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On a variational inequality of a micropolar fluids system

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Dedicated to Professor M. Milla Miranda
for his enthusiasm by research.

Abstract. In this paper we investigate the problem for a model of the micropolar fluid coupled system. We consider that the Newtonian viscosity depends on the velocity of the fluid. Making use of the Faedo-Galerkin's approximation and some basic results of the theory of monotone operators and an appropriate penalization, we obtain a variational inequality for the micropolar fluid coupled system. Regularity and Uniqueness of solutions for $n = 2$ are also analyzed.

Keywords: micropolar fluids, variable viscosity.

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1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^3$ with the boundary $\partial\Omega$ of class C^2. For $T > 0$, we denote by Q_T the cylinder $(0, T) \times \Omega$, with lateral boundary

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$\Sigma_T = (0,T) \times \partial\Omega$. By $\langle .,. \rangle$ we will represent the duality pairing between X and X', X' being the topological dual of the space X, and by C we denote various positive constants. The equations that describe the motion of micropolar fluids are given by

$$\begin{aligned}
&u' - (\nu + \nu_r)\Delta u + (u \cdot \nabla)u + \nabla p = 2\nu_r \text{ rot } w + f \text{ in } Q_T,\\
&w' - (c_a + c_d)\Delta w + (u \cdot \nabla)w - (c_0 + c_d - c_a)\nabla(\nabla \cdot w)\\
&= 2\nu_r \text{ rot } u + g, \text{ in } Q_T,\\
&\operatorname{div} u = 0 \quad \text{in} \quad Q_T,\\
&u = 0 \quad \text{on} \quad \Sigma_T,\\
&w = 0 \quad \text{on} \quad \Sigma_T,\\
&u(x,0) = u_0(x) \text{ in } \Omega,\\
&w(x,0) = w_0(x) \text{ in } \Omega,
\end{aligned} \tag{1.1}$$

where $u(x,t) \in \mathbb{R}^3$, $w(x,t) \in \mathbb{R}^3$ and $p(x,t) \in \mathbb{R}$ denotes, for $(x,t) \in Q_T$, respectively, the unknown velocity, the microrotational velocity and the hydrostatic pressure of the fluid. The constants ν and ν_r are, respectively, the *Newtonian* and *micro-rotational viscosity*; the positive constants c_0, c_a and c_d are called *coefficients of angular viscosities* and satisfies $c_0 + c_d > c_a$.

The main difference with respect to modeled fluids by the Navier-Stokes is that the rotation of the particles is taken into account. The above approach was introduced by *A. C. Eringen* [4]. The nonlinear coupled system (1.1) can be used to model the behavior of liquid crystals, polymeric fluids and blood under some circumstances (see for instance [5]). These systems have been mainly analyzed in the book of *G. Lukaszewicz* [8].

Following an idea of *J-L Lions* in [6], pp. 208, the authors studied (to appear) the following system, where existence ($n = 2,3$), regularity and uniqueness of solutions ($n = 2$) are obtained:

$$\begin{aligned}
u' &- (\nu_0 + \nu_1 \|u\|^2)\Delta u + (u.\nabla)\, u + \nabla p = \operatorname{rot} w + f \text{ in } Q_T,\\
w' &- \Delta w - \nabla(\nabla \cdot w) + (u.\nabla)\, w = \operatorname{rot} u + g \text{ in } Q_T, \qquad (1.2)\\
&\operatorname{div} u = 0 \quad \text{in} \quad Q_T,\\
&\quad u = 0 \quad \text{on} \quad \Sigma_T,\\
&\quad w = 0 \quad \text{on} \quad \Sigma_T,
\end{aligned}$$

$$u(x,0) = u_0(x) \quad \text{in} \quad \Omega,$$
$$w(x,0) = w_0(x) \quad \text{in} \quad \Omega,$$

where ν_0, ν_1 constants, as another version of the system (1.1). In the latter, we are assuming that the viscosity $\nu + \nu_r$ in the first equation of (1.1) is changed to one of the type $\nu_0 + \nu_1 \|u\|^2$, that depends of the velocity u, and all other coefficients are equal to 1. In Brézis [1], 1989, we find investigation for a unilateral problem for Navier-Stokes operator $(\nu_1 = 0)$, that is, constant viscosity.

In the present work we consider a unilateral problem with first inequality similar to Brézis [1], for the case non constant viscosity, that is, $\nu_0 + \nu_1 \|u(t)\|^2$, $\nu_1 > 0$. More precisely, in this paper we study a unilateral problem or a variational inequality, cf. Lions [6], for the system (1.2) under standard hypotheses on f, g u_0 and w_0. Making use of the penalty method and Galerkin's approximations, we establish existence, regularity and uniqueness theorems.

This work is organized as follows: in section 2 we introduce the notations and main results. In Section 3 we show the proofs of the results. Finally, in Section 4, we prove a simple result of uniqueness.

2 Notation and main results

We propose the following variational inequality associated to problem (1.2)

$$\begin{aligned}
&u' - (\nu_0 + \nu_1 \|u\|^2)\Delta u + (u \cdot \nabla)\, u + \nabla p \geq \operatorname{rot} w + f \quad \text{in} \quad Q_T \\
&w' - \Delta w - \nabla(\nabla . w) + (u \cdot \nabla)\, w \geq \operatorname{rot} u + g \quad \text{in} \quad Q_T, \\
&\operatorname{div} u = 0 \quad \text{in} \quad Q_T, \\
&u = 0 \quad \text{on} \quad \Sigma_T, \\
&w = 0 \quad \text{on} \quad \Sigma_T, \\
&u(x,0) = u_0(x) \quad \text{in} \quad \Omega, \\
&w(x,0) = w_0(x) \quad \text{in} \quad \Omega,
\end{aligned} \tag{2.1}$$

In order to formulate problem (2.1) we need some notations about Sobolev spaces. We use standard notation of $L^2(\Omega)$, $L^p(\Omega)$, $W^{m,p}(\Omega)$ and $C^p(\Omega)$ for functions that are defined on Ω and range in $\mathbb{R}$, and the notation $L^2(\Omega)^n$, $L^p(\Omega)^n$, $W^{m,p}(\Omega)^n$ and $C^p(\Omega)^n$ for functions that range in $\mathbb{R}^n$. Besides, we work also with the spaces $L^p(0,T;H^m(\Omega))$ or $L^p(0,T;H^m(\Omega))^n$. To complete this recall on functional spaces, see for instance, *Lions* [6].

Also we define the following spaces

$$\mathcal{V} = \{\varphi \in \mathcal{D}(\Omega)^n; \quad \operatorname{div} \varphi = 0\},$$

$V = V(\Omega)$ is the closure of $\mathcal{V}$ in the space $H_0^1(\Omega)^n$ with inner product and norm denoted, respectively by

$$((u,z)) = \sum_{i,j=1}^{n} \int_\Omega \frac{\partial u_i}{\partial x_j}(x) \frac{\partial z_i}{\partial x_j}(x)\, dx, \quad \|u\|^2 = \sum_{i,j=1}^{n} \int_\Omega \left(\frac{\partial u_i}{\partial x_j}(x) \right)^2 dx,$$

$H = H(\Omega)$ is the closure of $\mathcal{V}$ in the space $L^2(\Omega)^n$ with inner product and norm defined, respectively, by

$$(u,v) = \sum_{i=1}^{n} \int_\Omega u_i(x) v_i(x)\, dx, \quad |u|^2 = \sum_{i=1}^{n} \int_\Omega |u_i(x)|^2\, dx.$$

Remark 2.1. V and H are Hilbert's spaces, $V \hookrightarrow H \hookrightarrow V'$ with embedding dense and continuous.

