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Fixed Point Theorems for Hypersequences and the Foundation of Generalized Differential Geometry: The Simplified Algebra

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Abstract. We lay the foundations of a generalized geometry and prove a fixed point theorem for hypersequences in this generalized context. Given a classical riemannian manifold M we prove that it can be discretely embedded in a generalized manifold M^* in such a way that the differential structure of the latter is a natural extension of the differential structure of the former. Furthermore, if $\Omega \subset \mathbb{R}^n$ then $\mathcal{D}'(\Omega)$ is discretely embedded in $\mathcal{G}(\Omega)$ and the elements of $C^\infty(\Omega)$ form a grid of equidistant points in $\mathcal{G}(\Omega)$. Ergo, classical solutions to differential equations are scarce and association in $\mathcal{G}(\Omega)$ is a topological and not an algebraic notion.

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1 Introduction

The Theory of Colombeau Generalized Functions is a nonlinear Theory of Generalized Functions which includes Schwartz' Theory of Linear Generalized Functions, i.e., Schwartz Distribution Theory. Colombeau's Theory is well documented by now. Excellent textbooks and articles exist that are pitstops to appreciate and understand the theory and the wide spectra of applications. We refer the reader to some of these excellent text, [1, 2, 14, 16, 17, 18, 19, 32, 37, 38, 39, 40, 43, 44, 45, 46, 48, 49, 50, 51, 53, 61, 55, 56, 57, 58, 59, 62, 63, 64, 65], to get the zest of the basics and relish advanced parts of this theory.

In the Colombeau environments, proving existence for differential equations involving products of distributions in their data has always leaned on classical results to guarantee existence. To achieve this, classical existence results are used, proceeding to prove moderateness and conclude existence of solutions in the environments of Colombeau Algebras. This can be highly nontrivial. One of the setbacks is that most tools used are not intrinsic to these environments. The development of Generalized Differential Calculus (see [3, 9]) envisaged the buildout of tools, intrinsic to the generalized environments, making it possible to pin less faith on the classical ones.

Let M be an $n-$dimensional manifold. The idea of linking a generalized objected $\widetilde{M}_c$ to M was first employed in [41] where a blueprint was given how to use these objects to solve important problems in General Relativity. Based on this pivotal idea, in [9], the notion of a generalized manifold was introduced. The definition is exactly the same as the classical one the difference being that local charts take values in open subsets of $\overline{\mathbb{R}}^n$ and differentiability is checked using the Generalized Differential Calculus. In [3], more details of Generalized Differential Calculus were worked out, showing that it extends and behaves very similar to classical Calculus and an example of a generalized manifold, different from $\widetilde{M}_c$, was also given (actually it is a subset of $\widetilde{M}_c$). As far as we know, other examples were not given yet and it remained unclear whether $\widetilde{M}_c$ was a generalized manifold and whether there existed other examples.

A pursue in another direction was the construction of an diffeomorphism invariant Colombeau algebra. This was early undertaken in [20, 34] and was settled in the definite in [40] well afore Generalized Differential Calculus was proposed. These are top-notch papers which show that the main obstruction to the construction of such an algebra has a topological nature: Colombeau algebras are ultrametric spaces which naturally mismatch with the classically used topologies. This translate into a highly non-trivial endeavor the creation of an algebra that can be attached to classical manifolds. The last example given in [40] shows that having such an algebra does not necessarily make things much easier when applying the theory to obtain existence of solutions of differential equations having products of distributions in their data. It is essential to observe that an algebra of generalized functions that can be attached to a manifold was also achieved in [60]. Amazingly enough, in this case, technicalities are not that involved.

In [9], all necessary machinery of Generalized Differential Calculus (such as the Inverse Mapping Theorem, The Implicit Function Theorem and others) were proved so that a consistent basis could be laid for a Generalized Differential Geometry. At first, definitions given and results obtained are exactly the same as the classical ones but extend the latter in a non-trivial way. Howbeit, much has yet to be accomplished before this Generalized Differential Geometry unveils its smoldering potential. The development of this new Calculus is based on key ideas developed over the years by all prominent researchers in the field but the decisive ideas are due to Kuzinger-Oberguggenberger ([38]) and Biagioni-Oberguggenberger-Scarpalézos ([14, 61]). The topology in use (see [4, 5, 6]) is a slight modification of the sharp topology introduced by Biagioni-Oberguggenberger and Scarpalézos (see [14, 61]), yet equivalent to it, is more natural and in sync with the algebraic structure (see for example [6, 7]) of the Colombeau algebras. An interesting fact is that, in the sharp topology, $\mathbb{R}^n$ embeds as a discrete subset of $\overline{\mathbb{R}}^n$ and yet the Generalized Differential Calculus is a near perfect extension of the Newtonian Differential Calculus (see the Embedding Theorem in [9]). In particular, Classical Space-Time becomes a grid of equidistant points in Generalized Space-Time, a possibility that was raised along time by many physicists and more recently also in [68]. The common distance betwixt grid points could well be glossed as the Planck length l_p or the Plank time t_p depending wether we measure space or time.

Can this discontinuity in classical space-time be perceived experimentally? Or at least, can one be convinced that we do have an issue in this direction? Since classical space-time is a grid of equidistant points it is impossible for classical sequences to converge in this new environment. In particular, it is no longer true that the sequence $(\frac{1}{n})_{n\in\mathbb{N}}$ converges in the ring of Colombeau generalized numbers. To remedy this discontinuity is where the notion of hypersequences steps into the picture. These histories of sequences fill in the spaces between the grid points. The hypersequence generated by the classical $(\frac{1}{n})_{n\in\mathbb{N}}$ is of the form $(\frac{1}{n})_{n\in\widetilde{\mathbb{N}}}$ and now does converge to zero in the generalized environment. From the point of view of someone living in generalized environment, classical convergence of a sequence (x_n) is equivalent to the existence of a $n_0 \in \widetilde{\mathbb{N}}$ such that $x_n - x_m \in V_1(0)$ if $n, m > n_0$ and classically we cannot distinguish anymore between x_n and x_m. So classically we only measure upto scale $\alpha = [\varepsilon \longrightarrow \varepsilon]$, which is the reason for calling α our *natural gauge*, the latter being first introduced in [10]. A similar problem occurs when proving existence of differential equations using classical tools to prove moderateness and existence in environments that are like chalk and cheese, in the topological sense, compared to the environment where these tools come from.

The paper is structured as follows. In the next section we recall the necessary machinery needed to understand the context and prove subsequent results. In the third section we prove a fixed point theorem for hypersequences, prove that association is a topological and not an algebraic concept and that $\mathcal{D}'(\Omega)$ is discretely embedded in $\mathcal{G}(\Omega)$ thus proving that classical functions are extremely rare. In the fourth section we prove that $\widetilde{M_c}$ is a generalized manifold and devote the last section to examples and the enumeration of some results in this new generalized geometry. This is the first of two papers. The second paper is in the context of the full algebra of Colombeau Generalized Functions (see [35]) thus completing the proposal of this new Generalized Differential Geometry as a roundabout route to define generalized functions on manifolds.

The notation $\overline{\mathbb{K}}$, for the ring of Colombeau generalized numbers, was introduced by Colombeau. However, developments overtime show that it is more reasonable to use the notation $\widetilde{\mathbb{K}}$ to denote this ring. Here we will still be using Colombeau's original notation to be consistent with the notation in [3, 4, 5, 9].

This paper was written while the second author held a pos-doc position at IME-USP, the University of São Paulo-Brazil. The dimension invariance theorem of section four and some of the examples of the last section are part of his Ph.D. thesis ([52]) written under the supervision of the first author.

2 Preliminaries

We shall mainly work over the field $\mathbb{R}$ of real numbers but all results also hold for $\mathbb{C}$. This is the reason why sometimes we use $\mathbb{K}$ to denote either of these fields. One could rightfully ask "why not consider the field $\mathbb{Q}$?" The answer is simples: The Colombeau Theory constructed using $\mathbb{R}$ is the same as the Colombeau Theory constructed using $\mathbb{Q}$ since real numbers can be seen as nets of rational numbers. The reason why we end up with a bigger structure, which is not a field, but is never the less very interesting, is because we mod out some, but not all, nets converging to zero. These surviving nets, converging to zero, are the infinitesimals which inhabit the halos of the elements of the newly formed environments.

Set $I =]0,1]$, $I_\eta =]0,\eta]$, for $\eta \in]0,1[$ and let α be the identity map

$\alpha : I \longrightarrow \mathbb{R}$, $\alpha(\varepsilon) = \epsilon$. We shall denote, once in a while, $\alpha_n = \alpha^n$ and call α the *standard* or *natural gauge*. Nearly all results in this paper can be proved for other gauges using the already existing results for these gauges (see [54, 66]).

A map (also called a net) $x : I \longrightarrow \mathbb{R}$ is moderate if $|x| < \alpha^r$, for some $r \in \mathbb{R}$, i.e. $|x(\varepsilon)| < \varepsilon^r, \forall \varepsilon \in I_\eta =]0, \eta], \exists\, \eta < 1$. Denote the set of moderate maps by $\mathcal{E}_M$ and by $\mathcal{I} = \{x : x \text{ is moderate and } \ |x| < \alpha^n, \forall n \in \mathbb{N}\}$. For $x \in \mathcal{E}_M$, denote by $V(x) = Sup\{r \in \mathbb{R} : |x| < \alpha^r\}$ and set $\|x\| = e^{-V(x)}$. Then $\mathcal{I}$ is a radical ideal of the ring $\mathcal{E}_M$ and setting $\overline{\mathbb{R}} := \frac{\mathcal{E}_M}{\mathcal{I}}$, we have that $(\overline{\mathbb{R}}, \|\ \|)$ is an ultrametric partially ordered topological ring whose group of units, $Inv(\overline{\mathbb{R}})$, is open and dense (see [10]). The latter property is essential in developing the Generalized Differential Calculus ([3, 9]). This topological ring, $(\overline{\mathbb{R}}, \|\ \|)$, is called the ring of Colombeau Generalized (real) numbers. A generalized number $x \in \overline{\mathbb{R}}$ is a unit if and only if $|x| > \alpha^r$ for some $r \in \mathbb{R}$ and it is a non-unit if and only if there exist a nontrivial idempotent $e \in \mathcal{B}(\overline{\mathbb{K}})$ such that $e \cdot x = 0$ (see [10, 12]). In particular, a generalized number is either a unit or a zero divisor. The ring $\overline{\mathbb{R}}$, contains $\mathbb{R}$ as a discrete subfield. Actually, $\mathbb{R}$ is a grid of equidistant points in $\overline{\mathbb{R}}$. The latter is a partially ordered ring whose maximal ideals and idempotents have been completely determined (see [9, 10, 12, 64]). The partial order is not intrinsic but stems from the order of $\mathbb{R}$. This is maybe the only definition that is not, yet, intrinsic. Distance emerges from this order and that is why it is important to understand order. In the references we just mentioned, one finds the following facts: the Jacobson Radical of $\overline{\mathbb{R}}$ is trivial, its ideals are convex, its Krull dimension is infinite and it has a minimal prime which is also a maximal ideal. Its Boolean algebra, $\mathcal{B}(\overline{\mathbb{R}})$, consists of $\{0, 1\}$ and positive elements each of which is a characteristic functions of a subset $S \subset I$, such that $0 \in \overline{S} \cap \overline{S^c}$, where the last two bars stand for topological closure in $\mathbb{R}$. The set of these subsets is denoted by $\mathcal{S}$ and was defined in [10]. Ultrafilters of $\mathcal{S}$ parametrize prime and maximal ideals of $\overline{\mathbb{K}}$. It also holds that $\mathcal{B}(\overline{\mathbb{R}}) = \mathcal{B}(\overline{\mathbb{C}})$ (see [12]). In particular, the Heaviside function $H \notin \mathcal{B}(\overline{\mathbb{R}})$, i.e., $H^2 \neq H$ (see [25]).

