0(p,n,g)-\alpha)||u_m||_{L^p(M)}^q \\ &\le K(n,q)^q J_\alpha(u_m) + \varepsilon v_g(M)^{\frac{q}{p}-\frac{q}{q^*}}||u_m||_{L^{q^*}(M)}^q, \end{aligned}$$

there exists $M > 0$, $M > K(n,q)^q$ such that

$$J_\alpha(u_m) \ge 1/M$$

for all $m \ge 1$. So, the positivity of λ_α follows by passing the limit on m in the inequality above. In particular, from (2.7), one has $\mu_k > 0$ if, and only if, $\nu_k > 0$. In this case, from $K(n,q)^q\mu_k \ge \nu_k^{q/q^*}$, one gets

$$\mu_k \ge \frac{1}{K(n,q)^n \lambda_\alpha^{n/q^*}}\ .$$

This implies that $\mathcal{T}$ is a finite set, since μ is a bounded measure. We claim that $\mathcal{T} = \emptyset$. Otherwise, if $k \in \mathcal{T}$, then

$$\begin{aligned}
\lambda_\alpha &= \lim_{m\to+\infty} \left(\int_M |\nabla_g u_m|^q \, dv_g + \alpha K(n,q)^{-q} \left(\int_M |u_m|^p \, dv_g \right)^{\frac{q}{p}} \right) \\
&\geq \int_M |\nabla_g u_\alpha|^q \, dv_g + \alpha K(n,2)^{-2} \left(\int_M |u_\alpha|^p \, dv_g \right)^{\frac{q}{p}} + \sum_{j\in\mathcal{T}} \mu_j \\
&\geq \sum_{j\in\mathcal{T}} \mu_j \geq \mu_k \geq \frac{1}{K(n,q)^n \lambda_\alpha^{n/q^*}} \,,
\end{aligned}$$

so that

$$\lambda_\alpha \geq K(n,q)^{-q} \, .$$

However, this last inequality contradicts (2.1). Therefore, $||u_m||_{L^{q^*}(M)}$ converges to $||u_\alpha||_{L^{q^*}(M)}$. Brezis-Lieb lemma [6] then guarantees that (u_m) converges strongly to u_α in $L^{q^*}(M)$ and, in particular, $u_\alpha \in \Lambda_\alpha$. Moreover, u_α is a minimizer of J_α on Λ_α. We can assume that u_α is a nonnegative minimizer, since J_α and Λ_α are $\mathbb{Z}^2$-invariant. So, we find a nontrivial nonnegative weak solution u_α to (2.2). Its positivity and regularity follow directly from well-known results of the elliptic PDEs theory (Rabinowitz [37]).

3 Blow-up analysis

Let $q < p < q^*$ and $(\alpha) \subset \mathbb{R}$ be a sequence of positive real numbers converging to $B_0(p,n,g)$ with $\alpha < B_0(p,n,g)$. Assume that, for any α,

$$\lambda_\alpha = \inf_{\Lambda_\alpha} J_\alpha(u) < K(n,q)^{-q} \,,$$

where J_α and Λ_α are as in Section 2. In the previous section, we construct positive functions $u_\alpha \in C^\infty(M)$ satisfying

$$-\Delta_q u_\alpha + \alpha K(n,q)^{-q} ||u_\alpha||_p^{q-p} u_\alpha^{p-1} = \lambda_\alpha u_\alpha^{q^*-1} \qquad (3.1)$$

and $\int_M |u_\alpha|^{q^*} dv_g = 1$.

Since the sequence (u_α) is bounded in $H^{1,q}(M)$, there exists a nonnegative function $u \in H^{1,q}(M)$ such that, up to a subsequence, (u_α) converges weakly to u in $H^{1,q}(M)$ and strongly in $L^q(M)$ for any $2 \leq q < q^*$ as $\alpha \to B_0(p,n,g)$. Assume also that (λ_α) converges to λ. If $u \not\equiv 0$, then by letting $\alpha \to B_0(p,n,g)$ in (3.1), one has

$$-\Delta_q u + B_0(p,n,g)K(n,q)^{-q}||u||_p^{q-p}u^{p-1} = \lambda u^{q^*-1}. \tag{3.2}$$