We introduce the following bilinear and the trilinear forms

$$a(u,v) = \sum_{i,j=1}^{n} \int_\Omega \frac{\partial u_i}{\partial x_j}(x) \frac{\partial v_i}{\partial x_j}(x)\, dx = ((u,v)),$$

$$b(u,v,w) = \sum_{i,j=1}^{n} \int_\Omega u_i(x) \frac{\partial v_j}{\partial x_i}(x) w_j(x)\, dx.$$

Remark 2.2. We denote by $\mathcal{A}$ the monotonous, hemicontinuous and bounded operator $\mathcal{A} : V \longrightarrow V'$, $\langle \mathcal{A}u, v\rangle = \|u\|^2 a(u,v)$ (see, for example, Lions [6], p. 218). We have that $\mathcal{A}u = -\nu_1 \|u\|^2 \Delta u$.

Let K and $\widetilde{K}$ be a closed and convex subset of V and $H_0^1(\Omega)^n$ with $0 \in K$ and $\widetilde{K}$. We write

$\mathbb{V} = L^4(0,T;V)),\ \mathbb{V}' = L^{4/3}(0,T;V'),$
$\mathbb{H} = L^2(0,T;H),\ \mathbb{K} = \{v|v \in \mathbb{V}, v(t) \in K \text{ a.e. }, v(0)=0\},$
$D\left(\frac{d}{dt};\mathbb{V}'\right) = \left\{v|v \in L^4(0,T;V),\ v' \in L^{4/3}(0,T;V')\right\}$

Remark 2.3. $\mathbb{V}$ is a reflexive Banach space, $\mathbb{H}$ is a Hilbert space, $\mathbb{K}$ is a closed and convex subset of $\mathbb{V}$.

We consider also the "Compatibility Hypothesis" (see Lions [6], p. 269)

$$\begin{aligned} &\forall v \in \mathbb{K} \text{ there exists a mollifiers } v_j \text{ verify}\\ &i) \quad v_j \in \mathbb{K} \cap D\left(\frac{d}{dt};\mathbb{V}'\right),\\ &ii) \quad v_j \longrightarrow v \text{ in } \mathbb{V},\ j \longrightarrow \infty,\\ &iii) \ \limsup_{j\to\infty}\left(\frac{dv_j}{dt}, v_j - v\right) \le 0. \end{aligned} \tag{H1}$$

Now, we present the main results of this work.

Theorem 2.4. *If $n \le 3$, $f \in L^{4/3}(0,T;V'), g \in (L^2(0,T;H^{-1}(\Omega)^3)$ and hypothesis $(H1)$ hold, then there exists a pair of functions $\{u,w\}$ such that*

$u \in L^4(0,T;V) \cap L^\infty(0,T;H), w \in L^2(0,T;H_0^1(\Omega)^3) \cap L^\infty(0,T;L^2(\Omega)^3)$

$$u(t) \in K, a.e., w(t) \in \widetilde{K}, a.e. \tag{2.2}$$

$$\begin{aligned} \int_0^T [\langle \varphi', \varphi - u\rangle \quad &+ \quad \nu_0 a(u,\varphi-u) + b(u,u,\varphi-u) + \nu_1\|u\|^2 a(u,\varphi-u)]dt\\ &\ge \quad \int_0^T [(\operatorname{rot} w, \varphi-u) + \langle f, \varphi-u\rangle]dt, \end{aligned} \tag{2.3}$$

$\forall \varphi \in L^4(0,T;V), \varphi' \in L^{4/3}(0,T;V'), \varphi(0)=0,\ \varphi(t) \in K$ *a.e.*

$$\begin{aligned} \int_0^T [\langle \phi', \phi - w\rangle \quad &+ \quad a(w,\phi-w) + b(u,w,\phi-w) - (\nabla \operatorname{div} w, \phi-w)]dt\\ &\ge \quad \int_0^T [(\operatorname{rot} u, \phi-w) + \langle f, \phi-w\rangle\, dt], \end{aligned} \tag{2.4}$$

$\forall \phi \in L^4(0,T;H_0^1(\Omega)^3), \phi' \in L^2(0,T;H^{-1}(\Omega)^3), \phi(0)=0,\ \phi(t) \in \widetilde{K}$ *a.e.*

Theorem 2.5. *Under the assumptions* $(H1)$, *suppose that* $n = 2$ *and*

$$f, f', g, g' \in L^2(0,T; L^2(\Omega)^2) \tag{2.5}$$

$$u_0 \in K, w_0 \in \widetilde{K}. \tag{2.6}$$

Suppose also that

$$\begin{aligned}(f(0), v) &+ (\operatorname{rot} w_0, v) - \nu_0 a(u_0, v) - b(u_0, u_0, v) - \nu_1 \|u_0\|^2 a(u_0, v) \quad (2.7)\\ &= (u_1, v) \text{ for all } v \in V \quad \text{and for some } u_1 \in H,\end{aligned}$$

$$\begin{aligned}(g(0), \widetilde{v}) &+ (\operatorname{rot} u_0, \widetilde{v}) - \nu_0 a(w_0, \widetilde{v}) - b(u_0, w_0, \widetilde{v}) - (\operatorname{div} w_0, \operatorname{div} \widetilde{v}) \quad (2.8)\\ &= (w_1, \widetilde{v}) \text{ for all } \widetilde{v} \in H_0^1(\Omega)^2 \quad \text{and for some } w_1 \in L^2(\Omega)^2\end{aligned}$$

Then there exists a unique pair of functions $\{u, w\}$ *such that*

$$\begin{aligned}&u \in L^2(0,T; V),\ u' \in L^2(0,T;V) \cap L^\infty(0,T;H),\\ &w \in L^2(0,T; H_0^1(\Omega)^2),\ w' \in L^2(0,T; H_0^1(\Omega)^2) \cap L^\infty(0,T; L^2(\Omega)^2) \quad (2.9)\\ &u(t) \in K, w(t) \in \widetilde{K} \quad \forall t \in [0,T], \text{ satisfying}\end{aligned}$$

$$\begin{aligned}&(u'(t), v - u(t)) + \nu_0 a(u(t), v - u(t))\\ &+ \nu_1 \langle \mathcal{A} u(t), v - u(t)\rangle + b(u(t), u(t), v - u(t))\\ &\geq (\operatorname{rot} w(t), v - u(t)) + (f(t), v - u(t)) \ \ \forall v \in K, \quad a.e. \text{ in } t, \quad (2.10)\end{aligned}$$

$$\begin{aligned}&(w'(t), \widetilde{v} - w(t)) + a(w(t), \widetilde{v} - w(t)) + b(u(t), w(t), \widetilde{v} - w(t))\\ &\geq (\operatorname{rot} u(t), \widetilde{v} - w(t)) + (g(t), \widetilde{v} - w(t)) \ \ \forall \widetilde{v} \in \widetilde{K}, \quad a.e. \text{ in } t, \quad (2.11)\end{aligned}$$

$$u(0) = u_0,\ w(0) = w_0. \tag{2.12}$$

The proof of Theorems 2.4 and 2.5 will be given in Section 3 by the penalty method. It consists in considering a perturbation of the system (1.2) adding a singular term called penalization, depending on a parameter $\epsilon, \varepsilon > 0$. We solve the mixed problem in Q for the penalized operator and the estimates obtained for the local solution of the penalized equation, allow us pass to limits, when ϵ, ε goes to zero, in order to obtain a pair of functions $\{u, w\}$ which is the solution of our problem.