Biagioni-Oberguggenberger and Scarpalézos were the first to suggest the topology, defined above, for $\overline{\mathbb{R}}$. It came to be known as the *sharp topology* turning $\overline{\mathbb{R}}$ into a complete ultrametric algebra and hence, its topology is generated by balls. In [4, 5] it was shown that this topology was also generated by the sets $V_r[x] = \{y \in \overline{\mathbb{R}} : |y-x| < \alpha^r\}$, balls with generalized numbers as radii , compatible with the ring structure. It is easily seen that $B_{2r}(0) \subset V_r[0] \subset B_{r/2}(0)$ if $r > 0$.

Let $\Omega \subset \mathbb{R}^n$ be an open subset with an exhaustion by relatively compact subset $\Omega_m \subset \Omega$. Consider nets $p = (p_\varepsilon), p_\varepsilon \in \Omega_{m(p)}\ \forall \varepsilon \in I_\eta$, $m(p) \in \mathbb{N}$, such that the net $(\|p_\varepsilon\|) \in \mathcal{E}_M$. Factoring out nets p for which also $(\|p_\varepsilon\|) \in \mathcal{I}$, give rise to a subset of $\overline{\mathbb{R}}^n$ which is denote by $\widetilde{\Omega}_c$ (see [37, 38]). The notation $\widetilde{\Omega}$ is used if one does not require the existence of $m(p)$. The algebra of Colombeau generalized functions $\mathcal{G}(\Omega)$ (see [2, 16, 17, 37] for the original definition), defined on the open subset $\Omega \subset \mathbb{R}^n$, can be viewed as $\mathcal{C}^\infty-$functions defined on $\widetilde{\Omega}_c \subset \overline{\mathbb{R}}^n$ and taking values in $\overline{\mathbb{R}}$ (see [37, 38, 3, 9]). In [9] (see also [3]) the foundation of Colombeau Generalized Calculus is laid and shown that $\mathcal{G}(\Omega)$ can be embedded into $C^\infty(\widetilde{\Omega}_c, \overline{\mathbb{R}})$. In particular, Schwartz space of linear distributions, $\mathcal{D}'(\Omega)$ can be seen as infinitely differentiable functions where differentiability is defined a la Newton. So we have come full circle from seeing elements of $\mathcal{D}'(\Omega)$ as linear maps, and hence not undergoing variation, to seeing them as functions undergoing variation (note that, classically, derivation in $\mathcal{D}'(\Omega)$ is defined without the use of variation). An interesting fact is that, in the presence of moderateness, negligibility only has to be checked at level 0. This is mentioned and proved in several references. See, for example, [40, paragraphs after I.Theorem 7.13]. Generalized Differential Calculus allows to give an easy proof of this fact. In fact, in the presence of moderateness negligibility at level 0 is a statement about point values: If $\hat{f}(\varepsilon, x)$ is moderate, then it defines an element $f \in C^\infty(\widetilde{\Omega}_c, \overline{\mathbb{R}})$. Given $x \in \widetilde{\Omega}_c$ there exists a compact subset $K \subset \Omega$ containing a representative of x. Moderateness at level 0 implies that $\|(f)_{|K}\|_\infty = 0$ (note that this is exactly the uniformity on K). It follows that $f(x) = 0$, proving that $f = 0$ in $\widetilde{\Omega}_c$. Generalized Differential Calculus gives that $\partial^\beta f = 0, \forall\ \beta \in \mathbb{N}^n$. Since the Embedding Theorem [9, Theorem 4.1] tells us that derivations commutes with the embedding, it follows that $f = 0$ in $\mathcal{G}(\Omega)$. So there is no need to check other levels. The same proof holds for the full algebra and should also work for the invariant algebra once we have at hand a Generalized Differential Calculus for the latter. Note however that one must have negligibility at level 0 and not just point values of elements of Ω being zero. To see why, consider $f = x\delta \in \mathcal{G}(\mathbb{R})$, where $\delta = [(\rho_\varepsilon)]$, ρ a mollifier. We have that $f(x) = 0, \forall x \in \mathbb{R}$. Also, for $x_0 \in \mathbb{R}$, $f(x_0\alpha) = x_0 \cdot \rho(x_0)$, for $\varphi \in \mathcal{D}(\Omega)$ we have that $\int\limits_{\mathbb{R}} f(x)\varphi(x)dx = [(\int\limits_{\mathbb{R}}(x\varphi)(x)\rho_\varepsilon)]$ and hence, since (ρ_ε) is a delta-net, $\int\limits_{\mathbb{R}} f(x)\varphi(x)dx = (\int\limits_{\mathbb{R}} x\varphi(x)\rho_\varepsilon(x)dx) \approx (x\varphi)(0) = 0$. This proves that $f \approx 0$ but $f \neq 0$. This example also shows an interesting phenomena: $f(0) = 0$ and thus for $x \in V_r(0)$, a small enough sharp neighborhood of 0, $f(x) \in V_1(0)$. For classical mathematics (and hence measurements) $f(x) = 0$ and thus seemingly does not interfere with physical reality. But for histories of the form $x = x_0\alpha, x_0 \in \mathbb{R}$ we have that $f(x) = x_0 \cdot \rho(x_0) \in \mathbb{R}$ and thus interfere with physical reality. These "waves" of appearing and disappearing from physical reality are the source of the turbulence effects we see in physical reality. And it can be worse. Consider $g(x) = f^k$, $k \in \mathbb{N}$. Then $g(0) = 0, g(\alpha) = (\rho(1))^k$. So if $\rho(1) > 1$ and k is large, then these "waves" coming from $V_r(0)$ which we cannot measure, can effect in a non-trivial way physical reality. Note also that $\delta(0) = \alpha^{-1} \cdot \rho(0)$ is an infinity we can not measure but it is cancelled out on histories, $x_0\alpha$ near 0, giving us a real number, $f(x_0\alpha) = x_0 \cdot \rho(x_0) \in \mathbb{R}$, that we can measure. Even though the history $x_0\alpha$ is near 0, the position in physical reality where we observe the effect can be faraway from 0 (in this case at $x_0 \in \mathbb{R}$) and the result of the measurement $x_0\rho(x_0)$ becomes small as x_0 goes to infinity. So turbulence should be the interaction of elements of $B_1(0)$ and infinities, i.e., elements of norm greater then 1, producing a measurable but not predictable effect on physical reality. The non-predictability stems from the fact that spheres in generalized environments are clopen sets and classical Euclidean-Space is a grid of equidistant points. Jumps from one sphere to another sphere occur by multiplying with the α^r, $r \in \mathbb{R}$, which form a discrete chain of quantas.

The construction carried out above with $\Omega \subset \mathbb{R}^n$ can also be carried out with any subset $X \subset \mathbb{R}^n$. In fact, consider X with the induced topology, consider an exhaustion (X_n) of X by relatively compact subsets and proceed as before. The set obtained in $\overline{\mathbb{R}}^n$ will be denoted by $\widetilde{X}_c$. We embed X into $\widetilde{X}_c$ using constant nets $p = (p_\varepsilon), p_\varepsilon = x \in X, \varepsilon \in I_\eta$. It is clear that, in the sharp topology, X is a discrete grid of equidistant points contained in $\widetilde{X}_c$. The remarkable thing is that, in case of a submanifold M of $\mathbb{R}^n$, Generalized Differential Calculus on $\widetilde{M}_c$ will be a generalization of the Classical Differential Calculus on M, although M is discretely embedded in $\widetilde{M}_c$.

Let q be any norm on $\mathbb{R}^n$. Extending it in the obvious way to $\overline{\mathbb{R}}^n$, we define for $x = (x_1, \cdots, x_n) \in \overline{\mathbb{R}}^n$, $\|x\|_q = q(x) \in \overline{\mathbb{R}}$ and ${}_q\|x\| \in \mathbb{R}$ to be the norm of $q(x)$ as an element of $\overline{\mathbb{R}}$. If $q(x) = \sqrt{x_1^2 + \cdots + x_n^2}$ then we write $\|x\|_q = \|x\|_2$ (see [3, 12]). Since all norms on $\mathbb{R}^n$ are equivalent, it is easily seen that ${}_q\|x\|$ does not depend on the norm q and thus we shall write it as $\|x\|$.

The positive cone, $\overline{\mathbb{R}}_+$, of $\overline{\mathbb{R}}$ is not an open subset. In fact, let e be an idempotent and set $x_n = e - (1-e) \cdot \alpha^n$. Then $|x_n| = e + (1-e) \cdot \alpha^n$. We clearly have that x_n is not in the positive cone but $x_n \longrightarrow e$. However if we let $Inv(\overline{\mathbb{R}})_+ = Inv(\overline{\mathbb{R}}) \cap \overline{\mathbb{R}}_+$ then we have:

Lemma 2.1. *Let* $Inv(\overline{\mathbb{R}})_+ = Inv(\overline{\mathbb{R}}) \cap \overline{\mathbb{R}}_+$.

1. $Inv(\overline{\mathbb{R}})_+$ *is an open subgroup of* $\overline{\mathbb{R}}$.

2. *Let* $t \in \widetilde{[0,1]}$ *and* $x, y \in Inv(\overline{\mathbb{R}})_+$. *Then* $tx + (1-t)y \in Inv(\overline{\mathbb{R}})_+$.

Proof. The fact that $Inv(\overline{\mathbb{R}})_+$ is a subgroup of $\overline{\mathbb{R}}$ is clear. So let $x \in Inv(\overline{\mathbb{R}}) \cap \overline{\mathbb{R}}_+$. Since x is invertible, there exists α^r such that $x > \alpha^r$. If $y \in V_r(x)$ then $|y - x| < \alpha^r$ and thus $0 < x - \alpha^r < y$. On the other hand, since $Inv(\overline{\mathbb{R}})$ is open and $\{\alpha_t \ : \ t \in \mathbb{R}\}$ form a totally ordered set, we may take r such that $V_r(x) \subset Inv(\overline{\mathbb{R}})$.

To prove the second part, take $m > 0$ such that $\alpha^m < min\{x, y\}$. Then if follows that $tx + (1-t)y \geq t\alpha^m + (1-t)\alpha^m = \alpha^m$ and thus, $tx + (1-t)y \in Inv(\overline{\mathbb{R}})$. $\square$

The negative cone $Inv(\overline{\mathbb{R}})_+ = Inv(\overline{\mathbb{R}}) \cap \overline{\mathbb{R}}_-$ is also an open subset of $\overline{\mathbb{R}}$. It follows that $Inv(\overline{\mathbb{R}})$ has two connected component which are both open subsets of $\overline{\mathbb{R}}$. Moreover, $Inv(\overline{\mathbb{R}})_+ \cap Inv(\overline{\mathbb{R}})_- = \emptyset$ and 0 is in the topological closure of both $Inv(\overline{\mathbb{R}})_+$ and $Inv(\overline{\mathbb{R}})_-$. Both are closed under addition and interleaven (see the end of this section or [46]).

The lemma shows that there can not exist a continuous curve whose initial value is negative, its final value is positive and at all other instants its values are comparable with 0. This is exactly what is needed to define the notion of orientation on generalized manifolds.

The idea to consider nets of point in $\mathbb{R}^n$ was introduced in [32, 37, 38]. This was used in [3] to define the notion of membranes and histories in $\overline{\mathbb{R}}^n$. Subsequently, in [46], the notions of internal and strong internal sets (internal sets are generalization of membranes) were introduced, inspired also by concepts of nonstandard analysis. In this same paper, ([46]), very strong and relevant properties involving these notions were proved. For example, it is proved that strongly internal sets are open subsets of $\overline{\mathbb{R}}^n$ whereas internal sets are closed subsets of the same space. We will be using freely the results contained in these references.