From (1.2) and (3.2), it follows that

$$\begin{aligned}
\left(\int_M |u|^{q^*}\, dv_g\right)^{\frac{q}{q^*}} \leq& \\
&\leq K(n,q)^q \left(\int_M |\nabla_g u|^q\, dv_g\right) + B_0(p,n,g)\left(\int_M |u|^p\, dv_g\right)^{\frac{q}{p}} \\
&= K(n,q)^q \lambda \int_M |u|^{q^*}\, dv_g \leq \int_M |u|^{q^*}\, dv_g,
\end{aligned}$$

since $0 \leq \lambda \leq K(n,q)^{-q}$. This implies that $||u||_{L^{q^*}(M)} \geq 1$. On the other hand, we get $||u||_{L^{q^*}(M)} \leq \liminf ||u_\alpha||_{L^{q^*}(M)} = 1$, so that $||u||_{L^{q^*}(M)} = 1$. Using this information in the inequality above, it follows that $\lambda = K(n,q)^{-q}$ and, in particular, u is an extremal function of (1.2) .

The rest of the paper is dedicated to a detailed study of the sequence (u_α) when $u = 0$ on M. Such a study consists of blow-up analysis and PDEs estimates to the sequence (u_α). By definition of u_α, we have

$$\lambda_\alpha = \int_M |\nabla_g u_\alpha|^q dv_g + \alpha K(n,q)^{-q}\left(\int_M |u_\alpha|^p\, dv_g\right)^{\frac{q}{p}}.$$

On the other hand, by (1.2),

$$1 \leq K(n,q)^q \int_M |\nabla_g u_\alpha|^q dv_g + B_0(p,n,g)\left(\int_M |u_\alpha|^p\, dv_g\right)^{\frac{q}{p}}.$$

So, we get

$$1 \leq \lambda_\alpha K(n,q)^q + (B_0(p,n,g) - \alpha)\left(\int_M |u_\alpha|^p\, dv_g\right)^{\frac{q}{p}}$$

and it readily follows from this inequality that

$$\lambda_\alpha \to K(n,q)^{-q} \tag{3.3}$$

as $\alpha \to B_0(p,n,g)$. In particular, we have

$$\int_M |\nabla_g u_\alpha|^q \, dv_g \to K(n,q)^{-q} \,. \tag{3.4}$$

Note also that

$$||u_\alpha||_{L^\infty(M)} \to +\infty \,, \tag{3.5}$$

what can be seen from

$$1 = ||u_\alpha||^{q^*}_{L^{q^*}(M)} \leq ||u_\alpha||^{q^*-q}_{L^\infty(M)} ||u_\alpha||^q_{L^q(M)} \,.$$

We let $x_\alpha \in M$ and $\mu_\alpha > 0$ be such that

$$u_\alpha(x_\alpha) = ||u_\alpha||_{L^\infty(M)} = \mu_\alpha^{-\frac{n-q}{q}} \tag{3.6}$$

and μ_α converges to 0 as $\alpha \to B_0(p,n,g)$.

Let $\delta_0 > 0$ be a number less than the injectivity radius of (M,g). For $\beta > 0$ fixed, consider the function $B_{x_\alpha,\beta} : M \to \mathbb{R}$ given by

$$B_{x_\alpha,\beta}(x) = \beta^{\frac{n}{q^*}} \left(1 + (\beta\overline{\beta})^{\frac{q}{q-1}} d_g(x,x_\alpha)^{\frac{q}{q-1}}\right)^{-\frac{n}{q^*}} \,,$$

where $\overline{\beta} = (n(n-2))^{-1} K(n,q)^{-q}$.

The blow-up analysis on the sequence (u_α) is made in order to establish the following estimates:

Estimate 1. *For each $\delta_0/2 \leq \delta_\alpha \leq \delta_0$, we have*

$$\int_{B_{\delta_\alpha}(x_\alpha)} \left(|\nabla_g(u_\alpha - B_{x_\alpha,\mu_\alpha^{-1}})|^q + |u_\alpha - B_{x_\alpha,\mu_\alpha^{-1}})|^{q^*}\right) \, dv_g \to 0$$

as $\alpha \to B_0(p,n,g)$.

Proof. Consider the following rescaling of (u_α) defined on the geodesic ball $B_{\delta_\alpha}(x_\alpha)$:

$$v_\alpha(y) = \mu_\alpha^{\frac{n}{q^*}} u_\alpha(exp_{x_\alpha}(\mu_\alpha y))\,, \quad y \in \Omega_\alpha\,,$$

where

$$\Omega_\alpha = \mu_\alpha^{-1} exp_{x_\alpha}^{-1}(B_{\delta_\alpha}(x_\alpha)) = \mu_\alpha^{-1} B_{\delta_\alpha}(0)$$

and $exp_{x_\alpha}(\mu_\alpha y)$ is an exponential map.

