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Internal exact-approximate controllability for the thermoelastic Bresse system

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Abstract. The goal of this work is to obtain the exact-approximate controllability for the thermoelastic Bresse system, posed on a bounded interval $(0, L)$, with controls acting in (l_1, l_2) with $(l_1, l_2) \subset (0, L)$. The controls are obtained by minimizing a functional J associated to the thermoelastic Bresse system as in [1, 4]. The so called observability inequality is an important property to obtain the controllability of the system. In order to obtain such a inequality we proceed as in [1], [4] and [11]. The main result is obtained as in [1] and [4].

Keywords: Thermoelastic Bresse system, exact controllability, approximate controllability, observability inequality.

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1 Introduction

In this work, we will obtain the exact-approximate controllability for the following thermoelastic Bresse system, posed on a bounded interval $(0, L)$:

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$$
\begin{cases}
\rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - k_0 l(w_x - l\varphi) = f_1\chi_{(l_1,l_2)}, \\
in\ (0,L)\times(0,T) \\
\rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw) + \gamma\theta_x = f_2\chi_{(l_1,l_2)}, \\
in\ (0,L)\times(0,T) \\
\rho_1 w_{tt} - k_0(w_x - l\varphi)_x + kl(\varphi_x + \psi + lw) = f_3\chi_{(l_1,l_2)}, \\
in\ (0,L)\times(0,T) \\
\theta_t - k_1\theta_{xx} + m\psi_{xt} = 0, \quad in \quad (0,L)\times(0,T) \\
\varphi(0,t) = \varphi(L,t) = \psi(0,t) = \psi(L,t) = w(0,t) = w(L,t) \\
= \theta(0,t) = \theta(L,t) = 0, \quad t \in (0,T) \\
\varphi(.,0) = \varphi_0, \quad \varphi_t(.,0) = \varphi_1, \quad in \quad (0,L) \\
\psi(.,0) = \psi_0, \quad \psi_t(.,0) = \psi_1, \quad in \quad (0,L) \\
w(.,0) = w_0, \quad w_t(.,0) = w_1, \quad in \quad (0,L) \\
\theta(.,0) = \theta_0, \quad in \quad (0,L),
\end{cases}
\tag{1.1}
$$

where $\rho_1, \rho_2, k, b, k_0, l, \gamma, k_1, m$ are positive constants, and $\rho_1, \rho_2, k, b, k_0$ are related to the composition of the material, w, φ and ψ denote the longitudinal, vertical, and shear angle displacements, respectively (see [11]). The non homogeneous terms f_1, f_2 and f_3 will play the role of control inputs.

The exact approximate control problem, consists in finding a Hilbert space $\mathcal{H}$, ($\mathcal{H} = H_0^1(0,L) \times L^2(0,L) \times H_0^1(0,L) \times L^2(0,L) \times H_0^1(0,L) \times L^2(0,L) \times L^2(0,L)$) such that for each initial and final data $(\varphi_0, \varphi_1, \psi_0, \psi_1, w_0, w_1, \theta_0), (\Phi_0, \Phi_1, \Psi_0, \Psi_1, W_0, W_1, \eta_0) \in \mathcal{H}$ and $\varepsilon > 0$, one can find controls f_1, f_2, f_3, such that, the solution of (1.1) satisfies

$$
\begin{aligned}
&\varphi(T) = \Phi_0, \quad \varphi_t(T) = \Phi_1, \\
&\psi(T) = \Psi_0, \quad \psi_t(T) = \Psi_1, \\
&w(T) = W_0, \quad w_t(T) = W_1, \\
&|\theta(T) - \eta_0|_{L^2(0,L)} \le \varepsilon.
\end{aligned}
\tag{1.2}
$$

To obtain such result, we proceed as in [1], [4] and [11]. More precisely, the proof the of exact-approximate controllability consists in finding an observability inequality, given in Proposition 2.7, in order to minimize the functional J associated with the thermoelastic Bresse system, as it was done in [1] and [4].

We will organize the paper as follows: in Section 2 we will present some important results obtained in [1, 7, 8, 9, 10, 12], in Section 2.1 we will show uniqueness results [1, 3, 6] and in Section 2.2 we will prove the observability inequality and the internal controllability result [1, 4, 5, 6, 11].