Given a net (A_ε) in $\mathbb{R}^n$ we shall write it also as A_α, being $\alpha = [\varepsilon \to \varepsilon]$ our natural or standard gauge. For an idempotent $e \in \mathcal{B}(\overline{\mathbb{K}})$ we define $e\alpha = [\varepsilon \to e(\varepsilon)\varepsilon]$, meaning that when $e(\varepsilon) = 0$ this index will be omitted. We also write eA_α for the net $A_{e\alpha}$, ∂A_α for the net (∂A_ε) (the boundary) and $int(U)$ for the set of interior point of a subset $U \subset \mathbb{R}^n$. Given a net $A_\alpha \subset \mathbb{R}^n$ of subset of $\mathbb{R}^n$, we denote the membrane, or internal set, it originates by $[A_\alpha]$ and the strongly internal set it originates by $\langle A_\alpha \rangle$. As mentioned above, internal sets are closed in the sharp topology while strongly internal sets are open in the sharp topology. We say that (A_α) is *regular* if there exists $k \in \mathbb{N}$ such that at each boundary point A_ε we can inscribe balls of radius ε^k tangent to ∂A_ε and contained in $int(A_\varepsilon)$ and another ball of the same radius tangent at the same point but contained in $(int(A_\varepsilon))^c$. As a result, the volume of a regular net is a unit since $vol(A_\alpha) \geq vol(V_k(0)) = \pi\alpha^{2k} \in Inv(\overline{\mathbb{K}})$ (see [3]). For example, this is the case if the boundaries are compact hyper surfaces whose encompassing volumes are not shrinking too fast.

Lemma 2.2. *Let* (A_α) *be a net in* $\mathbb{R}^n$ *and* $U = \langle A_\alpha \rangle$ *its strong internal set. Then* $\partial\langle A_\alpha\rangle = [\partial A_\alpha]$.

Proof. For $z \in \partial\langle A_\alpha\rangle$, there exist sequences $(z_n) \subset \langle A_\alpha\rangle$ and $(p_n) \subset (\langle A_\alpha\rangle)^c$ both converging to z in $\overline{\mathbb{K}}^n$. Let $w = dist(z, [\partial A_\alpha])$ be the distance of z to the membrane $[\partial A_\alpha]$. If $w = 0$ then $z \in [\partial A_\alpha]$. If not, then there exist $e \in \mathcal{B}(\overline{\mathbb{K}})$ and $t >> k$ such that $e \cdot w > e \cdot \alpha^t$. In particular, $dist(z_\varepsilon, \partial A_\varepsilon) > \varepsilon^t \neq 0$ if $e(\varepsilon) = 1$. Hence, for ε such that $e(\varepsilon) = 1$, we have that either $z_\varepsilon \in int(A_\varepsilon)$ or $z_\varepsilon \in int((A_\varepsilon)^c)$. Consequently, we may write $e = e_1 + e_2$, a sum of orthogonal idempotents, such that $e_1 \cdot z \in e_1(int(A_\alpha))$ and $e_2 \cdot z \in e_2(int(A_\alpha)^c)$. On the other hand, since $p_n \to z$, there exists n_0 such that if $n > n_0$ we have that $dist(e_1 \cdot p_n, e_1 \cdot [(\partial A_\alpha)]) > e_1 \cdot \alpha^{7t}$. But since $e_1 \cdot z \in e_1 \cdot \partial\langle A_\alpha\rangle$ this implies that $e_1 \cdot p_n \in \langle e_1 \cdot A_\alpha\rangle$, a contradiction, unless $e_1 = 0$. If so, then revers the roles of z_n and p_n obtaining another contradiction. Thus we have that $w = 0$ and the result is proved. □

In case A_α consists of intervals $J_\varepsilon =]a_\varepsilon, b_\varepsilon[\subset \mathbb{R}$, uniformly bounded, we have that $\partial\langle A_\alpha\rangle = Interleaven\{a, b\}$, where $a = [(a_\varepsilon)]$ and $b = [(b_\varepsilon)]$. The notion of interleaven is defined in [46] which is as follows: the interleaved of a set $X \subset \overline{\mathbb{R}}^n$ is the set of all finite sums $\sum_{i=1}^{m} e_i \cdot x_i$, with $x_i \in X$ and $\{e_1, \cdots, e_m\}$ a complete set of mutually orthogonal idempotents in $\overline{\mathbb{R}}$, i.e., $e_i \cdot e_j = 0$ if $i \neq j$ and $\sum_{i=1}^{m} e_i = 1$. We extend the definition of interleaving allowing that the number of idempotents involved in the sum of the interleaving is countable and not necessarily finite. Interleavings can also be done with hypersequences and elements of $C^\infty(\Omega)$ (see the next section).

The expressions *entanglement* or *intertwine* express the same idea, since several points are connected in the same net that cannot be undone since it is a point in generalized Euclidean-Space. This is actually the way that new points in generalized Euclidean-Space are created. Observing a point x corresponds to the creation of the point $x \cdot \alpha^n$, $n \in \widetilde{\mathbb{N}} \cup \{\infty\}$, where α^n is defined in the second paragraph of the next section. Hence, observing is seeing a part of the interleaving x. An observation does not change the part of the point that it observes if and only if α^n is an idempotent.

Consider again $f(x) = x\delta$. The $halo(0) = B_1(0)$, or the halo of any other point, contains information of any subset of $\mathbb{R}^n$ via the histories $\mathbb{R}^n \cdot \alpha^r$, $r > 0$ or, in general $\mathbb{R}^n \cdot y$, $y \in B_1(0)$ and their interleavings. In the same way, information of any subset of $\mathbb{R}^n$ is contained in the complement of $\overline{B}_1(0)$ via histories $\mathbb{R}^n \cdot \alpha^r$, $r < 0$. This can also be seen using the homeomorphisms of $\overline{\mathbb{K}}^n$ like those whose existence are proved in [10, Theorem 3.3, Theorem 3.4]. The same holds for subsets of $\overline{\mathbb{K}}$. We can *intertwine the history* of points $x_0 \neq x_1 \in \mathbb{R}$ using the notion of interleaving: $x = (x_0 e_1 + x_1 e_2)\alpha, e_1 + e_2 = 1$, the latter being idempotents. As a result, the measurement, $f(x)$, is also an *intertwine*: $f(x) = x_0\rho(x_0)e_1 + x_1\rho(x_1)e_2$ which is what is measured in physical reality. We proceed to give an interpretation of this measurement in probabilistic terms.

Consider an intertwine $\sum_i e_i \cdot x_i$. For each idempotent $e \in \mathcal{B}(\overline{\mathbb{R}})$ involved in the sum, there exists a set $S_e \in \mathcal{S}$, (see [10, Definition 4.1]), such that $e = \chi_{S_e}$ is the characteristic function of S_e (see [10, 12]). If there exists $\eta_0 > 0$ such that $]0, \eta_0] \cap S_e$ is measurable, define $\mu(e) := \lim_{\eta \to 0} \left(\frac{1}{\eta} \int_0^\eta \chi_{S_e} d\mu \right)$. Since, in an interleaving, the idempotents involved form a complete set of mutually orthogonal idempotents, it follows that $\sum_i \mu(e_i) = 1$. Hence the $\mu(e_i)$'s can be seen as probabilities and, being countable in number, there exists i_0 such that $\mu(e_{i_0}) > 0$. The interpretation is that whenever $\mu(e_i) > 0$ the measurement at the corresponding point x_i is more likely to be obtained because $f(x_i)$ will appear with the same probability in the resulting measurement (see the example in the previous paragraph). We say that the entanglement is a *complete intertwine* if $\mu(e_i) > 0,\ \forall\, i$ and is a *perfect intertwine* if $\mu(e_i) = \mu(e_j),\ \forall\, i,\ j$. In the latter case the number of idempotents involved must be finite. Since measurements involve the same generalized function f, the whole history of measurement is determent and cannot be changed unless the entanglement is undone. One can let the x_i's in an interleaving take values in disjoint subsets $X_i \subset \mathbb{R}^n$ letting the probabilities relate to the sets X_i which, for example, can be regions in physical reality. In this case, an interleaving can be seen as a function from $x : I \longrightarrow \bigcup\limits_i X_i$. For example, rolling a dice produces an intertwine $x = \sum\limits_{i=1}^{6} e_i \cdot i$, with $X_i = \{i\}$, being perfect only if the dice is honest.

3 A Generalized Fixed Point Theorem

In this section we prove a fixed point theorem which is one more piece of the Generalized Differential Calculus whose development started in [3, 9]. All the features of this calculus have been extended in [30] to the context of Robinson-Colombeau rings of generalized numbers which includes the fields $\overline{\mathbb{K}}/\mathcal{M}$, where $\mathcal{M} \lhd \overline{\mathbb{K}}$ is maximal (see [10, 64]).

In the sequel, ideas contained in [29, 42] will be used. Let $\widetilde{\mathbb{N}} \subset \overline{\mathbb{R}}$ be the set of generalized numbers with a representative in $\mathbb{N}^I$ and $\widetilde{\mathbb{N} \cup \{\infty\}}$ elements of the form $e \cdot n + (1-e) \cdot \infty$, $e^2 = e$, $n \in \widetilde{\mathbb{N}}$. Another way to view these elements is to consider $\mathcal{E}_M(\mathbb{N})$ and factor by the ideal $\mathcal{I}$ defined in the previous section. The elements of $\widetilde{\mathbb{N}}$ are called *hyper natural numbers.* In the same way one can define the ring of *hyper integers* $\widetilde{\mathbb{Z}}$. We extend the notation $\alpha^n, n \in \mathbb{N}$, introduced in [10], to the case when $n \in \widetilde{\mathbb{N} \cup \{\infty\}}$: if $n = [(n_\varepsilon)]$ then $\alpha^n = [(\varepsilon^{n_\varepsilon})]$. We can extend this definition to $n \in \widetilde{\mathbb{Z}}$ as long as the set $\{n_\varepsilon \ :\ \varepsilon < 0\}$ is bounded, being obvious the reason to require this condition. With this notation, idempotents are also of the form α^n, where, in this case, n consists of a string of 0's and ∞'s.

A *hypersequence* is a map $x : \widetilde{\mathbb{N}} \longrightarrow \mathcal{G}(\Omega)$ and is denoted by (x_n). If $x(\widetilde{N}) \subset \overline{\mathbb{K}}$, we say that (x_n) *converges* to $L \in \overline{\mathbb{K}}$ if given $r > 0$, there exists $n_0 \in \widetilde{\mathbb{N}}$ such that if $n > n_0$ then $L - x_n \in V_r(0)$. Since we are in a Hausdorff space, limits are unique whenever they exist. Such a hypersequence (x_n) is a *Cauchy hypersequence* if given $r > 0$ there exists $n_0 \in \widetilde{\mathbb{N}}$ such that if $m, n > n_0$ then $x_m - x_n \in V_r(0)$. $\overline{\mathbb{K}}$ being a complete metric space, we have:

Lemma 3.1. *Let $x = (x_n)$ be a hypersequence. Then x is a convergent sequence if and only if it is a Cauchy sequence if and only if for each $r \in \mathbb{R}_+^*$ there exists $n_0 \in \widetilde{\mathbb{N}}$ such that $n > m > n_0$ implies that $x_n - x_m \in V_r(0)$.*

The sequence $x = (\frac{1}{n})_{n \in \mathbb{N}}$ does not converge in $\overline{\mathbb{K}}$ because its elements form a grid of equidistant points. However, the hypersequence $x = (\frac{1}{n})_{n \in \widetilde{\mathbb{N}}}$, converges to $0 \in \overline{\mathbb{K}}$. In fact, given $r > 0$, take $\alpha^{-r} \leq n_0 = [(\lfloor \varepsilon^{-r} + 1.5 \rfloor)] \leq \alpha^{-(r+1)}$. If $n > n_0$, we have that $\frac{1}{n} \in V_r(0)$. We also know that $\sum\limits_{n \in \mathbb{N}} \frac{1}{n}$ diverges. However if one sums over a countabel subset of $\widetilde{\mathbb{N}}$ containing a finite number of elements of norm one then the sum converges, since, in this setting, a series $\sum\limits_{n \in \mathbb{N}} a_n$ converges if and only if $a_n \longrightarrow 0$.