Clearly, v_α satisfies

$$-\Delta_{g_\alpha,q} v_\alpha + \eta_\alpha v_\alpha^{p-1} = \lambda_\alpha v_\alpha^{q^*-1} \quad \text{in} \quad \Omega_\alpha\,, \tag{3.7}$$

where

$$g_\alpha(y) = g(exp_{x_\alpha}(\mu_\alpha y))$$

and

$$\eta_\alpha = \alpha \mu_\alpha^{n - \frac{p(n-q)}{q}} ||u_\alpha||_{L^p(M)}^{q-p},$$

with a constant $C > 0$ independent of α. Remark that the sequence of metrics (g_α) converges to the Euclidean metric ξ on compact subsets of $\mathbb{R}^n$ in the C^1-topology. We claim that (η_α) converges to 0 as $\alpha \to B_0(p,n,g)$. In fact, from the definition of μ_α in (3.6), one gets

$$\eta_\alpha = \frac{\alpha ||u_\alpha||_{L^p(M)}^q}{||u_\alpha||_{L^\infty(M)}^{q^*-p} ||u_\alpha||_{L^p(M)}^p}$$

and

$$1 = \int_M |u_\alpha|^{q^*}\, dv_g \le ||u_\alpha||_{L^\infty(M)}^{q^*-p} ||u_\alpha||_{L^p(M)}^p\,,$$

so that

$$\eta_\alpha \le \alpha ||u_\alpha||_{L^p(M)}^q \to 0$$

as $\alpha \to B_0(p,n,g)$.

A simple change of variable furnishes

$$\int_{\Omega_\alpha} v_\alpha^{q^*}\, dv_{g_\alpha} = \int_{B_\alpha} u_\alpha^{q^*}\, dv_g$$

and

$$\int_{\Omega_\alpha} |\nabla_{g_\alpha} v_\alpha|^q \, dv_{g_\alpha} = \int_{B_\alpha} |\nabla_g u_\alpha|^q \, dv_g,$$

Consequently,

$$\limsup_{\alpha \to B_0} \int_{\Omega_\alpha} v_\alpha^{q^*} \, dv_{g_\alpha} \le 1 \tag{3.8}$$

and

$$\limsup_{\alpha \to B_0} \int_{\Omega_\alpha} |\nabla_{g_\alpha} v_\alpha|^q \, dv_{g_\alpha} \le K(n,q)^{-q} \,. \tag{3.9}$$

Since $0 \le v_\alpha \le 1$ in Ω_α and (g_α) converges to the Euclidean metric ξ_{ij} in $C^1_{loc}(\mathbb{R}^n)$, it follows from standard elliptic estimates that (v_α) converges, modulo a subsequence, to a function v in $C^1_{loc}(\mathbb{R}^n)$. Note that $v(0) = 1$ since $v_\alpha(0) = 1$. Moreover, by (3.4), v satisfies

$$-\Delta_q v = K(n,q)^{-q} v^{q^*-1} \quad \text{in} \quad \mathbb{R}^n \,. \tag{3.10}$$

Besides, by (3.9), for any $R > 0$,

$$\int_{B_R} |v|^{q^*} \, dy = \lim_{\alpha \to B_0} \int_{B_R} |v_\alpha|^{q^*} \, dv_{g_\alpha} \le 1 \,. \tag{3.11}$$

and

$$\int_{B_R} |\nabla v|^q \, dx = \lim_{\alpha \to B_0} \int_{B_R} |\nabla_{g_\alpha} v_\alpha|^q \, dv_{g_\alpha} \le K(n,q)^{-q}, \tag{3.12}$$

so that $v \in \mathcal{D}^{1,q}(\mathbb{R}^n)$. Multiplying (3.10) by v, integrating by parts and using the definition of $K(n,q)$, we have

$$K(n,q)^{-q} \int_{\mathbb{R}^n} v^{q^*} \, dx = \int_{\mathbb{R}^n} |\nabla v|^q \, dx \ge K(n,q)^{-q} \left(\int_{\mathbb{R}^n} |v|^{q^*} \, dx \right)^{\frac{q}{q^*}} .$$

Thus, $||v||_{L^{q^*}(\mathbb{R}^n)} \ge 1$ and, by (3.11), we get $||v||_{L^{q^*}(\mathbb{R}^n)} = 1$ and $||\nabla v||_{L^q(\mathbb{R}^n)} = K(n,q)^{-q}$. Therefore, necessarily $v = u_0$, where u_0 was defined in the introduction.