First of all, let us consider the penalty operators $\beta : V \longrightarrow V'$ and $\widetilde{\beta} : H_0^1(\Omega)^3 \longrightarrow H^{-1}(\Omega)^3$ associated to the closed convex sets K and $\widetilde{K}$, cf. Lions [6], pp. 370. The operators β and $\widetilde{\beta}$ are monotonous, hemicontinuous, takes bounded sets of V and $H_0^1(\Omega)^3$ into bounded sets of V' and $H^{-1}(\Omega)^3$, its kernel are K and $\widetilde{K}$ and $\beta : L^4(0,T;V) \longrightarrow L^{4/3}(0,T;V')$ and $\widetilde{\beta} : L^2(0,T;H_0^1(\Omega)^3) \longrightarrow L^2(0,T;H^{-1}(\Omega)^3)$ are equally monotone and hemicontinuous.

The penalized problem associated with the variational inequalities (2.1) consists in given
$0 < \epsilon, \varepsilon < 1$, find a pair $\{u_\epsilon, w_\epsilon\}$ solution in Q of the mixed problem

$$
\begin{aligned}
u'_\epsilon \;&-\; (\nu_0 + \nu_1\|u_\epsilon\|^2)\Delta u_\epsilon + (u_\epsilon.\nabla)\, u_\epsilon + \frac{1}{\epsilon}\beta u_\epsilon + \nabla p = \operatorname{rot} w_\varepsilon + f \;\text{ in }\; Q_T,\\
w'_\varepsilon \;&-\; \Delta w_\varepsilon - \nabla(\nabla.w_\varepsilon) + (u_\epsilon.\nabla)\, w_\varepsilon + \frac{1}{\varepsilon}\widetilde{\beta} w_\varepsilon = \operatorname{rot} u_\epsilon + g \;\text{ in }\; Q_T,\\
&\operatorname{div} u_\epsilon = 0 \quad \text{in} \quad Q_T,\\
&u_\epsilon = 0 \quad \text{on} \quad \Sigma_T,\\
&w_\varepsilon = 0 \quad \text{on} \quad \Sigma_T,\\
&u_\epsilon(x,0) = u_{\epsilon 0}(x) \quad \text{in} \quad \Omega,\\
&w_\varepsilon(x,0) = w_{\varepsilon 0}(x) \quad \text{in} \quad \Omega.
\end{aligned}
\tag{2.13}
$$

Definition 2.6. Let $u_{\epsilon 0} \in H$, $w_{\varepsilon 0} \in L^2(\Omega)^3$, $f \in L^{4/3}(0,T;V'), g \in L^2(0,T;H^{-1}(\Omega)^3)$. A weak solution to the boundary value problem (2.13) is a pair of functions $\{u_\epsilon, w_\varepsilon\}$, such that $u_\epsilon \in L^4(0,T;V) \cap L^\infty(0,T;H)$, $w_\varepsilon \in L^4(0,T;H_0^1(\Omega)^3) \cap L^\infty(0,T;L^2(\Omega)^3)$, for $T > 0$, satisfying the following identity

$$
\begin{aligned}
(u'_\epsilon, \varphi) \;&+\; \nu_0 a(u_\epsilon, \varphi) + b(u_\epsilon, u_\epsilon, \varphi) + \langle \mathcal{A} u_\epsilon, \varphi\rangle + \frac{1}{\epsilon}(\beta u_\epsilon, \varphi)\\
&=\; (\operatorname{rot} w_\varepsilon, \varphi) + (f, \varphi), \forall \varphi \in V,
\end{aligned}
\tag{2.14}
$$

$$
(w'_\varepsilon, \phi) \;+\; a(w_\varepsilon, \phi) + (\operatorname{div} w_\varepsilon, \operatorname{div} \phi) + b(u_\epsilon, w_\varepsilon, \phi) + \frac{1}{\varepsilon}(\widetilde{\beta} w_\varepsilon, \phi)
$$

$$
\begin{aligned}
&= (\operatorname{rot} u_\epsilon, \phi) + (g, \phi), \ \forall \phi \in H_0^1(\Omega)^3, \\
&u_\epsilon(0) = u_{\epsilon 0}, \ w_\varepsilon(0) = w_{\varepsilon 0}, \\
&\operatorname{div} \varphi = 0.
\end{aligned}
$$

The solution of this problem is given by the following theorem:

Theorem 2.7. *If $f \in L^{4/3}(0,T;V')$, $g \in L^2(0,T;H^{-1}(\Omega)^3)$, $u_{\epsilon 0} \in V$ and $w_{\varepsilon 0} \in H_0^1(\Omega)^3$, then for each $0 < \epsilon, \varepsilon < 1$ there exists a pair of functions $\{u, w\}$ defined for $(x,t) \in Q_T$, solution to the problem (2.13) in the sense of Definition 2.6.*

Theorem 2.8. *Assume that $n = 2$ and $f, f', g, g' \in L^2(0,T;L^2(\Omega)^2)$. Then for each $0 < \epsilon, \varepsilon < 1$ and $u_{\epsilon_0} \in V, w_{\varepsilon 0} \in H_0^1(\Omega)^2$, there exists a pair of functions $\{u_\epsilon, w_\varepsilon\}$ defined for $(x,t) \in Q_T$, solution to the problem (2.13) in the sense of Definition 2.6.*

3 Proof of the results

Proof of Theorem 2.7

In order to prove Theorem 2.4, we first prove the Theorem 2.7.

We represent by $V_m = [\varphi_1, \varphi_2, ..., \varphi_m]$ the V subspace generated by the vectors $\varphi_1, \varphi_2, ..., \varphi_m$ and $\widetilde{V}_k = [\phi_1, \phi_2, ..., \phi_k]$ the $H_0^1(\Omega)^3$ subspace generated by the vectors $\phi_1, \phi_2, ..., \phi_m$. We employ the Faedo-Galerkin method with a Hilbertian basis $(\varphi_\nu)_{\nu \in \mathbb{N}}$ and $(\phi_\mu)_{\mu \in \mathbb{N}}$ of Sobolev space V and $H_0^1(\Omega)^3$, cf. Brézis [2], defined as the solution to the eigenvalue problem

$$
\begin{aligned}
&((\varphi_\nu, v)) = \lambda_\nu(\varphi_\nu, v) \ \forall v \in V, \ \nu \in \mathbb{N}, \\
&((\phi_\mu, \widetilde{v})) = \lambda_\mu(\phi_\mu, \widetilde{v}) \ \forall \widetilde{v} \in H_0^1(\Omega)^3, \ \mu \in \mathbb{N}
\end{aligned}
$$

Let us consider

$$
u_{\epsilon_m}(t) = \sum_{j=1}^{m} g_{j_m}(t)\varphi_j
$$

and

$$
w_{\varepsilon_m}(t) = \sum_{j=1}^{m} h_{j_m}(t)\phi_j
$$

solution of approximate problem

$$
\begin{aligned}
&(u'_{\epsilon_m},\varphi_j)+\nu_0 a(u_{\epsilon_m},\varphi_j)+\nu_1\|u_{\epsilon_m}\|^2 a(u_{\epsilon_m},\varphi_j)+b(u_{\epsilon_m},u_{\epsilon_m},\varphi_j)\\
&+(\nabla p,\varphi_j)+\frac{1}{\epsilon}(\beta u_{\epsilon_m},\varphi_j)=(\operatorname{rot} w_{\varepsilon_m},\varphi_j)+\langle f(t),\varphi_j\rangle,\ j=1,2,...m\\
&(w'_{\varepsilon_m},\phi_j)+a(w_{\varepsilon_m},\phi_j)+(\operatorname{div} w_{\varepsilon_m},\operatorname{div}\phi_j)+b(u_{\epsilon_m},w_{\varepsilon_m},\phi_j)\\
&+\frac{1}{\varepsilon}(\widetilde{\beta} w_{\varepsilon_m},\phi_j)=(\operatorname{rot} u_{\epsilon_m},\phi_j)+\langle g(t),\phi_j\rangle,\ j=1,2,...m\\
&u_{\epsilon_m}(x,0)\to u_\epsilon(x,0)\text{ strongly in } V,\\
&w_{\varepsilon_m}(x,0)\to w_\varepsilon(x,0)\ \text{ strongly in } H_0^1(\Omega)^3
\end{aligned}
\tag{3.1}
$$

The system of ordinary differential equation (3.1) has a solution on a interval $[0,t_m[,\ 0<t_m<T$. The first estimate permits us to extend this solution to the whole interval $[0,T]$.