Let $r \in \,]0,1[$ be fixed and suppose that we want $r^n \in V_t(0)$, where $n = [(n_\varepsilon)]$. For this to occur one must have $r^{n_\varepsilon} < \varepsilon^t$. From this it follows that $n_\varepsilon > (\frac{-t}{|\ln(r)|}) \cdot \ln(\varepsilon)$. Hence we may take $n_\varepsilon = 2 \cdot \left\lfloor \frac{-t}{|\ln(r)|} \cdot \ln(\varepsilon)) \right\rfloor$. Note that $n < \alpha^{-1}$ and hence is moderate (actually its norm equals 1). Clearly, for any $m > n$ we have $r^m \in V_r(0)$. This proves that the hypersequence (r^n) converges to 0.

Any sequence $\hat{x}_0 : \mathbb{N} \longrightarrow \mathbb{K}$ defines a map $\hat{x} : \widetilde{\mathbb{N}} \longrightarrow \mathbb{K}^I$ in the obvious way: $\hat{x}_n = (\varepsilon \longrightarrow \hat{x}_{0n_\varepsilon})$. If $\hat{x}$ is moderate then it defines a hypersequence x. If there exists $n_0 \in \widetilde{\mathbb{N}}$ and $L \in \overline{\mathbb{K}}_{as} = \mathbb{K} + \overline{\mathbb{K}}_0$ (see [10]) such that $n > n_0$ implies $x(n) - x(n_0) \in V_1(0)$, then the sequence $(\hat{x}_{0n})_{n \in \mathbb{N}}$ converges to $L_0 \in \mathbb{K}$, with $L_0 \approx L$, in the classical sense, and the whole history of measuring this convergence is contained in the sentence "$n > n_0$ implies $x(n) - x(n_0) \in V_1(0)$". Conversely, if $(x_{0n})_{n \in \mathbb{N}}$ converges to the real number L_0 and for each ε one choses n_ε minimal such that $n > n_\varepsilon$ implies $|\hat{x}_n - L| < \varepsilon$ then $n = [(n_\varepsilon)]$ must be an element of $\widetilde{\mathbb{N}}$ if x were to converge. This shows that a sequence of measurements can have precision α^{k_0}, for some k_0, but not for all $k \in \mathbb{N}$. It might be possible to infer from this if classical space-time is discontinuous (unless we declare it continuous and stop measuring beyond $V_1(0)$). Since it is not at all clear that the convergence of $\hat{x}_0$ implies de convergence of x, one might question the definition of the notion of integral given in [3, 9, 37]. We shall prove that, at least in this case, limits do exists and are equal. It is important that this is true if we want that classical theories also hold in the generalized environment.

Let $f \in \mathcal{G}(\Omega)$ and let $[K_\varepsilon]$ be a membrane (see [3]). To keep things simple we suppose that $K_\varepsilon = K$ is compact for all ε and contained in an open and relatively compact subset $\Omega_m \subset \Omega$. Note however that the result obtained also holds for a general membrane. Given a fixed $dV \in \{\frac{1}{n},\ \alpha^r\ :\ r \in I, n \in \widetilde{N}\}$, consider the partition P of norm dV of K contained in Ω_m, i.e., for each $\varepsilon \in I$ and $dV < \frac{2}{\mu(\Omega_m)}$, P_ε has norm $dV_\varepsilon \in \{\frac{1}{n_\varepsilon},\ \varepsilon^r\}$, where $\mu(\Omega_m)$ denotes the Lebesgue measure of Ω_m. Since K is compact, there exists $x_0, x_1 \in \widetilde{K}$ such that $m = f(x_0) \leq f(x) \leq M = f(x_1), \forall x \in \widetilde{K}$. Let $s(f, dV), S(f, dV)$ and $S(f, dV, *)$ be, respectively the lower and upper Riemann sum and any other starred Riemann sum with this dV as the norm of the partition. Then $s(f, n) \leq S(f, dV, *) \leq S(f, dV)$ and $|S(f, dV) - s(f, dV)| \leq \alpha_{-N} \cdot \mu(\Omega_m) \cdot dV$, where $N > 0$ is such that $\|(\nabla f)_{|K}\|_\infty \leq \alpha^{-N}$. Choosing $dV < \alpha^{N+r+1}$ (this is possible because the hypersequence $(\frac{1}{n})$ converges to zero), we have that $|S(f, dV) - s(f, dV)| \in V_{r+1}(0)$ and thus $\{s(f, dV), S(f, dV), S(f, dV, *)\} \subset V_r(\int\limits_K f(x)dx)$, where $\int\limits_K f(x)dx)$ is as defined in [37, 3, 9]. From this it follows that the classical and generalized limits are the same and if $r = 1$ then we already have that $s(f, dV)$, and $S(f, dV), S(f, dV, *)$ are all associated to $\int\limits_K f(x)dx$ and thus, in the classical sense, they are all equal. In case we take $dV = \frac{1}{n}$ we have a hypersequence and we just proved its convergence. For each

$s(f,dV)$, $S(f,dV), S(f,dV,*)$ and $\int\limits_K f(x)dx)$ there exists an element $c \in \widetilde{K}$, for each one of them there exists one such element, so that it is equal to $f(c)\cdot\mu(K)$. If $\varphi \in \mathcal{D}(\Omega)$ is non-negative, then $\int\limits_K f\varphi(x)dx) = f(c)\cdot \int\limits_K \varphi(x)dx$, for some $c \in \widetilde{K}$. In particular, if $\varphi \in A_0(\Omega)$ (see [2]) is positive, then $\int\limits_K f\varphi(x)dx) = f(c)$. This is useful when looking at the notion of association in $\mathcal{G}(\Omega)$.

Fix $k \in \mathbb{R}^*_+$ and, in $\mathcal{E}_M(\mathbb{N})$, define $\hat{x}(n)(\varepsilon) = \chi_{S(n)}\cdot\varepsilon^k$, where $\chi_{S(n)}$ is the characteristic function of the set $S(n) = \{(n_\varepsilon)^{-1} : \varepsilon \in I\}$. If $n \in \mathbb{N}$ then $S(n) = \{n^{-1}\}$ and $\hat{x}(n)$ would be non-zero only when $\varepsilon = \frac{1}{n}$, having value $\frac{1}{n^k}$. If these were measurements or observations, and since we cannot make infinitely many measurements or observations, the result would be points of the sequence $(\frac{1}{n^k})_{n\in\mathbb{N}}$ and hence converge to 0. However, if we look at the hypersequence x then $x(n) = e_n \cdot \alpha^k$, where $e_n = [\chi_{S(n)}] \in \mathcal{B}(\overline{\mathbb{K}})$. Hence $x(n) = 0, \forall\ n \in \mathbb{N}$ and $\|x(n) - x(m)\| = e^{-k}\cdot\|e_n - e_m\| \in \{0, e^{-k}\}$. Consequently, this is not a converging hypersequence.

In the introduction of [33] there are two examples that are worth rewriting into this context. The first is that of a point mass of weight 1 at a point x_0 on the real axis. Consider the membrane $M = V_1(x_0)$. Then the corresponding functionals can be written as

$$L(\varphi) = \frac{1}{vol(M)}\int\limits_M \varphi(x)dx = 2\alpha^{-1}\int\limits_M \varphi(x)dx$$

By [3, Proposition 3] we have that $L(\varphi) = \varphi(c)$, for some $c \in M$. Since $\varphi \in \mathcal{D}(\mathbb{R})$, we have that $\varphi(M) \subset B_1(\varphi(x_0)) = halo(\varphi(x_0))$ and hence $\varphi(c) \approx \varphi(x_0)$, the latter being what measurements give us, but $\varphi(c) \in \overline{\mathbb{R}}$ being the actual value. We can extend this to $\varphi \in \mathcal{G}(\mathbb{R})_{as}$ (which will be defined yet in this section) by taking $M = V_k(x_0)$ such that $\varphi(M) \subset B_1(\varphi(x_0))$.

The other example is that of a dipole at 0 with moment 1. In this case, the functionals can be written as

$$M(\varphi) = \alpha^{-1} \cdot (\varphi(\alpha) - \varphi(0))$$

This amounts to $\varphi(\alpha) = \varphi(0) + M(\varphi) \cdot \alpha$. Expanding around 0 gives us that $M(\varphi) - \varphi'(0) \in V_1(0)$, i.e., $M(\varphi) \approx \varphi'(0)$, the latter being what we measure, the actual value being $M(\varphi) \in \overline{\mathbb{R}}$. Again, this can be extended to $\mathcal{G}(\mathbb{R})$. In both cases, our measurements are the only "real" point in the halo of the actual values. Howbeit, classically we cannot differentiate among points in a halo. From the classical point of view, each halo contains only one point but, as we already saw, they do interfere in physical reality. In both examples, the measurable effect in physical reality is a product of an infinity, $2\alpha^{-1}$ respectively α^{-1}, and an infinitesimal, $\int\limits_M \varphi(x)dx$ respectively $\varphi(\alpha) - \varphi(0)$, both accredited and coexisting in harmony in this generalized milieu.

In the generalized environments the history of measurements and of convergence is captured and not each measurement separately (however arbitrarily). If the problem lies in the classical mathematical tools that we use and not in space-time its self, then we hope that these examples help to convince that the generalized environments, in particular generalized space-time, are environments that perhaps should be considered. See also [37, Section 1.6] and the references mentioned therein.

For the reader's sake, we recall the basics of the sharp topology, or Biagioni-Oberguggenberg topology (see [61, 14, 4, 5, 6]). Let (Ω_m) be an exhaustion of relatively compact and open subsets of $\Omega \subset \mathbb{R}^n$. Given $f \in \mathcal{G}(\Omega)$ and $(m, p) \in \mathbb{N}^2$, define $V_{mp} := Sup\{a \in \mathbb{R} \mid \forall \beta \in \mathbb{N}^n,\ |\beta| \leq p, \|\partial^\beta f(\varepsilon, \cdot\,)\|_{\Omega_m} = o(\varepsilon^a),\ \text{for}\ \ \varepsilon \text{ small }\}$ and $D_{mp}(f, g) := exp(-V_{mp}(\hat{f} - \hat{g}))$, where "^" stands for representative. The latter are pseudo-ultrametrics defining the Biagioni-Oberguggenberger sharp topology on $\mathcal{G}(\Omega)$ (see [1, Definition 1.9, Proposition 1.10 and 1.11]). This topology is proved to be equivalent to the topologies given in [4, 5, 6]. As observed in these references, this topology is metrizible and, with the notation given above, an ultrametric in $\mathcal{G}(\Omega)$ is given by

$$D(f,g) = sup\left\{ \frac{2 \cdot D_{mm}(f,g)}{1 + D_{mm}(f,g)} \; : \; m \in \mathbb{N} \right\}.$$

With the notation of [4, 5, 6], $W^k_{m,r}(0) = \{f \in \mathcal{G}(\Omega) \mid |\partial^\beta f(x)| \in V_r(0),\ \forall\ |\beta| \le k,\ \forall\ x \in \widetilde{\Omega_m}\}$, $V_r(0) = \{x \in \overline{\mathbb{K}} \ : \ |x| < \alpha^r\}$ and $\|\partial^\beta f\|_{\beta,m} := [\varepsilon \longrightarrow \|(\partial^\beta f_\varepsilon)_{|\Omega_m}\|_\infty]$. By [5, Theorem 3.6], the sets $W^k_{m,r}(0)$ form a filtered basis of the sharp topology of $\mathcal{G}(\Omega)$.