2 Main results

Considering the thermoelastic Bresse system (1.1), the transposed problem is

$$\begin{cases} \rho_1 u_{tt} - k(u_x + v + lz)_x - k_0 l[z_x - lu] = 0, \quad in \quad (0,L)\times(0,T) \\ \rho_2 v_{tt} - bv_{xx} + k(u_x + v + lz) + mp_{xt} = 0, \quad in \quad (0,L)\times(0,T) \\ \rho_1 z_{tt} - k_0[z_x - lu]_x + kl(u_x + v + lz) = 0, \quad in \quad (0,L)\times(0,T) \\ -p_t - k_1 p_{xx} - \gamma v_x = 0, \quad in \quad (0,L)\times(0,T) \\ u(0,t) = u(L,t) = v(0,t) = v(L,t) = z(0,t) = z(L,t) \\ = p(0,t) = p(L,t) = 0, \quad t \in (0,T) \\ u(.,T) = u_0, \quad u_t(.,T) = u_1, \quad in \quad (0,L) \\ v(.,T) = v_0, \quad v_t(.,T) = v_1, \quad in \quad (0,L) \\ z(.,T) = z_0, \quad z_t(.,T) = z_1, \quad in \quad (0,L) \\ p(.,T) = p_0, \quad in \quad (0,L), \end{cases} \tag{2.1}$$

with $(u_0, u_1, v_0, v_1, z_0, z_1, p_0) \in L^2(0,L)\times H^{-1}(0,L)\times L^2(0,L)\times H^{-1}(0,L)\times L^2(0,L)\times H^{-1}(0,L)\times L^2(0,L)$. Making the change of variables

$$U(x,t) = -\int_t^T u(x,s)ds + \chi_1(x),$$

$$V(x,t) = -\int_t^T v(x,s)ds + \chi_2(x),$$

$$Z(x,t) = -\int_t^T z(x,s)ds + \chi_3(x),$$

with χ_1, χ_2 and χ_3 solution of

$$\begin{cases} -k(\chi_{1x}+\chi_2+l\chi_3)_x + k_0 l(\chi_{3x}-l\chi_1) = -\rho_1 u_1, \\ -b\chi_{2xx} + k(\chi_{1x}+\chi_2+l\chi_3) = -\rho_2 v_1 - m p_{0x}, \\ -k_0(\chi_{3x}-l\chi_1)_x + kl(\chi_{1x}+\chi_2+l\chi_3) = 0, \\ \chi_1(0)=\chi_1(L)=\chi_2(0)=\chi_2(L)=\chi_3(0)=\chi_3(L)=0, \end{cases} \tag{2.2}$$

it follows that U, V and Z satisfy

$$\begin{cases} \rho_1 U_{tt} - k(U_x+V+lZ)_x - k_0 l[Z_x - lU] = 0, \quad in \quad (0,L)\times(0,T) \\ \rho_2 V_{tt} - bV_{xx} + k(U_x+V+lZ) + m p_x = 0, \quad in \quad (0,L)\times(0,T) \\ \rho_1 Z_{tt} - k_0[Z_x - lU]_x + kl(U_x+V+lZ) = 0, \quad in \quad (0,L)\times(0,T) \\ -p_t - k_1 p_{xx} - \gamma V_{xt} = 0, \quad in \quad (0,L)\times(0,T) \\ U(.,T)=\chi_1, \quad U_t(.,T)=u_0, \quad in \quad (0,L) \\ V(.,T)=\chi_2, \quad V_t(.,T)=v_0, \quad in \quad (0,L) \\ Z(.,T)=\chi_3, \quad Z_t(.,T)=z_0, \quad in \quad (0,L) \\ p(.,T)=p_0, \quad in \quad (0,L). \end{cases} \tag{2.3}$$

System (2.2) has a unique solution in $(H_0^1(0,L))^3$ and

$$\|(\chi_1,\chi_2,\chi_3)\|_{(H_0^1(0,L))^3} \approx \|(\rho_1 u, \rho_2 v_1 + m p_{0x}, \rho_1 z_1)\|_{(H^{-1}(0,L))^3}.$$

The well-posedness for the previous systems were obtained in [1] and [4].