Remark 3.1. In order to obtain a better notation, we omit the parameters $\{\epsilon,\varepsilon\}$ in the approximate solutions.

FIRST ESTIMATE

Multiplying both sides of $(3.1)_1$ by g_{j_m} and $(3.1)_2$ by h_{j_k} adding from $j=1$ to $j=m,k$, we obtain that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|u_m(t)|^2+\nu_0\|u_m(t)\|^2+\nu_1\|u_m(t)\|^4\\
&\leq(\operatorname{rot} w_m,u_m)+\langle f(t),u_m(t)\rangle,
\end{aligned}
\tag{3.2}
$$

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|w_m(t)|^2+\|w_m(t)\|^2+|\operatorname{div} w_m(t)|^2\\
&\leq(\operatorname{rot} u_m,w_m)+\langle g(t),w_m(t)\rangle,
\end{aligned}
\tag{3.3}
$$

since $b(u_m,u_m,u_m)=0$ (see J.L.Lions [6]) and $(\beta u_m(t),u_m(t))\geq 0$ because β is monotone and $0\in K$. Follows from (3.2) and (3.3) that

$$u_{\epsilon_m} \rightharpoonup u_\epsilon \text{ weak star } \text{ in } L^\infty(0,T;H) \tag{3.4}$$

$$u_{\epsilon_m} \rightharpoonup u_\epsilon \text{ weak in } L^2(0,T;V) \tag{3.5}$$

$$u_{\epsilon_m} \rightharpoonup u_\epsilon \text{ weak in } L^4(0,T;V) \tag{3.6}$$

$$w_{\varepsilon_m} \rightharpoonup w_\varepsilon \text{ weak star } \text{ in } L^\infty(0,T;L^2(\Omega)^3) \tag{3.7}$$

$$w_{\varepsilon_m} \rightharpoonup w_\varepsilon \text{ weak in } L^2(0,T;H_0^1(\Omega)^3) \tag{3.8}$$

SECOND ESTIMATE

Using projector operator (see [6]), we obtain returning to the notation $\{u_{\epsilon_m}, w_{\epsilon_m}\}$, it follows from estimates above that

$$u_{\epsilon_m}(T) \rightharpoonup \xi \text{ weak in } H \tag{3.9}$$

$$\mathcal{A}u_{\epsilon_m} \rightharpoonup \chi \text{ weak in } L^{4/3}(0,T;V') \tag{3.10}$$

$$u'_{\epsilon_m} \rightharpoonup u'_\epsilon \text{ weak in } L^{4/3}(0,T;V'). \tag{3.11}$$

$$\beta u_{\epsilon_m} \rightharpoonup \zeta \text{ weak in } L^{4/3}(0,T;V'). \tag{3.12}$$

$$u_{\epsilon_{m_i}} u_{\epsilon_{m_j}} \rightharpoonup u_{\epsilon_i} u_{\epsilon_j} \text{ weak in } L^2(0,T;L^2(\Omega)) \tag{3.13}$$

$$u_{\epsilon_m} \longrightarrow u_\epsilon \text{ strong in } L^2(0,T;H) \text{ and a.e in } Q_T. \tag{3.14}$$

Similar convergences we obtain for (w_{ε_m}). From the convergence above, letting $m \to \infty$ in (3.1), we have

$$\begin{aligned} &u'_\epsilon + Au_\epsilon + \chi + \frac{1}{\epsilon}\zeta + Bu_\epsilon = \operatorname{rot} w_\varepsilon + f \text{ in } L^{4/3}(0,T;V') \\ &w'_\varepsilon + Aw_\varepsilon - \nabla(\nabla . w_\varepsilon) + \frac{1}{\varepsilon}\widetilde{\zeta} + \widetilde{B}u_\epsilon = \operatorname{rot} u_\varepsilon + g \\ &\text{in } L^2(0,T;H^{-1}(\Omega)^3). \end{aligned} \tag{3.15}$$

Here,

$$Au = -\nu_0 \Delta u, \;\; Aw = -\Delta w,$$

$$Bu = \sum_{i=1}^{3} u_i \frac{\partial u}{\partial x_i} \text{ and } \widetilde{B}u = \sum_{i=1}^{3} u_i \frac{\partial w}{\partial x_i}.$$

It is necessary to prove that $\chi + \frac{1}{\epsilon}\zeta = \mathcal{A}u_\epsilon + \frac{1}{\epsilon}\beta u_\epsilon$ and $\frac{1}{\varepsilon}\widetilde{\zeta} = \frac{1}{\varepsilon}\widetilde{\beta} w_\varepsilon$. We make this using the monotony of the operators $\mathcal{A}+\dfrac{1}{\epsilon}\beta$ and $\dfrac{1}{\varepsilon}\widetilde{\beta}$ (see [6], Chap. 2). Theorem 2.7 is proved.

Proof of Theorem 2.4

From the convergences (3.4)-(3.14), convergences similar to (w_{ε_m}) and Banach-Steinhauss theorem, it follows that there exists subnet $(u_\epsilon)_{0<\epsilon<1}$ and $(w_\varepsilon)_{0<\varepsilon<1}$, such that they converge to u and w as $\epsilon,\varepsilon\to 0$ in the sense of (3.4)-(3.14). These functions satisfies (2.13) in the sense Definition 2.6. For another way, we have from $(3.15)_1$ that

$$\beta u_\epsilon = \epsilon[\operatorname{rot} w_\varepsilon + f - u'_\epsilon - Au_\epsilon - \mathcal{A}u_\epsilon - Bu_\epsilon]. \tag{3.16}$$

Then

$$\beta u_\epsilon \longrightarrow 0 \text{ in } \mathcal{D}'(0,T;V'). \tag{3.17}$$

It results from (3.16) that

$$\beta u_\epsilon \text{ is bounded in } L^{4/3}(0,T;V'). \tag{3.18}$$

Therefore,

$$\beta u_\epsilon \longrightarrow 0 \;\text{ weak in }\; L^{4/3}(0,T;V'). \tag{3.19}$$

On the other hand we deduce from $(3.15)_1$ that

$$0 \le \int_0^T \langle \beta u_\epsilon, u_\epsilon\rangle\, dt \le \epsilon\, C. \tag{3.20}$$

So,

$$\int_0^T \langle \beta u_\epsilon, u_\epsilon\rangle dt \longrightarrow 0. \tag{3.21}$$

As β is a monotonous operator, we have

$$\int_0^T \langle \beta u_\epsilon, u_\epsilon\rangle\, dt - \int_0^T \langle \beta u_\epsilon, \varphi\rangle\, dt - \int_0^T \langle \beta\varphi, u_\epsilon - \varphi\rangle\, dt \ge 0. \tag{3.22}$$

We have from (3.19), (3.21) and (3.22) that

$$\int_0^T \langle \beta\varphi, u_\epsilon(t) - \varphi\rangle\, dt \le 0. \tag{3.23}$$

Taking $\varphi = u_\epsilon - \lambda v$, with $v \in L^4(0,T;V)$ and $\lambda > 0$, we deduce using the hemicontinuity of β that

$$\beta u_\epsilon(t) = 0, \tag{3.24}$$

and this implies that $u_\epsilon(t) \in K$ a. e. Similarly, we obtain for $(3.15)_2$ that,

$$\widetilde{\beta} w_\varepsilon(t) = 0 \tag{3.25}$$

and this implies that $w_\varepsilon \in \widetilde{K}$ a.e.