If $f, g \in \mathcal{G}(\Omega)$ then $f \approx g$ iff $\int\limits_\Omega (f-g)\varphi dx \in \overline{\mathbb{K}}_0$ and $f \sim g$ iff $\int\limits_\Omega (f-g)\varphi dx = 0 \in \overline{\mathbb{K}}$, $\forall \varphi \in \mathcal{D}(\Omega)$ (see [37]). In the latter case, one says that f and g are *test equivalent* and in the former case one says that they are *associated*. In [10], $\overline{\mathbb{K}}_0 = \{x \in \overline{\mathbb{K}} \ : \ x \approx 0\}$ was first formally given a notation as was $\overline{\mathbb{K}}_{as} = \mathbb{K} + \overline{\mathbb{K}}_0$ (see [37, Definition 1.6.5] where they were first defined). Here, we introduce the notation $\mathcal{G}(\Omega)_0 = \{f \in \mathcal{G}(\Omega) \ : \ f \approx 0\}$ and $\mathcal{G}(\Omega)_{as} = \{f \in \mathcal{G}(\Omega) \ : \ \exists\, T \in \mathcal{D}'(\Omega)$, such that $f \approx T\}$. Clearly, $\overline{\mathbb{K}}_0 \subset \mathcal{G}(\Omega)_0$. Define the *halo* of $f \in \mathcal{G}(\Omega)$ as $halo(f) := f + B_1(0) = B_1(f)$, where $B_1(0)$ is the ball $\{f \in \mathcal{G}(\Omega) \ : \ D(f,0) < 1\} \subset \mathcal{G}(\Omega)$ (see also [25]). Confusion should not arise between $\mathcal{G}(\Omega)_0$ and $\mathcal{G}_0(\Omega)$, the original Colombeau algebra (see [37]). Association in Colombeau algebras has been presented as an algebraic notion substituting equality in some sense. We proceed to prove that it is in fact a topological notion and use this to prove that, in a topological sense, Schwartz generalized functions, and hence classical solutions of differential equations, are scarce.

Proposition 3.2. *Let $f, g \in \mathcal{G}(\Omega)$. Then the following hold.*

1. *If $g \in halo(f)$ then $g \approx f$.*

2. *$B_1(0) = halo(0) \subset \mathcal{G}(\Omega)_0$.*

3. *If $\|f\|_m \in \overline{\mathbb{K}}_0$, $\forall\ m$, then $f \in \mathcal{G}(\Omega)_0$.*

4. *If $Im(f) \subset \overline{\mathbb{K}}_0$ then $f \in \mathcal{G}(\Omega)_0$.*

5. *$\mathcal{G}(\Omega)_{as} = \mathcal{D}'(\Omega) + \mathcal{G}(\Omega)_0$.*

Proof. To prove the first two items, by hypothesis, $h := f - g \in B_1(0)$. Since $D(h,0) < 1$, it follows that $D_{mm}(h,0) < 1$, $\forall m \in \mathbb{N}$ and hence there exists a decreasing sequence $((r_m)\ , r_m > 0, \ \ \forall\ m)$, such that $V_{mm}(h) = r_m > 0, \forall m \in \mathbb{N}$. Consequently, $h \in \bigcap\limits_{m\in\mathbb{N},} W^m_{m,r_m}$. In particular, $h(x) \in V_{r_m}(0), \forall x \in \widetilde{\Omega_m}$, i.e., $|h(x)| < \alpha^{r_m}, \forall x \in \widetilde{\Omega_m}$. Let $\varphi \in \mathcal{D}(\Omega)$ and $m \in \mathbb{N}$ be such that $supp(\varphi) \subset \Omega_m$; then $|\int\limits_\Omega h\varphi dx| \ = |\int\limits_{\Omega_m} h\varphi dx| \leq \alpha^{r_m} \cdot \|\varphi\|_\infty \cdot \mu(\Omega_m)$. Hence $\int\limits_\Omega h\varphi dx \in \overline{\mathbb{K}}_0$.

If $\varphi \in \mathcal{D}(\Omega)$ then there exists m such that $supp(\varphi) \subset \Omega_m$. Hence $|\int\limits_\Omega f(x)\varphi(x)dx| = |\int\limits_{\Omega_m} f(x)\varphi(x)dx| \leq \|f\|_m \cdot \|\varphi\|_\infty \cdot \mu(\Omega_m) \in \overline{\mathbb{K}}_0$, proving the third item.

To prove the fourth item, use an appropriate Riemann sum, as was shown to exist in the examples preceding the proposition, and the fact that $\overline{\mathbb{K}}_0$ is a ring. It also follows by the previous item. The fifth item is an obvious statement. □

If $f \approx 0$ then for each $x_0 \in \Omega$ and each $B_r(x_0) \subset \Omega$ we have that there exists $c_r \in \widetilde{B_r(x_0)}$ such that $f(c_r) \in \overline{\mathbb{K}}_0$. This follows from the observation at the end of the paragraph about Riemann sums.

In [9] it was proved that $\mathbb{R}^n$ is a discrete subset of $\overline{\mathbb{R}}^n$ and that if $r \in \mathbb{K}^*$ and $x \in \overline{\mathbb{K}}$ then $\|rx\| = \|x\|$. Our next results state that the same is true for $D'(\Omega)$ and $C^\infty(\Omega)$. The first part of the next corollary is in fact nothing more than a topological interpretation of [37, Proposition 1.6.3] and [37, Proposition 1.7.28]. The second part looks at the building blocks of $\mathcal{G}(\Omega)$ and equate them with the building blocks of $\overline{\mathbb{K}}$.

Corollary 3.3. *Let $\Omega \subset \mathbb{R}^n$ be an open subset. The embedding of $D'(\Omega)$ in $\mathcal{G}(\Omega)$ is a discrete embedding. Moreover, if $h \in C^\infty(\Omega)^*$ and $f \in \mathcal{G}(\Omega)$ then $\|hf\| = \|f\|$.*

Proof. Since the embedding is linear, we just have to prove discreteness at the origin. Let $f \in B_1(0) \cap \mathcal{D}'(\Omega)$. By the previous proposition, $f \approx 0$ and hence, by [37, Proposition 1.6.3], $f = 0$. The proof of the second part is straightforward. □

Since $g \approx f$ if $g \in halo(f)$, it follows that uniqueness of solutions and association can be seen as statements about halos. That is, the weaker notion of equality called association is a topological notion! Equality is an algebraic notion! Elements of $\mathcal{G}(\Omega)$ that are associated are indistinguishable one from another from the classical point of view (See also [37, Section 1.6]). An element of $\mathcal{G}(\Omega)$, not in the halo of any point of $\mathcal{D}'(\Omega)$, is called a vampier in [37]: it has no distributional shadow. Note however that it can be that, multiplying it with an $e \in \mathcal{B}(\overline{\mathbb{R}})$ it has a shadow, even infinitely many and thus, it can flaunt omnipresence (see the notion of intertwine or of support in the next section). This shows that the construction of $\widetilde{\Omega}_c$ starting from Ω is the same as the construction of $\mathcal{G}(\Omega)$ starting from $\mathcal{D}(\Omega)$. All that is classical becomes discrete in the generalized environments. Once again, no sequence converging in $D'(\Omega)$ can converge in $\mathcal{G}(\Omega)$ and, classically, measuring convergence is stopped at the sets $W^m_{p,1}$! Again, it appears that one has to call upon hypersequences to fix this.

That being so, even in the Colombeau Algebras there is just one notion of equality, i.e., the classical one! Association is not equality in any sense but a topological statement and test equivalent looks more like classical equality but is not. In fact, once again, let $f = x\delta$, $q \in \mathbb{N}$ and let the delta-net be induced by the mollifier ρ. Then $\int\limits_{\Omega} f_\varepsilon \varphi dx = \varepsilon \cdot \int\limits_{\Omega} z\rho(z)\varphi(\varepsilon z)dz$ $= \varepsilon \cdot \int\limits_{\Omega} z\rho(z)[\varphi(\varepsilon z) - \varphi(0)]dz = \frac{\varepsilon^q}{(q-1)!} \cdot \int\limits_{\Omega} z\rho(z)\varphi(\theta \cdot \varepsilon z)dz = o(\varepsilon^q)$. This proves that $\int_\Omega f\varphi dx = 0 \in \overline{\mathbb{K}}$ and thus, $f \sim 0$ but $f \neq 0$. For this f, $f(x) = 0$, $\forall\ x \in \mathbb{R}$ but it is not true that it is zero uniformly on compact subset of $\mathbb{R}$, i.e., it is not negligible at level zero and even more, $f \notin W^0_{m,r}(0)$ with $r > 0$.

There should be no strangeness in the fact that $f(x) = x\delta$ is identically zero on $\mathbb{R}$ but is not in $\widetilde{\mathbb{R}}_c$. Recall that $\mathbb{R}$ forms a grid of equidistant points in $\overline{\mathbb{R}}$ and thus $f = 0$ on a discrete grid with no accumulation points. However, $f'(x) = \delta + x\delta'$ is not necessarily identically zero in $\mathbb{R}$ because $f'(0) = \rho(0) \cdot \alpha^{-1}$. For comparison, $g(x) = sin(\pi x)$, $x \in \mathbb{Z}$, is identically zero, but $g'(x) = \pi cos(\pi x),\ x \in \mathbb{Z}$, is not.

In case of $\overline{\mathbb{K}}$, nets of $\mathbb{K}$ are its building blocks and one knows that $\mathbb{K}$ embeds as a grid of equidistant points into $\overline{\mathbb{K}}$. In case of $\mathcal{G}(\Omega)$, the building blocks are nets of elements of $C^\infty(\Omega)$ (see also [2, 14, 17, 37] about how to construct intrinsically $D'(\Omega)$ starting with the embedding of $C^\infty(\Omega)$). Hence the following results should not come as a surprise (see also [12, Theorem 5.8]).

Corollary 3.4. *The elements of* $C^\infty(\Omega)$ *form a grid of equidistant points in* $\mathcal{G}(\Omega)$. *In particular,* $\mathcal{G}(\Omega)$ *is a fractal.*

Proof. Since $C^\infty(\Omega)$ is diagonally embedded it follows that $V_{mp} = 0,\ \forall m, p$. By [10, Lemma 3.6], it follows that the inductive dimension $dim(\mathcal{G}(\Omega)) = \infty$ and, being an ultrametric space, $Ind(\mathcal{G}(\Omega)) = 0$. Consequently, $\mathcal{G}(\Omega)$ is a fractal. $\square$

Given a net of maps (T_ε) and $n = [(n_\varepsilon)] \in \widetilde{\mathbb{N}}$, we denote by T^n the net $(T_\varepsilon^{n_\varepsilon})$ acting on nets $f = (f_\varepsilon)$ as $T^n(f) = (T_\varepsilon^{n_\varepsilon}(f_\varepsilon))$. Suppose that T^n is well defined in $\mathcal{G}(\Omega)$, with $\Omega \subset \mathbb{R}^n$ an open subset, and denote by T the net when $n = 1$. Let $\mathcal{A} \subset \mathcal{G}(\Omega)$ be a closed subspace and suppose that the restriction $T_{|\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}$. The map $T : \mathcal{A} \longrightarrow \mathcal{A}$ is said to be a *contraction* if there exist $L \in \overline{\mathbb{R}}_+^*, \lambda \in]0,1[$ such that $L < \lambda$ and $|T(f) - T(g)|(x) \leq L \cdot |f - g|(x)$. It easily follows that $|T^n(f) - T^n(g)|(x) \leq L^n \cdot |f - g|(x) \leq \lambda^n \cdot |f - g|(x)$. Our interest is when the hypersequence $(T^n(f))$ converges in $\mathcal{G}(\Omega)$. We look at some examples inspired by the classical analog.

Let $\Omega \subset \mathbb{R}^n$ be an open subset, $L \in \mathbb{R}_+$ and $x_0 \in \widetilde{\Omega}_c$. Define $\mathcal{A} = \{f \in \mathcal{G}(\Omega) \ : \ |f(x) - x_0| \leq L,\ x \in \widetilde{\Omega}_c\}$ and consider it with the induced sharp topology. Then $\mathcal{A}$ is a closed subset of $\mathcal{G}(\Omega)$. In fact, let (f_n) be a Cauchy sequence in $\mathcal{A}$. Since $\mathcal{G}(\Omega)$ is complete (see [4, 5, 6]), $f_n \longrightarrow f$, for some f, and thus $x_n := |f_n(x) - x_0| \longrightarrow |f(x) - x_0| =: a$. Since x_n converges to a, for each $r > 0$ there exists n_0 such that if $n > n_0$ then $a - x_n \in V_r(0)$ and thus $|a - x_n| < \alpha^r$. Hence $a < x_n + \alpha^r \leq L + \alpha^r$. It follows that $a \leq L$, i.e., $f \in \mathcal{A}$, thus proving that $\mathcal{A}$ is a closed subset of

$\mathcal{G}(\Omega)$. We did not use the fact that $L \in \mathbb{R}$ but the reason why it appears here is that when considering some differential equations, compositions of generalized maps is necessary. In [9] it was shown that this results in the classical composition of maps. Since domains of generalized functions are $\widetilde{\Omega}_c$, the real bound L will guarantee that compositions of maps are allowed. Consequently, if $f \in \mathcal{A}$ then $|f(x)| \leq L + |x_0|$ proving that $\|f\| \leq 1$, i.e., $\mathcal{A} \subset \overline{B}_1(0)$.