Independently, for any $a, b \in \mathbb{R}$ and $s \ge 1$, one has

$$||a+b|^s - |a|^s - |b|^s| \le C(s)\Big(|a|^{s-1}|b| + |a||b|^{s-1}\Big).$$

So, choosing $s = q^*$, $a = v_\alpha - v$ and $b = v$ in the inequality above, we obtain

$$\begin{aligned}
&\int_{\Omega_\alpha} |v_\alpha - v|^{q^*} \, dv_{g_\alpha} \\
&\le \int_{\Omega_\alpha} |v_\alpha|^{q^*} \, dv_{g_\alpha} - \int_{\Omega_\alpha} |v|^{q^*} \, dv_{g_\alpha} + \\
&\qquad + C\left(\int_{\Omega_\alpha} |v_\alpha - v|^{q^*-1}|v| \, dv_{g_\alpha} + \int_{\Omega_\alpha} |v_\alpha - v||v|^{q^*-1} \, dv_{g_\alpha}\right) \\
&\le o(1) + C\left(\int_{\Omega_\alpha} |v_\alpha - v|^{q^*-1}|v| \, dv_{g_\alpha} + \int_{\Omega_\alpha} |v_\alpha - v||v|^{q^*-1} \, dv_{g_\alpha}\right) ,
\end{aligned}$$

since $||v||_{L^{q^*}(\mathbb{R}^n)} = 1$ and (3.11) clearly implies that both $||v_\alpha||_{L^{q^*}(\Omega_\alpha)}$ and $||v||_{L^{q^*}(\Omega_\alpha)}$ converge to 1 as $\alpha \to B_0(p, n, g)$.

The remaining right-hand side is easily seen to tend to 0 as $\alpha \to B_0(p, n, g)$:

$$\begin{aligned}
&\int_{\Omega_\alpha} |v_\alpha - v|^{q^*-1}|v| \, dv_{g_\alpha} \\
&= \int_{B_R} |v_\alpha - v|^{q^*-1}|v| \, dv_{g_\alpha} + \int_{\Omega_\alpha \setminus B_R} |v_\alpha - v|^{q^*-1}|v| \, dv_{g_\alpha} \\
&\le \int_{B_R} |v_\alpha - v|^{q^*-1}|v| \, dv_{g_\alpha} + \\
&\qquad + \left(\int_{\Omega_\alpha \setminus B_R} |v_\alpha - v|^{q^*} \, dv_{g_\alpha}\right)^{\frac{n(q-1)+q}{nq}} \left(\int_{\Omega_\alpha \setminus B_R} |v|^{q^*} \, dv_{g_\alpha}\right)^{\frac{1}{q^*}} \\
&\le \int_{B_R} |v_\alpha - v|^{q^*-1}|v| \, dv_{g_\alpha} + C\left(\int_{\mathbb{R}^n \setminus B_R} |v|^{q^*} \, dx\right)^{\frac{1}{q^*}} .
\end{aligned}$$

By taking $R > 0$ large, the last integral can be made arbitrarily small, so that the C^1_{loc}-convergence of (v_α) lead to

$$\int_{\Omega_\alpha} |v_\alpha - v|^{q^*-1}|v| \, dv_{g_\alpha} \to 0 \, .$$

Similarly, one easily checks that

$$\int_{\Omega_\alpha} |v_\alpha - v||v|^{q^*-1} \, dv_{g_\alpha} \to 0 \, .$$

Therefore,

$$\int_{\Omega_\alpha} |v_\alpha - v|^{q^*}\, dv_{g_\alpha} \to 0 \quad \text{as} \quad \alpha \to B_0(p,n,g)\ .$$