Next, we prove that $\{u, w\}$ are solution of inequalities (2.3) and (2.4).

Initially, we show that

$$\limsup_{\epsilon \to 0} \int_0^T \langle \mathcal{A} u_\epsilon, u_\epsilon - u \rangle dt \leq 0. \tag{3.26}$$

In fact, we have that $\forall v \in \mathbb{K} \cap D\left(\dfrac{d}{dt}; \mathbb{V}'\right)$

$$\begin{aligned}
&\int_0^T \langle v' + Au_\epsilon + Bu_\epsilon + \mathcal{A}u_\epsilon - \operatorname{rot} w_\varepsilon - f, v - u_\epsilon \rangle \, dt \\
&= \int_0^T \langle u'_\epsilon + Au_\epsilon + Bu_\epsilon + \mathcal{A}u_\epsilon - \operatorname{rot} w_\varepsilon - f, v - u_\epsilon \rangle \, dt \\
&+ \int_0^T \langle v' - u'_\epsilon, v - u_\epsilon \rangle \, dt = \frac{1}{\epsilon} \int_0^T \langle \beta v - \beta u_\epsilon, v - u_\epsilon \rangle \, dt \\
&+ \int_0^T \langle v' - u'_\epsilon, v - u_\epsilon \rangle \, dt \geq 0.
\end{aligned} \tag{3.27}$$

Remark 3.2. $\displaystyle\int_0^T \langle \beta v - \beta u_\epsilon, v - u_\epsilon \rangle \, dt \geq 0$ and $\displaystyle\int_0^T \langle v' - u'_\epsilon, v - u_\epsilon \rangle \geq 0$ because β is a monotonous operator, $v \in \mathbb{K} \cap D\left(\dfrac{d}{dt}, \mathbb{V}'\right)$, $v(0) = 0$.

It follows from (3.27) that

$$\begin{aligned}
&\int_0^T \langle \mathcal{A}u_\epsilon, u_\epsilon\rangle\, dt \le \int_0^T \langle \mathcal{A}u_\epsilon, v\rangle\, dt + \int_0^T \langle Au_\epsilon, v-u_\epsilon\rangle\, dt \\
&+ \int_0^T \langle Bu_\epsilon, v-u_\epsilon\rangle\, dt + \int_0^T \langle v' - \operatorname{rot} w_\varepsilon - f, v-u_\epsilon\rangle\, dt \\
&= \int_0^T \langle \mathcal{A}u_\epsilon, v\rangle\, dt + \int_0^T \langle Au_\epsilon, v\rangle\, dt - \int_0^T \langle Au_\epsilon, u_\epsilon\rangle\, dt \\
&+ \int_0^T \langle Bu_\epsilon, v\rangle\, dt - \int_0^T \langle Bu_\epsilon, u_\epsilon\rangle\, dt + \int_0^T \langle v' - \operatorname{rot} w_\varepsilon - f, v-u_\epsilon\rangle\, dt.
\end{aligned} \tag{3.28}$$

Taking the lim sup in (3.28), we obtain

$$\begin{aligned}
&\limsup \int_0^T \langle \mathcal{A}u_\epsilon, u_\epsilon\rangle\, dt \le \limsup \int_0^T \langle \mathcal{A}u_\epsilon, v\rangle\, dt \\
&+ \limsup\left(- \int_0^T \|u_\epsilon(t)\|^2\, dt\right) + \limsup \int_0^T \langle Au_\epsilon, v\rangle\, dt \\
&+ \limsup \int_0^T \langle Bu_\epsilon, v\rangle\, dt + \limsup \int_0^T \langle v' - \operatorname{rot} w_\varepsilon - f, v-u_\epsilon\rangle\, dt.
\end{aligned} \tag{3.29}$$

Reminding that $\operatorname{rot} w_\varepsilon \to \operatorname{rot} w$ weakly in $L^2(0,T;L^2(\Omega)^n)$ and $u_\epsilon \to u$ strongly in $L^2(0,T;H)$, it follows from (3.29) that

$$\begin{aligned}
&\limsup \int_0^T \langle \mathcal{A}u_\epsilon, u_\epsilon\rangle\, dt \le \int_0^T \langle \chi, v\rangle\, dt \\
&- \int_0^T \|u(t)\|^2\, dt + \int_0^T \langle Au, v\rangle\, dt \\
&+ \int_0^T \langle Bu, v\rangle\, dt + \int_0^T \langle v' - \operatorname{rot} w - f, v-u\rangle\, dt,
\end{aligned} \tag{3.30}$$

$\forall v \in \mathbb{K} \cap D\left(\dfrac{d}{dt}, \mathbb{V}'\right).$

But, in view $(H1)$, we can take $u_j \in \mathbb{K} \cap D\left(\dfrac{d}{dt}, \mathbb{V}'\right)$ such that

$u_j \longrightarrow u$ in $\mathbb{V}$ and $\limsup_{j\to\infty}(u'_j, u_j - u) \le 0.$

Taking $v = u_j$ in (3.30) we obtain

$$
\begin{aligned}
&\limsup \int_0^T \langle \mathcal{A}u_\epsilon, u_\epsilon\rangle \, dt \le \int_0^T \langle \chi, u_j\rangle \, dt \\
&- \int_0^T \|u(t)\|^2 \, dt + \int_0^T \langle Au, u_j\rangle \, dt \qquad (3.31)\\
&+ \int_0^T \langle Bu, u_j\rangle \, dt + \int_0^T \langle u'_j - \operatorname{rot} w - f, u_j - u\rangle \, dt.
\end{aligned}
$$

Taking $\limsup$ in (3.31) with $j \longrightarrow \infty$, we obtain

$$
\limsup \int_0^T \langle \mathcal{A}u_\epsilon, u_\epsilon\rangle \, dt \le \int_0^T \langle \chi, u\rangle \, dt, \qquad (3.32)
$$

Then,

$$
\limsup \int_0^T \langle \mathcal{A}u_\epsilon, u_\epsilon\rangle \, dt - \limsup \int_0^T \langle \mathcal{A}u_\epsilon, u\rangle \, dt \le 0,
$$

that is, (3.26) is verified.