Suppose that for each $\varepsilon \in I$ we have a map $T_\varepsilon : \mathcal{A}_\varepsilon \longrightarrow \mathcal{A}_\varepsilon$, with $\mathcal{A}_\varepsilon = \{h \in C(\Omega, \mathbb{R}) : |h(x) - x_{0\varepsilon}| < L\}$, which is a contraction with Lipschitz constant $K_\varepsilon < \lambda \in]0,1[$, the latter being independent of ε. It is clear that with these settings T^n, with $n \in \widetilde{N}$, is a well defined and continuous map in $\mathcal{G}(\Omega)$ and $T^{n+m} = T^n \circ T^m, n, m \in \widetilde{N}$. In particular, $T^{n+1} = T \circ T^n$, observing that 1 must be seen as an element of $\widetilde{N}$.

Another classical situation is the following. Let $\Omega \subset \mathbb{R}^n$ be open and relatively compact. For each $\varepsilon \in I$, let $T_\varepsilon(x)(t) = x_{0\varepsilon} + \int\limits_{t_{0\varepsilon}}^{t} h(s, x(s))ds, t \in \Omega$, $h \in C^\infty(\mathbb{R}^{n+1})$ and $\mathcal{A}_\varepsilon = C(\Omega, \mathbb{R}^n)$ (see [22, 23] and [24, Theorem 3.1]). In this case, there exists $n_{0\varepsilon} \in \mathbb{N}$, such that $T_\varepsilon^{n_{0\varepsilon}}$ is a contraction with Lipschitz constant $K_\varepsilon \leq \lambda < 1$, the latter being real and fixed. If $n_0 := [(n_{0\varepsilon})]$, then define $T = [(T_\varepsilon^{n_{0\varepsilon}})]$ and thus reducing it to the case of the previous paragraph, a classical argument.

Theorem 3.5 (**The Generalized Fixed Point Theorem**)**.** *Let $\Omega \subset \mathbb{R}^N$, $\mathcal{A} \subset \mathcal{G}(\Omega) \cap \overline{B}_1(0)$ a close subset. For each $\varepsilon \in I$, let $\mathcal{A}_\varepsilon = C^\infty(\Omega) \cap \mathcal{A}$, initial conditions taken for that specific ε, and (T_ε) a net of functionals from $\mathcal{A}_\varepsilon$ to its self. If there exists $k \in \widetilde{\mathbb{N}}$ such that each $T_\varepsilon^{k_\varepsilon}$ is Lipschitz with Lipschitz constant $K_\varepsilon < \lambda \in]0,1[$, then $T = (T_\varepsilon^{k_\varepsilon})$ is well defined, continuous and has a unique fixed point $\tilde{x} \in \mathcal{A}$.*

Proof. The proof uses what was already discussed and also freely facts about the topology.

Choose any $x \in \mathcal{A}$ and consider the hypersequence $(T^n(x))$. Given $r > 0$ and $m \in \mathbb{N}$, choose $n_0 \in \widetilde{\mathbb{N}}$ such that $\lambda^{n_0} \cdot \alpha^{-1} \in V_{4^{mN}r}(0) = V_{4^N r_1}(0)$, where $r_1 = 4^{N(m-1)}r$. If $n, s > n_0$ then, writing $n = n_0 + k,\ s = n_0 + l$, we have that $|T^n(x) - T^s(x)|(t) \leq \lambda^{n_0} \cdot |T^k(x) - T^l(x)|(t) \leq \lambda^{n_0} \cdot \alpha^{-1} \in V_{4^N r_1}(0)$. Setting $F(t) = (T^k(x) - T^l(x))(t)$, with $t \in \Omega_m$, we have that $F \in \overline{B}_1(0) \cap W^0_{m,4^N r_1}(0)$ and hence $|F''(t)| \leq \alpha^{-0.5r_1}$. Without loss of generality, we may considering $N = 1$, obtaining, since $\widetilde{\Omega_m}$ is open, $F(t + \alpha^{2r_1}) - F(t) = F'(t) \cdot \alpha^{2r_1} + \frac{F''(c)}{2} \cdot \alpha^{4r_1}$ and thus $|F'(t)| = \left| \frac{F(t+\alpha^{2r_1}) - F(t)}{\alpha^{2r_1}} + \frac{F''(c)}{-2} \cdot \alpha^{2r_1} \right| \leq \frac{2\alpha^{4r_1}}{\alpha^{2r_1}} + \frac{\alpha^{-0.5r_1}}{2} \cdot \alpha^{2r_1} < \alpha^{r_1}$. This proves that $F \in W^1_{m,r_1}(0) = W^1_{m,4^{m-1}r}(0)$. By induction, we have that $F \in W^m_{m,r}(0)$. This proves that $(T^n(x))$ is a Cauchy hypersequence and hence converges. Since $T^{n+1} = T \circ T^n$ and T is continuous, the limit is a fixed point of T. $\square$

In case compositions of maps are not involved, the theorem still holds if $\mathcal{A} \subset B_R(0)$, $R > 1$ (It always holds since estimates are made in $\widetilde{\Omega_m}$). Together with the other tools, it makes the Generalized Differential Calculus developed in [3, 9] (see also [26]) a useful tool to generalize most classical results. For example, one can prove the local, and hence the global, existence of geodesics in $\widetilde{M}_c$ (see the next section). Let's formalize the argument used in the last part of the proof of the theorem since it is a useful tool to be considered in Generalized Analysis.

Definition 3.6. The Down Sequencing Argument
Let $f \in \mathcal{G}(\Omega)$, with $\Omega \subset \mathbb{R}^n$. If $f \in W^0_{m,r}(0)$ with $r > 0$ and $p_0 \in \mathbb{N}^n$ then $f \in W^{|p_0|}_{m,s}(0)$, where $s = 4^{-n|p_0|} \cdot r$, i.e., $W^0_{m,r}(0) \subset W^{|p_0|}_{m,s}(0)$.

Using the **DSA**, another proof of a fact already mentioned can be given: If f is moderate and negligible at level 0, then f is negligible. This can be considered a statement about rigidity. In fact, such an $f \in W^0_{m,r}$, $\forall\ m,\ r > 0$ and hence DSA gives what was claimed. In other words $\bigcap\limits_{r>0} W^0_{m,r} = \{0\}$, showing that these sets serve as a basis for the sharp topology. Let's consider the example that inspired the Generalized Fixed Point Theorem. Consider the following equation from [39, 40].

$$\ddot{x}(t) = f(x(t))\delta(t) + h(t),\ x(-1) = x_0,\ \dot{x}(-1) = \dot{x}_0$$

with $h, f \in C^\infty(\mathbb{R})$ and δ the Dirac function. This is a typical differential equations from Physics and Engineering having a product of distributions in its data and does not allow the use of classical tools to obtain a solution. Let $b > 0$, $C > 0$ a positive constant limiting the $\mathbb{L}_1$ norm of the $\delta-$net, $M := \int\limits_{-2}^{1}\int\limits_{-2}^{1} |h(r)|drds$, $L := b + M + |\dot{x}_0| + \|f\| \cdot C$ and $1 + a = min\{\frac{b}{C\cdot\|f\|+|\dot{x}_0|}, \frac{1}{2CK}, 2\}$, where K is a Lipschitz constant of f on a compact subset of $\mathbb{R}$ containing $\Omega =]-1-\frac{a}{2}, \frac{a}{2}[$. The norm $\|f\|$ is also taken over the same compact subset. Let $\mathcal{A} = \{x = (x_1, \cdots, x_n) \in C^\infty(\widetilde{\Omega}_c, \tilde{\mathbb{R}}^n) : x_i \in \mathcal{G}(\Omega)$ and $|x(t) - x_0| \leq L\}$. Since $C^\infty(\widetilde{\Omega}_c, \tilde{\mathbb{R}}^n) \cong (C^\infty(\widetilde{\Omega}_c, \tilde{\mathbb{R}}))^n$ we have that $\mathcal{A} \subset (\mathcal{G}(\Omega))^n$ and the latter is a complete metric space (see [4, 5, 6]). From what we already discussed, it follows that $\mathcal{A}$ is a closed subspace of a complete algebra. Define $\mathcal{A}_\varepsilon$ accordingly and let T_ε be defined in $\mathcal{A}_\varepsilon$ by $T_\varepsilon(x)(t) = x_{0\varepsilon} + \dot{x}_0(t+1) + \int_{-1}^{t}\int_{-1}^{s} f(x(r))\rho_\varepsilon(r)drds + \int\limits_{-1}^{1}\int\limits_{-1}^{s} h(r)drds$. We have that $|T_\varepsilon(x)(t) - x_{0\varepsilon}| = |\dot{x}_{0\varepsilon}(t+1) + \int_{-1}^{t}\int_{-1}^{s} f(x(r))\rho_\varepsilon(r)drds + \int\limits_{-1}^{t}\int\limits_{-1}^{s} h(r)drds| \leq |\dot{x}_{0\varepsilon}| \cdot (|t| + 1)$
$+ \int_{-1}^{t}\int_{-1}^{s} |f(x(r))\rho_\varepsilon(r)|drds + \int\limits_{-1}^{t}\int\limits_{-1}^{s} |h(r)|drds \leq |\dot{x}_{0\varepsilon}| \cdot (2 + a) + (2 + a) \cdot \|f\| \cdot C + M = |\dot{x}_{0\varepsilon}| + M + (1 + a)(|\dot{x}_{0\varepsilon}| + \|f\| \cdot C) \leq |\dot{x}_{0\varepsilon}| + M + b + \|f\| \cdot C) = L$. It is also Lipschitz: $|T_\varepsilon(x)(t) - T_\varepsilon(\tilde{x})(t)| = |\int_{-1}^{t}\int_{-1}^{s}(f(x(r)) - f(\tilde{x}(t)))\rho_\varepsilon(t)drds| \leq \|f\| \cdot KC \cdot (2 + a) \cdot |x(t) - \tilde{x}(t)| \leq 2\|f\|C \cdot (1 + a) \cdot K|x(t) - \tilde{x}(t)| \leq K|x(t) - \tilde{x}(t)|$. It follows that the T they define is Lipschitz and hence well defined and continuous in $\mathcal{G}(\Omega)$. The first part shows that T maps $\mathcal{A}$ into itself. Hence we are in the setting of the Generalized Fixed Point Theorem. We have a Lipschitz map and the corresponding hypersequence has a fixed point which is a solution for the system.

Supposing $h = 0$, expand δ at collision time $t = 0$ and chose the mollifier such that $\rho(0) = 1$. The equation to be solved is

$$\ddot{x}(t) = f(x(t))\alpha^{-1},\ x(-1) = x_0,\ \dot{x}(-1) = \dot{x}_0$$

Its solution is obtained just as in the classical case but using Generalized Differential Calculus. Albeit, the solution might not be a Colombeau generalized function since it might only be defined on a membrane. This happens because the Theory of Generalized Differential Calculus strictly contains the Theory of Colombeau Generalized Functions (see [3] for an explicit example). What classically is perceived as a singularity, might not be the case in the generalized environments.