In order to establish the strong convergence of the gradients, note first that

$$\begin{aligned}
&\left| \int_{\Omega_\alpha} \langle \nabla_{g_\alpha}(v_\alpha - v), \nabla_{g_\alpha} v\rangle^{\frac{q}{2}}\, dv_{g_\alpha} \right| \\
&\le \int_{B_R} |\nabla_{g_\alpha}(v_\alpha - v)|^{\frac{q}{2}} |\nabla_{g_\alpha} v|^{\frac{q}{2}}\, dv_{g_\alpha} + \\
&\qquad + \left(\int_{\Omega_\alpha \setminus B_R} |\nabla_{g_\alpha}(v_\alpha - v)|^{q}\, dv_{g_\alpha} \right)^{\frac{1}{2}} \left(\int_{\Omega_\alpha \setminus B_R} |\nabla_{g_\alpha} v|^{q}\, dv_{g_\alpha} \right)^{\frac{1}{2}} \\
&\le \int_{B_R} |\nabla_{g_\alpha}(v_\alpha - v)|^{\frac{q}{2}} |\nabla_{g_\alpha} v|^{\frac{q}{2}}\, dv_{g_\alpha} + C \left(\int_{\Omega_\alpha \setminus B_R} |\nabla v|^{q}\, dx \right)^{\frac{1}{2}}
\end{aligned}$$

which implies

$$\int_{\Omega_\alpha} \langle \nabla_{g_\alpha}(v_\alpha - v), \nabla_{g_\alpha} v\rangle^{\frac{q}{2}}\, dv_{g_\alpha} \to 0. \tag{3.13}$$

From simple computations, we have

$$\begin{aligned}
&\int_{\Omega_\alpha} |\nabla_{g_\alpha}(v_\alpha - v)|^2\, dv_{g_\alpha} = \int_{\Omega_\alpha} |\nabla_{g_\alpha} v_\alpha|^2\, dv_{g_\alpha} - \int_{\Omega_\alpha} |\nabla_{g_\alpha} v|^2\, dv_{g_\alpha} \\
&-2 \int_{\Omega_\alpha} \langle \nabla_{g_\alpha}(v_\alpha - v), \nabla_{g_\alpha} v\rangle\, dv_{g_\alpha}
\end{aligned} \tag{3.14}$$

Then, combining (3.12), (3.14) and $(A+B)^k \le 2^{k-1}(A^k + B^k)$, for $A, B \in \mathbb{R}^+$, $k \ge 1$ we have

$$\begin{aligned}
&\int_{\Omega_\alpha} |\nabla_{g_\alpha}(v_\alpha - v)|^{q}\, dv_{g_\alpha} \\
&\le 2^{q-1} \Big(\int_{\Omega_\alpha} |\nabla_{g_\alpha} v_\alpha|^{q}\, dv_{g_\alpha} + \int_{\Omega_\alpha} \langle \nabla_{g_\alpha}(v_\alpha - v), \nabla_{g_\alpha} v\rangle^{\frac{q}{2}}\, dv_{g_\alpha} \Big) \to 0
\end{aligned}$$

from (3.13). Estimate 1 follows after a change of variable. □

As a consequence of Estimate 1, $(u_\alpha)_\alpha$ possesses only one concentration point. Indeed, this follows directly from the next estimate.

Estimate 2. *For any $\delta > 0$ small enough, we have*

$$\int_{M\setminus B_\delta(x_\alpha)} \left(|\nabla_g u_\alpha|^q + u_\alpha^{q^*}\right) \, dv_g \to 0 \quad \text{as} \quad \alpha \to B_0(p,n,g).$$

Proof. Let $0 < \delta < \delta_0$. Given $\varepsilon > 0$, by Estimate 1 and a change of variable, there exists a constant $\alpha_1 > 0$ such that, for any $\alpha \geq \alpha_1$,

$$\int_{B_\delta(x_\alpha)} |u_\alpha|^{q^*} \, dv_g \geq \int_{\mathbb{R}^n} |v|^{q^*} \, dx - \frac{\varepsilon}{4} = 1 - \frac{\varepsilon}{4}$$

and

$$\int_{B_\delta(x_\alpha)} |\nabla_g u_\alpha|^q \, dv_g \geq \int_{\mathbb{R}^n} |\nabla v|^q \, dx - \frac{\varepsilon}{4} = K(n,q)^{-q} - \frac{\varepsilon}{4}.$$

Using that $||u_\alpha||_{L^{q^*}(M)} = 1$ and (3.4), we find a constant $\alpha_0 \geq \alpha_1$ such that, for all $\alpha \geq \alpha_0$,

$$\int_{M\setminus B_\delta(x_\alpha)} \left(|\nabla_g u_\alpha|^q + u_\alpha^{q^*} \, dv_g\right) \leq \varepsilon.$$