Let us consider $\mathbf{X}_\epsilon$ defined by

$$
\begin{aligned}
\mathbf{X}_\epsilon \;&=\; \int_0^T \langle \varphi', \varphi - u_\epsilon\rangle \, dt + \int_0^T a(u_\epsilon, \varphi - u_\epsilon) \, dt \\
&+\; \int_0^T \langle \mathcal{A}u_\epsilon, \varphi - u_\epsilon\rangle \, dt + \int_0^T b(u_\epsilon, u_\epsilon, \varphi - u_\epsilon) \, dt \qquad (3.33)\\
&-\; \int_0^T (\operatorname{rot} w_\varepsilon, \varphi - u_\epsilon) \, dt - \int_0^T \langle f, \varphi - u_\epsilon\rangle \, dt,
\end{aligned}
$$

with $\varphi \in L^4(0,T;V), \varphi' \in L^{4/3}(0,T;V'), \varphi(0) = 0, \;\; \varphi(t) \in K$ a.e. It follows from (3.33) that

$$
\mathbf{X}_\epsilon = \int_0^T \langle \varphi', \varphi\rangle \, dt - \int_0^T \langle \varphi', u_\epsilon\rangle \, dt + \int_0^T a(u_\epsilon, \varphi) \, dt
$$

$$
\begin{aligned}
- & \int_0^T a(u_\epsilon, u_\epsilon)\, dt + \int_0^T b(u_\epsilon, u_\epsilon, \varphi)\, dt \\
- & \int_0^T b(u_\epsilon, u_\epsilon, u_\epsilon) dt + \int_0^T \langle \mathcal{A} u_\epsilon, \varphi \rangle\, dt \\
- & \int_0^T \langle \mathcal{A} u_\epsilon, u_\epsilon \rangle\, dt - \int_0^T (\operatorname{rot} w_\varepsilon, \varphi)\, dt + \int_0^T (\operatorname{rot} w_\varepsilon, u_\epsilon)\, dt \\
- & \int_0^T \langle f, \varphi \rangle\, dt + \int_0^T \langle f, u_\epsilon \rangle\, dt.
\end{aligned} \tag{3.34}
$$

On the other hand, multiplying $(3.15)_1$ by $\varphi - u_\epsilon$ and integrating in $[0, T]$, we obtain that

$$
\begin{aligned}
& \int_0^T \langle u_\epsilon', \varphi \rangle\, dt - \int_0^T \langle u_\epsilon', u_\epsilon \rangle\, dt + \int_0^T a(u_\epsilon, \varphi)\, dt \\
& - \int_0^T a(u_\epsilon, u_\epsilon)\, dt + \int_0^T b(u_\epsilon, u_\epsilon, \varphi)\, dt \\
& - \int_0^T b(u_\epsilon, u_\epsilon, u_\epsilon) dt + \int_0^T \langle \mathcal{A} u_\epsilon, \varphi \rangle\, dt \\
& - \int_0^T \langle \mathcal{A} u_\epsilon, u_\epsilon \rangle\, dt + \frac{1}{\epsilon} \int_0^T \langle \beta u_\epsilon - \beta \varphi, \varphi - u_\epsilon \rangle \\
& - \int_0^T (\operatorname{rot} w_\varepsilon, \varphi)\, dt + \int_0^T (\operatorname{rot} w_\varepsilon, u_\epsilon)\, dt \\
& - \int_0^T \langle f, \varphi \rangle\, dt + \int_0^T \langle f, u_\epsilon \rangle\, dt = 0,
\end{aligned} \tag{3.35}
$$

because $\beta\varphi = 0$. Adding member to member (3.34) and (3.35), we obtain

$$
\begin{aligned}
\mathbf{X}_\epsilon = & \int_0^T \langle \varphi', \varphi \rangle\, dt - \int_0^T \langle \varphi', u_\epsilon \rangle\, dt - \int_0^T \langle u_\epsilon', \varphi \rangle\, dt \\
+ & \int_0^T \langle u_\epsilon', u_\epsilon \rangle\, dt + \frac{1}{\epsilon} \int_0^T \langle \beta \varphi - \beta u_\epsilon, \varphi - u_\epsilon \rangle\, dt \geq 0,
\end{aligned} \tag{3.36}
$$

because

$$\int_0^T \langle \varphi', \varphi \rangle \, dt - \int_0^T \langle \varphi', u_\epsilon \rangle \, dt - \int_0^T \langle u'_\epsilon, \varphi \rangle \, dt + \int_0^T \langle u'_\epsilon, u_\epsilon \rangle \, dt$$
$$= \int_0^T \langle \varphi' - u'_\epsilon, \varphi - u_\epsilon \rangle \geq 0.$$

From (3.36) and (3.33) it follows that

$$\begin{aligned} \mathbf{X}_\epsilon &= \int_0^T \langle \varphi', \varphi - u_\epsilon \rangle \, dt + \int_0^T a(u_\epsilon, \varphi) \, dt - \int_0^T a(u_\epsilon, u_\epsilon) dt \\ &+ \int_0^T b(u_\epsilon, u_\epsilon, \varphi) \, dt \int_0^T \langle \mathcal{A} u_\epsilon, \varphi - u_\epsilon \rangle \, dt \\ &\geq \int_0^T (\operatorname{rot} w_\varepsilon, \varphi - u_\epsilon) \, dt + \int_0^T \langle f, \varphi - u_\epsilon \rangle \, dt. \end{aligned} \tag{3.37}$$

As $\mathcal{A}$ is a pseudo-monotonous operator (see Lions [6] p. 179), it follows from (3.26) that

$$- \liminf \int_0^T \langle \mathcal{A} u_\epsilon, u_\epsilon - \varphi \rangle \, dt \leq \int_0^T \langle \mathcal{A} u, \varphi - u \rangle \, dt. \tag{3.38}$$

Taking lim sup in (3.37), it follows from (3.30), (3.38) and previous convergences that

$$\begin{aligned} \int_0^T [\langle \varphi', \varphi - u \rangle &+ a(u, \varphi - u) + b(u, u, \varphi - u) + \langle \mathcal{A} u, \varphi - u \rangle] \\ &\geq \int_0^T [(\operatorname{rot} w, \varphi - u) + \langle f, \varphi - u \rangle]. \end{aligned} \tag{3.39}$$

The inequality (2.4) is obtained in a similar way, observing that

$$\begin{aligned} - \liminf \int_0^T |\operatorname{div} w_\varepsilon(t)|^2 &= \limsup - \int_0^T |\operatorname{div} w_\varepsilon(t)|^2 \\ &\leq - \int_0^T |\operatorname{div} w(t)|^2, \end{aligned}$$

because $w_\varepsilon \to w$ weakly in $L^2(0, T; H_0^1(\Omega)^3)$. Theorem 2.4 is proved.

Proof of Theorem 2.5

In order to prove Theorem 2.5, we first prove the penalized Theorem 2.8. As in the proof of Theorem 2.7, we employ the Faedo-Galerkin Method.

Let $(\varphi_\nu)_{\nu\in\mathbb{N}}$ and $(\phi_\nu)_{\nu\in\mathbb{N}}$, V_m and $\widetilde{V}_m$ as in the proof of Theorem 2.7 and $u_{\epsilon_m}(t), w_{\epsilon_m}(t)$ solution of approximate penalized problem (3.1).

FIRST ESTIMATE

As in the proof of Theorem 2.7, omitting the parameters ϵ, ε and taking $\varphi_j = u_m,\ \phi_j = w_m$ in the approximate equation (3.1) we obtain

$$(u_m) \text{ is bounded in } L^\infty(0,T;H), \tag{3.40}$$

$$(u_m) \text{ is bounded in } L^2(0,T;V), \tag{3.41}$$

$$(u_m) \text{ is bounded in } L^4(0,T;V). \tag{3.42}$$

$$(w_m) \text{ is bounded in } L^\infty(0,T;L^2(\Omega)^3), \tag{3.43}$$

$$(w_m) \text{ is bounded in } L^2(0,T;H_0^1(\Omega)^3), \tag{3.44}$$

SECOND ESTIMATE

In both side of the equations (3.1) we take the derivatives with respect t and consider $\varphi_j = u'_m(t),\ \phi_j = w'_m(t)$. We obtain

$$\begin{aligned} & (u''_m(t), u'_m(t)) + (\nu_0+\nu_1\|u_m(t)\|^2)a(u'_m(t), u'_m(t)) + 2\nu_1 a(u'_m(t), u_m(t))^2 \\ + \ & b(u'_m(t), u_m(t), u'_m(t)) + \frac{1}{\epsilon}((\beta u_m(t))', u'_m(t)) = (\operatorname{rot} w'_m(t), u'_m(t)) \\ + \ & (f'(t), u'_m(t)) \end{aligned} \tag{3.45}$$

$$\begin{aligned} & (w''_m(t), w'_m(t)) + a(w'_m(t), w'_m(t)) \ + b(u'_m(t), w_m(t), w'_m(t)) \\ + \ & (\operatorname{div} w'_m(t), \operatorname{div} w'_m(t)) + \frac{1}{\epsilon}((\widetilde{\beta} w_m(t))', w'_m(t)) = (\operatorname{rot} u'_m(t), w'_m(t)) \\ + \ & (g'(t), w'_m(t)) \end{aligned}$$