4 Generalized Manifolds

In this section, α will stand for an index and not for our standard gauge defined in the previous section. In [9] the definition of a generalized manifold was given and proved that each such manifold had a maximal $\mathcal{G}-$atlas. These manifolds were denoted short by $\mathcal{G}-$manifolds and the atlas by $\mathcal{G}-$atlas. In case the underlying field is $\mathbb{R}$, we have a real generalized manifold and in case the underlying field is $\mathbb{C}$ we have a complex generalized manifold. The topology in $\overline{\mathbb{R}}^n$ is the sharp topology and differentiability is in the sense of Generalized Differential Calculus ([3, 9]). Other notions such as diffeomorphism and continuity are those defined in [3, 4, 5, 9].

Definition 4.1. Let M be a non-void set. A $\mathcal{G}$-atlas of dimension n and class C^∞ of M is a family $\mathcal{A} = \{(\mathcal{U}_\lambda, \varphi_\lambda)\}_{\lambda\in\Lambda}$ verifying the following conditions:

1. For every $\lambda \in \Lambda$ the map $\varphi_\lambda : \mathcal{U}_\lambda \longrightarrow \overline{\mathbb{R}}^n$ is a bijection of the open subset $\emptyset \neq \mathcal{U}_\lambda \subset M$ onto the open subset $\varphi_\lambda(\mathcal{U}_\lambda) \subset \overline{\mathbb{R}}^n$.

2. $M = \displaystyle\bigcup_{\lambda\in\Lambda} \mathcal{U}_\lambda$

3. For every pair $\alpha, \beta \in \Lambda$, with $\mathcal{U}_{\alpha,\beta} = \mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \emptyset$, the subsets $\varphi_\alpha(\mathcal{U}_{\alpha,\beta})$ and $\varphi_\beta(\mathcal{U}_{\alpha,\beta})$ are open contained in $\overline{\mathbb{R}}^n$ such that $\varphi_\beta \circ \varphi_\alpha^{-1}$: $\varphi_\alpha(\mathcal{U}_{\alpha\beta}) \longrightarrow \varphi_\beta(\mathcal{U}_{\alpha,\beta})$ is a diffeomorphism of class C^∞.

- The pair $(\mathcal{U}_\lambda, \varphi_\lambda)$ is denominated a local chart (or coordinate system) of $\mathcal{A}$.

- If $\mathcal{U} \subset M$ and $\varphi : \mathcal{U} \longrightarrow u(\mathcal{U})$ is a homeomorphism of $\mathcal{U}$, where $\varphi(\mathcal{U})$ is an open set of $\overline{\mathbb{R}}^n$, the pair $(\mathcal{U}, \varphi)$ is said to be compatible with $\mathcal{A}$ if for each pair $(\mathcal{U}_\lambda, \varphi_\lambda) \in \mathcal{A}$ with $\mathcal{W}_\lambda = \mathcal{U} \cap \mathcal{U}_\lambda \neq \emptyset$ we have that $\varphi \circ \varphi_\lambda^{-1} : \varphi_\lambda(\mathcal{W}_\lambda) \longrightarrow \varphi(\mathcal{W}_\lambda)$ is a diffeomorphism of class C^∞, where $\varphi_\lambda(\mathcal{W}_\lambda)$ and $\varphi(\mathcal{W}_\lambda)$ are open subsets of $\overline{\mathbb{R}}^n$.

A Generalized Manifold, or $\mathcal{G}-$manifold, is a set M with a $\mathcal{G}-$atlas defined on it. A $\mathcal{G}-$manifold M with a maximal $\mathcal{G}-$atlas is called a $\mathcal{G}-$differential structure of M. If clear from the context, the prefix $\mathcal{G}$ will be omitted. The topology on the $\mathcal{G}-$manifold is the one that makes all charts simultaneously homeomorphisms. Our first step in setting the foundations of the Generalized Differential Geometry is settling the invariance of the dimension of a $\mathcal{G}-$manifold.

Theorem 4.2. *[Dimension Invariance [52]] Let $(M, \mathcal{A})$ be a $\mathcal{G}$-manifold. Then, the dimension of a $\mathcal{G}$-atlas $\mathcal{A}$ is constant in each connected component of M.*

Proof. Suppose there are two intersecting local charts $(\mathcal{U}_\alpha, \phi_\alpha)$ and $(\mathcal{U}_\beta, \phi_\beta)$ belonging to the $\mathcal{G}$-atlas $\mathcal{A} = \{(\mathcal{U}_\alpha, \phi_\alpha)\}_{\alpha\in\Lambda}$, such that $\phi_\alpha(\mathcal{U}_\alpha) \subset \overline{\mathbb{R}}^m$, and $\phi_\beta(\mathcal{U}_\beta) \subset \overline{\mathbb{R}}^n$. If $\mathcal{U}_{\alpha,\beta} = \mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \emptyset$, then we have that $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(\mathcal{U}_{\alpha\beta}) \longrightarrow \phi_\beta(\mathcal{U}_{\alpha\beta})$ is a diffeomorphism and therefore its differential in a point $p \in \phi_\alpha(\mathcal{U}_{\alpha\beta})$, $D(\phi_\beta \circ \phi_\alpha^{-1})(p) : \overline{\mathbb{R}}^m \longrightarrow \overline{\mathbb{R}}^n$, is a $\overline{\mathbb{R}}-$isomorphism of $\overline{\mathbb{R}}-$modules. Since $\overline{\mathbb{R}}$ is a commutative ring with unity, it follows from a result of [13] that $n = m$. $\square$

Let $\mathcal{A} = \{(\mathcal{U}_\alpha, \phi_\alpha), \alpha \in \Lambda\}$, be an altlas of a C^∞ n-dimensional connected submanifold M of $\mathbb{R}^N$. Suppose that for each $\alpha \in \Lambda$ we have that $\phi_\alpha(U_\alpha) = \Omega_0 = B_r(0) \subset \mathbb{R}^n$, the open ball of fixed radius $r > 0$ centered at the origin. Denote by $\widetilde{M}_c$ the subset of $\widetilde{\mathbb{R}}_c^N \subset \overline{\mathbb{R}}^N$ constructed from M (see the previous sections). We saw that M is discretely embedded in $\widetilde{M}_c$ as constants nets. Recall from [9] that $\widetilde{\mathbb{R}}_c^N \subset \overline{B}_1(0)$, the ball of radius 1 centered at the origin and that the image of $\mathbb{R}^N$ under this map is a grid of equidistant points. We denote by $\widetilde{\Lambda}$ the set of maps from $I =]0,1]$ into Λ and, for $\lambda \in \widetilde{\Lambda}$, we denote by

$$U_\lambda = \text{ the strongly internal set } \langle U_{\lambda(\varepsilon)} \rangle \text{ contained in } \widetilde{\mathbb{R}}_c^N$$

$$\phi_\lambda = (\phi_{\lambda(\varepsilon)})_{\varepsilon \in I}$$

$$\phi_\lambda : U_\lambda \longrightarrow \widetilde{\mathbb{R}}_c^n, \text{defined by } \phi_\lambda([(p_\varepsilon)]) = [(\phi_{\lambda(\varepsilon)}(p_\varepsilon))]$$

For $p = [(p_\varepsilon)] \in \widetilde{M}_c$, consider the set $\{q \in M : \exists\ \varepsilon_n \to 0,\ p_{\varepsilon_n} \to q\}$. Algebraically this can be written as: Given $q_0 \in \mathbb{R}^n$, we have that $q_0 \in \{q \in M : \exists\ \varepsilon_n \to 0,\ p_{\varepsilon_n} \to q\}$ if and only if there exists $e \in \mathcal{B}(\overline{\mathbb{R}})$ such that $e \cdot p \approx e \cdot q_0$ (extending the notion of association to $\overline{\mathbb{K}}^n$ in the obvious way). This is a compact subset of M to which we shall refer as the *support of the point* p and denote it by $supp(p)$. It follows that there exists a complete set of orthogonal idempotents $\{e_x : x \in supp(p)\}$, such that

$$p = \sum_{x \in supp(p)} e_x \cdot p, \text{ with } e_x \cdot p \approx e_x \cdot x$$

For example, if $p = [(p_\varepsilon)] = [(sin(\frac{1}{\varepsilon}))]$ then $supp(p) = [-1,1]$. On the other hand, if $p \in B_1(0)$ then $supp(p) = \{0\}$. Although $supp(p)$ might be uncountable, nevertheless the sum above is well defined and can be thought of as a history of events: multiplying by idempotents one sees its behavior along a specific path. It generalizes the concept of interleaving. The support of elements belonging to a halo of a point in $\mathbb{R}^n$ consists of only that single point.

Proposition 4.3. *Suppose that for each $\alpha \in \Lambda$ the map α and its inverse are classical Lipschitz functions with respect to the norms of $\mathbb{R}^N$ and $\mathbb{R}^n$. Then the following hold.*

1. *The topology of* $\widetilde{M}_c$ *is induced by the topology of* $\overline{\mathbb{R}}^n$.

2. *If* $\lambda \in \widetilde{\Lambda}$ *has finite range then* U_λ *is an open subset of* $\widetilde{M}_c$.

3. $\langle\Omega_0\rangle$ *is an open subset of* $\widetilde{\mathbb{R}}^n_c$

4. *If* $\lambda \in \widetilde{\Lambda}$ *has finite range then* $\phi_\lambda : U_\lambda \longrightarrow \langle\Omega_0\rangle \subset \widetilde{\mathbb{R}}^n$ *is a isometry w.r.t to the sharp topologies of* $\overline{\mathbb{R}}^n$ *and* $\overline{\mathbb{R}}^N$.

5. $\widetilde{M}_c = \bigcup_{\lambda\in\widetilde{\Lambda}} U_\lambda$, *with* λ *of finite range.*

6. *If* $\lambda_1, \lambda_2 \in \widetilde{\Lambda}$ *are of finite range and* $U_{\lambda_1,\lambda_2} := U_{\lambda_1} \cap U_{\lambda_2} \neq \emptyset$ *then* $\phi_{\lambda_2} \circ \phi^{-1}_{\lambda_1}$ *is a* C^∞ *diffeomorphism on its domain* $\phi_{\lambda_1}(U_{\lambda_1,\lambda_2})$.

Proof. Let $p = [(p_\varepsilon)] \in \widetilde{M}_c$. Since $supp(p)$ is compact, there exist a finite number of $\alpha \in \Lambda$ such that $supp(p)$ is contained in the union of the corresponding U_α's. Let δ be a Lebesgue number of this covering and choose a finite number of $q_i \in supp(p)$ such that $supp(p) \subset \bigcup_i B_{\delta_1}(q_i)$, with $\delta_1 = \frac{\delta}{2}$. Starting with q_1, define $\lambda(\varepsilon) = \alpha_{q_1} \in \Lambda$, where α_{q_1} is such that $B_{\delta_1}(q_1) \subset U_{\alpha_{q_1}}$ and $p_\varepsilon \in B_{\delta_1}(q_1)$.

For $\lambda(\varepsilon)$ not yet defined, continue defining $\lambda(\varepsilon) = \alpha_{q_2} \in \Lambda$, where α_{q_2} is such that $B_{\delta_1}(q_2) \subset U_{\alpha_{q_2}}$ and $p_\varepsilon \in B_{\delta_1}(q_2)$. Since there are a finite number of q_i's, this process ends in a finite number of steps. If λ is not defined on I then there exists a sequence (ε_n), converging to 0, with $p_{\varepsilon_n} \to q \in supp(p)$. Since the balls $B_{\delta_1}(q_i)$ cover $supp(p)$, there exists a n_0 such that $n > n_0$ implies that p_{ε_n} is in the ball $B_\delta(q_{i_0})$, say. But this is a contradiction, since, for these ε's, $\lambda(\varepsilon)$ was already defined. Hence, we defined a $\lambda \in \widetilde{\Lambda}$ such that $p_\varepsilon \in U_{\lambda(\varepsilon)}, \varepsilon \in I$ and the distance to the boundary is bigger than $\frac{\delta}{2}$. It follows that $p \in U_\lambda$, with λ being of finite range.