□

Estimate 3. *For any $\delta > 0$ small enough,*

$$||u_\alpha||_{L^\infty(M\setminus B_\delta(x_\alpha))} \to 0 \quad \text{as} \quad \alpha \to B_0(p,n,g).$$

Proof. By Estimate 2, we have, for any ball $B_\rho(x) \subset M \setminus B_\delta(x_\alpha)$,

$$||u_\alpha^{q^*-q}||_{L^{\frac{q^*}{q^*-q}}(B_\rho(x))} = ||u_\alpha||^{q^*-q}_{L^{q^*}(B_\rho(x))} \leq ||u_\alpha||^{q^*-q}_{L^{q^*}(M\setminus B_\delta(x_\alpha))} \to 0\,.$$

So, by De Giorgi-Nash-Moser iterative scheme, we derive

$$||u_\alpha||_{L^\infty(B_{\rho/2}(x))} \leq C||u_\alpha||_{L^1(B_\rho(x))} \leq C||u_\alpha||_{L^1(M)}$$

for some constant $C > 0$ independent of α. So, the conclusion follows from the L^1-convergence and the fact that $u = 0$ on M. □

Let $x_0 \in M$ be the limit, up to a subsequence, of the sequence (x_α).

Estimate 4. *For any $\delta > 0$ small enough, we have*

$$\frac{\int_{M \setminus B_g(x_0,\delta)} u_\alpha^p \, dv_g}{\int_M u_\alpha^p \, dv_g} \to 0 \quad \text{as} \quad \alpha \to B_0(p,n,g)\,,$$

if either $p \in (q, q^*)$ and $n \geq 4$ or $p = 2$ and $n \geq 5$.

Proof. We recall that

$$- \Delta_q u_\alpha + \alpha K(n,q)^{-q} ||u_\alpha||_{L^p(M)}^{q-p} u_\alpha^{p-1} = \lambda_\alpha u_\alpha^{q^*-1}\,. \tag{3.15}$$

Since $B_0(p,n,g) > 0$, we have

$$-\Delta_q u_\alpha \leq \lambda_\alpha u_\alpha^{q^*-1},$$

so that Estimate 2 and the De Giorgi-Nash-Moser scheme provide a constant $C_1 > 0$, independent of α, such that

$$||u_\alpha||_{L^\infty(M \setminus B_g(x_0,\delta))} \leq C_1 ||u_\alpha||_{L^p(M)}\,.$$

Thus, from (3.15)

$$\begin{aligned} \int_{M \setminus B_g(x_0,\delta)} u_\alpha^p \, dv_g &\leq ||u_\alpha||_{L^\infty(M \setminus B_g(x_0,\delta))} \int_M u_\alpha^{p-1} \, dv_g \\ &\leq C_1 ||u_\alpha||_{L^p(M)} ||u_\alpha||_{L^p(M)}^{p-q} \int_M u_\alpha^{q^*-1} \, dv_g\,, \end{aligned}$$

so that

$$\frac{\int_{M \setminus B_g(x_0,\delta)} u_\alpha^p \, dv_g}{\int_M u_\alpha^p \, dv_g} \leq C_1 \frac{\int_M u_\alpha^{q^*-1} \, dv_g}{||u_\alpha||_{L^p(M)}^{q-1}}\,. \tag{3.16}$$

If $p \geq q^* - 1$, applying Holder's inequality in (3.16), we derive

$$\frac{\int_{M \setminus B_g(x_0,\delta)} u_\alpha^p \, dv_g}{\int_M u_\alpha^p \, dv_g} \leq C_2 ||u_\alpha||_{L^p(M)}^{q^*-q} \to 0.$$