We note that

$$\begin{aligned} & u'_m(0) \longrightarrow u_1 \text{ strongly in } H, \\ & w'_m(0) \longrightarrow w_1 \text{ strongly in } L^2(\Omega)^2. \end{aligned} \tag{3.46}$$

Indeed, (3.46) is obtained using $(3.1)_{1,2}$ with $t = 0$ and (2.7). Note that

$\beta u_0 = \widetilde{\beta} w_0 = 0$. Then

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}|u'_m(t)|^2 + \left(\nu_0 + \nu_1\|u_m(t)\|^2\right) a(u'_m(t), u'_m(t)) + \\
+ \quad & 2\nu_1 a(u'_m(t), u_m(t))^2 + b(u'_m(t), u_m(t), u'_m(t)) \\
+ \quad & \frac{1}{\epsilon}\langle(\beta u_m(t))', u'_m(t)\rangle = (\operatorname{rot} w'_m(t)), u'_m(t)) + (f'(t), u'_m(t), \qquad (3.47) \\
& \frac{1}{2}\frac{d}{dt}|w'_m(t)|^2 + \|u'_m(t)\|^2 + |\operatorname{div} w'_m(t)|^2 + b(u'_m(t), w_m(t), w'_m(t)) \\
+ \quad & \frac{1}{\epsilon}\langle(\widetilde{\beta} u_m(t))', u'_m(t)\rangle = (\operatorname{rot} u'_m(t)), w'_m(t)) + (g'(t), u'_m(t)
\end{aligned}
$$

Remark 3.3. The derivative with respect to t of $(\beta(v(t)), w)$ is only formal. The correct method is to consider the difference equation in $t + h$ and t, divided by h and take the limits when $h \to 0$. Here is fundamental the operator β to be monotonous. This justify the formal procedure of taking the derivative with respect to t, on both sides of (3.39) and take $v = u'_m(t)$. See Brezis [1], Browder [3] or Lions [7] for details.

Remark 3.4. We have to $n \leq 4$ that $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$ with continuous embedding.

Therefore,

$$|b(u, v, w)| \leq \|u\|_{(L^4(\Omega))^n}\|v\|_{(L^4(\Omega))^n}\|w\|. \qquad (3.48)$$

On the other hand, when $n = 2$ (see Lions [6] p. 220) we have

$$\|v\|_{L^4(\Omega)} \leq C\|v\|^{1/2}|v|^{1/2} \quad \forall v \in H_0^1(\Omega). \qquad (3.49)$$

It follows from (3.48) and (3.49) that

$$
\begin{aligned}
&|b(u'_m(t), u_m(t), u'_m(t))| \leq C\|u'_m(t)\|^{\frac{1}{2}}|u'_m(t)|^{\frac{1}{2}}\|u_m(t)\|^{\frac{1}{2}}|u_m(t)|^{\frac{1}{2}}\|u'_m(t)\|, \\
&|b(u'_m(t), w_m(t), w'_m(t))| \leq C\|u'_m(t)\|^{\frac{1}{2}}|u'_m(t)|^{\frac{1}{2}}\|w_m(t)\|^{\frac{1}{2}}|w_m(t)|^{\frac{1}{2}}\|w'_m(t)\|
\end{aligned} \qquad (3.50)
$$

It follows from (3.47)-(3.50) that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|u_m'(t)|^2+\nu_0\|u_m'(t)\|^2\le\frac{\nu_0}{10}\|u_m'(t)\|^2+C_{\nu_0}|w_m'(t)|^2+\frac{1}{2}|f'(t)|^2\\&+\quad\frac{1}{2}|u_m'(t)|^2+\frac{\nu_0}{10}\|u_m'(t)\|^2+C_{\nu_0}|u_m'(t)|^2+\frac{\nu_0}{10}\|u_m'(t)\|^2\\&+\quad C_{\nu_0}\|u_m(t)\|^2+\frac{\nu_0}{10}\|u_m'(t)\|^2+C_{\nu_0}|u_m'(t)|^2,\end{aligned}\tag{3.51}$$

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|w_m'(t)|^2+\|w_m'(t)\|^2+|\operatorname{div}w_m'(t)|\le\frac{\nu_0}{10}\|u_m'(t)\|^2+C_{\nu_0}|w_m'(t)|^2\\&+\quad\frac{1}{2}|g'(t)|^2+\frac{1}{2}\|w_m'(t)\|^2+\frac{\nu_0}{10}\|u_m'(t)\|^2+C_{\nu_0}|u_m'(t)|^2\\&+\quad\frac{1}{2}\|w_m(t)\|^2+\frac{1}{4}\|w_m'(t)\|^2+\frac{1}{4}\|w_m'(t)\|^2+C|w_m(t)|^2.\end{aligned}$$

because $\nu_1\|u_m(t)\|^2a(u_m'(t),u_m'(t))\ge 0$, and $((\beta u_m)',u_m')\ge 0$ (see J. L. Lions [6] p. 399), $(\operatorname{div}w_m'(t),u_m'(t))=(w_m'(t),\operatorname{div}u_m'(t))$ and $|\operatorname{div}u_m'(t)|=|\nabla u_m'(t)|$. Adding the above equations and integrating from 0 to t, with $0\le t\le T$, we conclude using (3.41) and (3.44) that

$$\begin{aligned}&|u_m'(t)|^2+|w_m'(t)|^2+\frac{2\nu_0}{5}\int_0^t\|u_m'(s)\|^2ds+\frac{1}{2}\int_0^t\|w_m'(s)\|^2ds\\&\qquad\le C+C\int_0^T|u_m'(t)|^2+|w_m'(t)|^2dt.\end{aligned}\tag{3.52}$$

Therefore, using Gronwall's inequality in (3.52), we obtain

$$(u_m')\text{ is bounded in }L^2(0,T;V),\tag{3.53}$$

$$(u_m')\text{ is bounded in }L^\infty(0,T;H),\tag{3.54}$$

$$(w_m')\text{ is bounded in }L^\infty(0,T;L^2(\Omega)^2),\tag{3.55}$$

$$(w_m')\text{ is bounded in }L^2(0,T;H_0^1(\Omega)^2),\tag{3.56}$$

To finish the proof of Theorem 2.8, we use the same argument used in the proof of Theorem 2.7. Theorem 2.8 is proved.

We shall now prove Theorem 2.5.

From the convergences (3.40)-(3.44), (3.53)-(3.56) and Banach-Steinhauss theorem, it follows that there exists a subnet $(u_\epsilon)_{0<\epsilon<1}$, such that it converges to u as $\epsilon \to 0$, in the sense of (3.40)-(3.42), (3.53)-(3.44). This function satisfies (2.9)-(2.10). Using the same arguments used in Theorem 2.4 we obtain that $\beta u = \widetilde{\beta} w = 0$. Therefore, u, w satisfy (2.10) of Theorem 2.5.