For each $\lambda(\varepsilon)$ there exists an open subset $U^{\lambda(\varepsilon)} \subset \mathbb{R}^N$ such that $U_{\lambda(\varepsilon)} = M \cap U^{\lambda(\varepsilon)}$. Setting $U^\lambda = \langle U^{\lambda(\varepsilon)}\rangle$, we have that $U_\lambda = \widetilde{M}_c \cap U^\lambda$, with U^λ an open subset of $\overline{\mathbb{R}}^N$, proving that $\widetilde{M}_c$ has the induced topology. The fact that strongly internal sets are open can be found in [46]. This settles the proof of the first three and the fifth items.

To finish the proof, we prove the forth and sixth items. We first prove that ϕ_λ is well defined. In fact, since local charts and their inverses are classical Lipschitz functions, and the λ's are of finite range, it follows easily that $\|p-q\| = \|\phi_\lambda(p) - \phi_\lambda(q)\|$, i.e., the ϕ_α's are isometries considered as maps from a subset of $\widetilde{\mathbb{R}}^N$ to a subset of $\widetilde{\mathbb{R}}^n$. From this it follows that if $dist(p,(U_\lambda)^c)$ is an invertible then $dist(\phi_\alpha(p),(\langle\Omega_0\rangle)^c)$ is an invertible, thus proving that ϕ_λ is well defined and its image is in U_λ. Surjectivity, injectivity and continuity are now obvious. The map resulting from a change of coordinates, is a homeomorphism that stems from a net whose elements are all infinitely differentiable taking value in a bounded subset of $\mathbb{R}^n$. Hence it originates a diffeomorphism. □

The condition that all charts have the same image is not necessary. We could just suppose that there exists $r > 0$ such that $B_r(0) \subset \mathbb{R}^n$ contains all of them. The $(U_\lambda, \phi_\lambda)$ are called *local charts* of $\widetilde{M}_c$. If λ is constant then U_λ is called a *principal* chart.

Theorem 4.4. *Let M be a submanifold of $\mathbb{R}^N$ of dimension n, and suppose that its local charts and their inverses are classical Lipschitz functions with respect to the norms of $\mathbb{R}^N$ and $\mathbb{R}^n$. Then $(\widetilde{M}_c, \mathcal{A})$, $\mathcal{A} = \{(U_\lambda, \phi_\lambda) : \lambda \in \widetilde{\Lambda}$ of finite range$\}$ is a generalized submanifold of $\overline{\mathbb{R}}^N$ of codimension $N-n$ containing M as a discrete subset. Moreover, each local chart is an isometry in the sharp topologies and the geometry of $\widetilde{M}_c$ extends in a natural way the geometry of M.*

Proof. Only the last part of the theorem needs to be proved. To see this, we use Lemma A.1 of [66] which state that for each compact subset of $K \subset M$ its Riemannian metric satisfies a $\|p-q\| \le dist_M(p,q) \le C \cdot \|p-q\|$, for some $C > 0$, $p, q \in K$ and $dist_M$ the Riemannian metric of M. This implies that local charts are isometries. □

Using Whitney's Embedding Theorem, the above theorem extends to abstract manifolds.

5 Differential Functions on $\mathcal{G}-$manifold

A one dimensional $\mathcal{G}-$manifold shall be referred to as a *curve*. This definition agrees with the notion of a history given in [9], only now we parametrize the history. If M is an $n-$dimensional $\mathcal{G}-$manifold and $p \in M$ then the tangent vectors and space at p are defined just as in the classical way (see [52]). The tangent space we shall also denote by T_pM. Furthermore, it is easily proved that T_pM is a free $\overline{\mathbb{R}}-$module which is $\overline{\mathbb{R}}-$isomorphic to $\overline{\mathbb{R}}^n$.

As in the classical case, one defines the tangent bundle of M and denote it by TM. It is easily seen that if M is a classical n-dimensional manifold then $\widetilde{(TM)}_c = T\widetilde{M}_c$ is a $\mathcal{G}-$manifold of dimension n^2.

Given another $\mathcal{G}-$manifold N, define a differential function between M and N using the same classical definition. In case $N = \overline{\mathbb{R}}$ we call a differentiable function a scaler field and if $N = \overline{\mathbb{R}}^n$, with $n > 1$, then we call such a map a vector valued map, being a vector field if also the dimension of M equals n.

The notions of an immersion and an embedding are defined completely analogous as in classical geometry. Using the Chain rule of Colombeau Generalized Calculus it follows easily that composition of differentiable maps between $\mathcal{G}-$manifolds also satisfy the Chain Rule (see [52]). We recall a Linear Algebra results (see [32]).

Lemma 5.1. *Let $A : \overline{\mathbb{R}}^n \longrightarrow \overline{\mathbb{R}}^n$ be a $\overline{\mathbb{R}}-$linear map. Then A is injective if and only if it is surjective if and only if* $\det(A) \in Inv(\overline{\mathbb{R}})$.

We sum up, without proofs, some of the most classical theorems that also hold for $\mathcal{G}-$manifolds. Some of the proofs rely on the previous lemma and the fact that $Inv(\overline{\mathbb{R}})$ is open (see [52]).

Theorem 5.2. *Let M_1, M_2 and M_3 be $\mathcal{G}$-manifolds. If $f : M_1 \longrightarrow M_2$ and $g : M_2 \longrightarrow M_3$ are differentiable applications at $p \in M_1$ and $f(p) \in M_2$, respectively, then $g \circ f : M_1 \longrightarrow M_3$ is differentiable at p and $D(g \circ f)_p = (Dg)_{f(p)} \circ Df_p$.*

Theorem 5.3. *Let M_1 and M_2 be n-dimensional $\mathcal{G}$-manifolds and $f : M_1 \longrightarrow M_2$ a map of class C^∞, such that for $p_0 \in M_1$ we have that $Df_{p_0} : T_{p_0}M_1 \longrightarrow T_{f(p_0)}M_2$, is an isomorphism. Then f is a local diffeomorphism of class C^∞.*

Theorem 5.4. *Let $f : M \longrightarrow N$ be an immersion at p of class C^∞, where M and N are $\mathcal{G}$-manifolds of dimension m and n, respectively. Then there exist local coordinate systems around p and $f(p)$, such that*

$$f(x_1, ..., x_m) = (x_1, ..., x_m, 0, ..., 0).$$

Theorem 5.5. *If $f : M \longrightarrow N$ is an generalized embedding, then $f(M)$ is an $\mathcal{G}$-submanifold of N.*

Definition 5.6. Let $f : \widetilde{\Omega}_c \subset \overline{\mathbb{R}}^n \longrightarrow \overline{\mathbb{R}}^m$ be a differentiable map, where Ω is an open subset of $\mathbb{R}^n$. A point $a \in \overline{\mathbb{R}}^m$ is called a regular value of f if for each $x \in f^{-1}(a)$ the derivative $f'(x) : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^m$ is surjective.

Theorem 5.7. *Let Ω be an open subset of $\mathbb{R}^m \times \mathbb{R}^n$ and $f : \widetilde{\Omega}_c \longrightarrow \overline{\mathbb{R}}^n$ be a application of class $\mathcal{C}^\infty$, where $\widetilde{\Omega}_c \subset \overline{\mathbb{R}}^m \times \overline{\mathbb{R}}^n$. If $a \in Im(f)$ is a regular value of f, then:*

1. *$f^{-1}(a)$ is an $m-$dimensional $\mathcal{G}$-submanifold of $\overline{\mathbb{R}}^m \times \overline{\mathbb{R}}^n$.*

2. *For each $p \in f^{-1}(a)$, we have that $T_p(f^{-1}(a)) = \ker(f^{'}(p))$.*

Let's look at some examples of $\mathcal{G}-$manifolds.

EXAMPLES

1. Consider $M = Graf(f)$, where $f \ : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}$, a $C^\infty-$function with bounded first derivate. Denote by $\phi : M \longrightarrow \Omega$ the projection $\phi(p) = x$, where $p = (x, f(x)) \in M$. If $q = (y, f(y)) \in M$ then $\|\phi(p) - \phi(q)\| = \|x - y\| < \|p - q\|$. On the other hand

   $\|\phi^{-1}(x) - \phi^{-1}(y)\|^2 = \|(x, f(x)) - (y, f(y))\|^2 = \|x - y\|^2 + |f(x) - f(y)|^2 \leq \|x - y\|^2 + \|\nabla f(p_0)\| \cdot \|x - y\|)^2 \leq (1 + C) \cdot \|x - y\|^2$

   This proves that the conditions of Proposition 4.3 are satisfied and thus $\widetilde{M_c}$ is a $\mathcal{G}-$submanifold of $\overline{\mathbb{R}}^{n+1}$.

2. Let $M \subset \mathbb{R}^n$ be a codimension one submanifold with an atlas whose elements are graphs. Then, by Theorem 3 of the previous section, we have that $\widetilde{M}_c$ is a $\mathcal{G}-$submanifold of $\overline{\mathbb{R}}^{n+1}$. Hence, this is true if M is a $m-$dimensional surface of $\mathbb{R}^n$. In particular, this holds if M is the pre image of a regular value of a C^∞ differentiable function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$.

3. Let $M = S_r^n \subset \mathbb{R}^n$ be an $n-$dimensional sphere of radius r. It can be parametrized by graphs such that derivates of the functions involved are bounded. Hence, by the previous example, we have that $\widetilde{M}_c$ is a $\mathcal{G}-$submanifold of $\overline{\mathbb{R}}^{n+1}$. One can cover M just with two local charts but this can not be done with $\widetilde{M}_c$. This example inspired the construction of the non principal charts and the notion of the support of a generalized point seen in the previous section.

4. The sphere $S = S_1(0)$ contained in $\overline{\mathbb{R}}^n$ is a generalized manifold whose local charts do not come from subsets of $\mathbb{R}^n$. In fact, S is an open subset of $\overline{\mathbb{R}}^n$ because given $x \in S$ and $y \in B_1(0)$ we have that $\|x + y\| = max\{\|x\|, \|y\|\} = 1$, since $\|x\| = 1 > \|y\|$. Consequently, we can take local charts to be the identity map with domain $B_1(x)$. Since these balls are either equal or disjoint, it follows that they form an atlas for S.

5. Let $f \in \mathcal{G}(\Omega)$ with $\Omega \subset \mathbb{R}^n$. We know (see [9]) that f can be viewed as a differentiable map $\widetilde{\Omega}_c \longrightarrow \overline{\mathbb{R}}$ and its differential at each point is a $\overline{\mathbb{R}}-$linear map from $\overline{\mathbb{R}}^n$ to $\overline{\mathbb{R}}$. A value $a \in Im(f)$ is said to be *a regular value* of f if for each $x \in f^{-1}(a)$ we have that Df is surjective. This only happens if, writing $\nabla f(x) = (z_1, \cdots, z_n)$, the ideal in $\overline{\mathbb{R}}$ generated by $z_1, \cdots, z_n$ equals $\overline{\mathbb{R}}$. In particular, this is the case if $\|\nabla f(x)\|_2^2 \in Inv(\overline{\mathbb{R}})$. If a is a regular value of f set $M = f^{-1}(a)$. We assert that M is a submanifold of $\overline{\mathbb{R}}^n$. In fact, just like in the classical case, we can use the Implicit Function Theorem (see [3]) to prove that at every point M is locally a graph over a subset of $\widetilde{\Omega}_c$. The standard classical argument still holds to complete the assertion.

An example of such a function is $f(x) = \|x\|_2^2$ and the value in question is $a = 1$. In this case we have that $\|\nabla f(x)\|_2^2 = 4\|x\|_2^2 = 4 \in Inv(\overline{\mathbb{R}})$.

6. The halo of any point in $\overline{\mathbb{K}}^n$ is a generalized manifold which is not classical. In [9] an example of a function $f \not\equiv 0$ is given such that $f' \equiv 0$. This function is $f(x) = \alpha_{-2\ln(\|x\|)}$ which is not a Colombeau generalized function. This f is constant on spheres $S_R(0)$ and the only point where it is not locally constant is $x_0 = \vec{0}$. This f is easily modified such that it is of class C^k. Since the origin is the only points where such spheres accumulate, and spheres are clopen, we have that $Graf(f) - \{\vec{0}\}$ is a generalized manifold but $Graf(f)$ is not a generalized manifold.

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