Otherwise, if $p < q^* - 1$, using an interpolated Holder's inequality, we get

$$\begin{aligned}
\frac{\int_{M\setminus B_g(x_0,\delta)} u_\alpha^p \, dv_g}{\int_M u_\alpha^p \, dv_g} &\leq C_1 \frac{\int_M u_\alpha^{q^*-1} \, dv_g}{||u_\alpha||_{L^p(M)}} \\
&\leq C_1 \frac{||u_\alpha||_{L^p(M)}^{\theta(q^*-1)} ||u_\alpha||_{L^{q^*}(M)}^{(1-\theta)(q^*-1)}}{||u_\alpha||_{L^p(M)}} \\
&= C_1 \frac{||u_\alpha||_{L^p(M)}^{\theta(q^*-1)}}{||u_\alpha||_{L^p(M)}},
\end{aligned}$$

where

$$\frac{1}{q^*-1} = \frac{\theta}{p} + \frac{1-\theta}{q^*}.$$

Note that the condition $\theta(q^* - 1) > 1$ is equivalent to $p > \frac{n}{n-q}$. On the other hand, $q \geq \frac{n}{n-q}$ for all $n \geq 4$ and equality holds only when $n = 4$. This ends the proof of Estimate 4. □

4 Proof of Theorem 1.2

In this section, we end the proof of the existence of extremal functions to (1.2) by deriving a contradiction to the fact that the sequence (u_α) converges weakly to 0.

Due to the Estimate 4, we now easily arrive in a contradiction. In the sequel, some possibly different positive constants independent of α and δ will be denoted by c. Let $0 < \delta < \delta_0$ be a fixed number and consider a smooth cutoff function η such that $0 \leq \eta \leq 1$, $\eta = 1$ in $B_g(x_0, \delta/2)$ and $\eta = 0$ in $M \setminus B_g(x_0, \delta)$. Taking $\varphi_\alpha = \eta u_\alpha$ as a test function in the sharp inequality (1.2), using the identity

$$\int_M |\nabla_g(\eta u_\alpha)|^q \, dv_g = -\int_M \eta^q u_\alpha \Delta_q u_\alpha \, dv_g + \int_M |\nabla_g \eta|^q u_\alpha^q \, dv_g,$$

and the equation (2.2), one arrives at

$$\left(\int_M |\eta u_\alpha|^{q^*} \, dv_g\right)^{q/q^*} - \int_M \eta^q |u_\alpha|^{q^*} \, dv_g$$

$$\begin{aligned}
&\leq B_0(q,n,g)\int_M \eta^q u_\alpha^q\, dv_g \\
&\qquad + \int_M |\nabla_g \eta|^q u_\alpha^q\, dv_g - \alpha K(n,q)^{-q}||u_\alpha||_p^{q-p}\int_M \eta^q u_\alpha^p\, dv_g \\
&\leq c\int_{B_g(x_0,\delta)} u_\alpha^q\, dv_g + c_\delta \int_{M\setminus B_g(x_0,\delta)} u_\alpha^q\, dv_g + \\
&\qquad -\alpha K(n,q)^{-q}||u_\alpha||_p^{q-p}\int_{B_g(x_0,\delta/2)} u_\alpha^p\, dv_g.
\end{aligned}$$

On the other hand, by Hölder's inequality,

$$\begin{aligned}
\int_M \eta^q |u_\alpha|^{q^*}\, dv_g &\leq \left(\int_M |\eta u_\alpha|^{q^*}\, dv_g\right)^{q/q^*}\left(\int_M |u_\alpha|^{q^*}\, dv_g\right)^{(q^*-q)/q^*} \\
&\leq \left(\int_M |\eta u_\alpha|^{q^*}\, dv_g\right)^{q/q^*}
\end{aligned}$$

$$\int_{B_g(x_0,\delta)} u_\alpha^q\, dv_g \leq c\delta^{n(p-q)/q}\left(\int_{B_g(x_0,\delta)} u_\alpha^p\, dv_g\right)^{q/p}$$

and

$$\int_{M\setminus B_g(x_0,\delta)} u_\alpha^q\, dv_g \leq c\left(\int_{M\setminus B_g(x_0,\delta)} u_\alpha^p\, dv_g\right)^{q/p},$$

so that

$$\begin{aligned}
0 \leq c\delta^{n(p-q)/q}&\left(\int_{B_g(x_0,\delta)} u_\alpha^p\, dv_g\right)^{q/p} + c_\delta\left(\int_{M\setminus B_g(x_0,\delta)} u_\alpha^p\, dv_g\right)^{q/p} + \\
&-\alpha K(n,q)^{-q}||u_\alpha||_p^{q-p}\int_{B_g(x_0,\delta/2)} u_\alpha^p\, dv_g.
\end{aligned}$$

Dividing both sides of this inequality by $||u_\alpha||_p^q$, letting $\alpha \to B_0(p,n,g)$ and applying Estimate 4, we clearly achieved the desired contradiction.

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