We need to show only that u is a solution of inequality $(2.11)_1$ a.e. in t. In fact, we have that u_ϵ satisfy

$$\begin{aligned} &(u'_\epsilon, \widehat{v}) + \nu_0 a(u_\epsilon, \widehat{v}) + \nu_1 \|u_\epsilon\|^2 a(u_\epsilon, \widehat{v}) + b(u_\epsilon, u_\epsilon, \widehat{v}) \\ &+ \frac{1}{\epsilon}(\beta u_\epsilon, \widehat{v}) = (f, \widehat{v}), \ \forall\, \widehat{v} \in V \\ &u_\epsilon(0) = u_0. \end{aligned} \tag{3.57}$$

Then from (3.57), with $\widehat{v} = v - u_\epsilon$, $v \in K$, we have

$$\begin{aligned} &(u'_\epsilon, v - u_\epsilon) + \nu_0 a(u_\epsilon, v) + b(u_\epsilon, u_\epsilon, v) + \nu_1 \langle \mathcal{A} u_\epsilon, v - u_\epsilon \rangle \\ &-(\operatorname{rot} w_\epsilon, v - u_\epsilon) - (f, v - u_\epsilon) \geq \nu_0 a(u_\epsilon, u_\epsilon), \ \forall v \in K, \end{aligned} \tag{3.58}$$

because $(\beta u_\epsilon - \beta v, u_\epsilon - v) \geq 0$. Let us denote

$$\begin{aligned} X_\epsilon^v &= (u'_\epsilon, v - u_\epsilon) + \nu_0 a(u_\epsilon, v) + b(u_\epsilon, u_\epsilon, v) + \nu_1 \langle \mathcal{A} u_\epsilon, v - u_\epsilon \rangle \\ &- (\operatorname{rot} w_\epsilon, v - u_\epsilon) - (f, v - u_\epsilon). \end{aligned}$$

We obtain

$$X_\epsilon^v \geq \nu_0 a(u_\epsilon, u_\epsilon), \forall v \in V. \tag{3.59}$$

Let $\psi \in C^0([0,T])$ with $\psi(t) \geq 0$. Then $v\psi \in C^0([0,T]; V)$ for all $v \in V$. It follows from (3.59) that

$$\begin{aligned} &\int_0^T \psi (u'_\epsilon, v - u_\epsilon)\, dt + \nu_0 \int_0^T \psi a(u_\epsilon, v)\, dt + \int_0^T \psi b(u_\epsilon, u_\epsilon, v)\, dt \\ &+\nu_1 \int_0^T \psi \langle \mathcal{A} u_\epsilon, v - u_\epsilon \rangle\, dt - \int_0^T \psi (\operatorname{rot} w_\epsilon, v - u_\epsilon)\, dt \\ &- \int_0^T \psi (f, v - u_\epsilon)\, dt \geq \nu_0 \int_0^T \psi a(u_\epsilon, u_\epsilon)\, dt. \end{aligned} \tag{3.60}$$

Taking lim sup in both side of inequality (3.60) we obtain

$$\int_0^T \psi(u', v-u)\,dt + \nu_0 \int_0^T \psi a(u,v)\,dt$$

$$-\int_0^T \psi b(u,u,v)\,dt + \limsup -\nu_1 \int_0^T \psi\langle \mathcal{A}u_\epsilon, u_\epsilon - v\rangle\,dt \tag{3.61}$$

$$-\int_0^T \psi(\operatorname{rot} w, v-u)\,dt - \int_0^T \psi(f, v-u)\,dt \geq \nu_0 \int_0^T \psi a(u,u)\,dt.$$

Using the same arguments as in the proof of Theorem 2.4 we have that

$$\limsup \int_0^T \psi\langle \mathcal{A}u_\epsilon, u_\epsilon - u\rangle\,dt \leq 0.$$

It follows of pseudo-monotony of operator $\mathcal{A}$ that

$$\liminf \int_0^T \psi\langle \mathcal{A}u_\epsilon, u_\epsilon - v\rangle\,dt \geq \int_0^T \psi\langle \mathcal{A}u, u - v\rangle\,dt \;\forall v \in V,$$

that is,

$$-\liminf \int_0^T \psi\langle \mathcal{A}u_\epsilon, u_\epsilon - v\rangle\,dt \leq \int_0^T \psi\langle \mathcal{A}u, v - u\rangle\,dt. \tag{3.62}$$

Therefore,

$$\limsup -\int_0^T \psi\langle \mathcal{A}u_\epsilon, u_\epsilon - v\rangle\,dt \leq \int_0^T \psi\langle \mathcal{A}u, v - u\rangle\,dt. \tag{3.63}$$

From (3.61) and (3.63), we obtain finally

$$\begin{aligned}&(u', v-u) + \nu_0 a(u, v-u) + b(u,u,v-u) + \nu_1\langle \mathcal{A}u, v-u\rangle \\ &\geq (\operatorname{rot} w, v-u) + (f, v-u) \quad \forall v \in K, \quad \text{a.e. in } \; t.\end{aligned} \tag{3.64}$$

Analogously we prove $(2.11)_2$. Theorem 2.5 is proved

4 Uniqueness

We now prove that when $n = 2$ we have uniqueness in Theorem 2.5. Indeed, suppose that $\{u_1, w_1\}$ and $\{u_2, w_2\}$ are two solutions of (2.11) and set $u = u_2 - u_1, w = w_2 - w_1$ and $t \in (0, T)$. Taking $v = u_1$ (resp.u_2) in the inequality (2.11) relative to v_2 (resp. v_1), analogously to $\widetilde{v}$ and adding up the results we obtain

$$\begin{aligned} &\frac{1}{2}\int_0^t \frac{d}{dt}|u(t)|^2\,dt + \nu_0 \int_0^t \|u(t)\|^2\,dt \\ &\leq \int_0^t |(\operatorname{rot} w, u)|\,dt + \int_0^t |b(u, u_2, u)|\,dt, \\ &\frac{1}{2}\int_0^t \frac{d}{dt}|w(t)|^2\,dt + \int_0^t \|w(t)\|^2\,dt \\ &\leq \int_0^t |(\operatorname{rot} u, w)|\,dt + \int_0^t |b(u, w_2, w)|\,dt \end{aligned} \tag{4.1}$$

because $\langle \mathcal{A}u_2 - \mathcal{A}u_1, u_2 - u_1\rangle \geq 0$ and $b(u_2, u_2, u) - b(u_1, u_1, u) = b(u, u_2, u)$. In the other hand, if $n = 2$, we have (see Lions [6], p. 70)

$$\begin{aligned} &|b(u(t), u_2(t), u(t))| \leq C\|u(t)\||u(t)|\|u_2(t)\|, \\ &|b(u(t), w_2(t), w(t))| \leq C\|u(t)\|^{\frac{1}{2}}|u(t)|^{\frac{1}{2}}\|w(t)^{\frac{1}{2}}|w(t)|^{\frac{1}{2}}\|w_2\| \end{aligned} \tag{4.2}$$

It follows from (4.1) and (4.2) that

$$\begin{aligned} |u(t)|^2 \quad &+ \quad |w(t)|^2 + \int_0^t \|u(t)\|^2\,dt + \int_0^t \|w(t)\|^2\,dt \\ &\leq \quad C\int_0^t (\|u_2(t)\|^2 + \|w_2(t)\|^2)(|u(t)|^2 + |w(t)|^2)\,dt. \end{aligned}$$

This implies, using Gronwall's inequality that $u(t) = w(t) = 0$, because $u_2 \in L^2(0, T; V)$ and $w_2 \in L^2(0, T; H_0^1(\Omega)^2)$. Uniqueness is proved.

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