Theorem 2.1. *Let P be the orthogonal projection from $L^2(0,L)$ into $F = \{\Psi_x \in H_0^1(0,L)\}$ and let us denote by $\{S^0(t)\}_{t\geq 0}$ the strongly continuous* *semigroup in $\mathcal{H}$, associated to the following decoupled system*

$$\begin{cases} \rho_1\widetilde{\varphi}_{tt} - k(\widetilde{\varphi}_x + \widetilde{\psi} + l\widetilde{w})_x - k_0 l[\widetilde{w}_x - l\widetilde{\varphi}] = 0, \quad in \quad (0,L)\times(0,T) \\ \rho_2\widetilde{\psi}_{tt} - b\widetilde{\psi}_{xx} + k(\widetilde{\varphi}_x + \widetilde{\psi} + l\widetilde{w}) + \frac{m\gamma}{k_1}P\widetilde{\psi}_t = 0, \quad in \quad (0,L)\times(0,T) \\ \rho_1\widetilde{w}_{tt} - k_0[\widetilde{w}_x - l\widetilde{\varphi}]_x + kl(\widetilde{\varphi}_x + \widetilde{\psi} + l\widetilde{w}) = 0, \quad in \quad (0,L)\times(0,T) \\ \widetilde{\theta}_t - k_1\widetilde{\theta}_{xx} + m\widetilde{\psi}_{xt} = 0, \quad in \quad (0,L)\times(0,T) \\ \widetilde{\varphi}(0,t) = \widetilde{\varphi}(L,t) = \widetilde{\psi}(0,t) = \widetilde{\psi}(L,t) = \widetilde{w}(0,t) = \widetilde{w}(L,t) \\ = \widetilde{\theta}(0,t) = \widetilde{\theta}(L,t) = 0, \quad t\in(0,T) \\ \widetilde{\varphi}(.,0) = \varphi_0, \quad \widetilde{\varphi}_t(.,0) = \varphi_1, \quad in \quad (0,L) \\ \widetilde{\psi}(.,0) = \psi_0, \quad \widetilde{\psi}_t(.,0) = \psi_1, \quad in \quad (0,L) \\ \widetilde{w}(.,0) = w_0, \quad \widetilde{w}_t(.,0) = w_1, \quad in \quad (0,L) \\ \widetilde{\theta}(.,0) = \theta_0, \quad in \quad (0,L). \end{cases} \tag{2.4}$$

Then, $S(t) - S^0(t) : \mathcal{H} \to C([0,T];\mathcal{H})$ is continuous and compact, where $P\widetilde{\psi}_t = \widetilde{\psi}_t - \dfrac{1}{L}\displaystyle\int_0^L \widetilde{\psi}_t dx$.

Proof. See [4] pages 109-117. □

Theorem 2.1 will be used to obtain the controllability of the system (1.1).

2.1 Uniqueness Results

In this section, we will present some important results to obtain the controllability of the thermoelastic Bresse system.

Lemma 2.2. *Suppose that the solution (φ,ψ,w,θ) of the homogeneous problem (1.1) satisfies $(\varphi,\psi,w,\theta) = (c_1,c_2,c_3,c_4)$ in $(l_1,l_2)\times(0,T)$, where c_1,c_2,c_3,c_4 are constant, then $(\varphi,\psi,w,\theta) = (c_1,c_2,c_3,c_4)$ in $(0,L)\times(0,T)$.*

Proof. Without loss of generality, we may assume that $c_1 = c_2 = c_3 = c_4 = 0$. For $\alpha \in \mathbb{Z}^2, \alpha = (\alpha_1,\alpha_2)$ with $|\alpha| = m$, we denote $\dfrac{\partial^m}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}}$. In Schwartz notation, the general form of a m order linear system of N differential equation in N unknowns takes the simple form

$$\sum_{|\alpha|\le m} A_\alpha(x)\partial^\alpha u = B(x),$$

where u and B are column vectors with N components and A_α are $N \times N$ square matrices.

Let $X = [\varphi, \psi, w, \theta]$ be a column vector. The main part of the homogeneous system (1.1) is given by $A_{(2,0)}\partial^{(2,0)}X + A_{(1,1)}\partial^{(1,1)}X + A_{(0,2)}\partial^{(0,2)}X$ with

$$A_{(0,2)} = \begin{pmatrix} \rho_1 & 0 & 0 & 0 \\ 0 & \rho_2 & 0 & 0 \\ 0 & 0 & \rho_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad A_{(1,1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 \end{pmatrix}$$

and

$$A_{(2,0)} = \begin{pmatrix} -k & 0 & 0 & 0 \\ 0 & -b & 0 & 0 \\ 0 & 0 & -k_0 & 0 \\ 0 & 0 & 0 & -k_1 \end{pmatrix}.$$

Therefore its characteristic matrix is

$$\Lambda((\xi, \eta, \tau)) = \begin{pmatrix} \rho_1\xi^2 - k\tau^2 & 0 & 0 & 0 \\ 0 & \rho_2\xi^2 - b\tau^2 & 0 & 0 \\ 0 & 0 & \rho_1\xi^2 - k_0\tau^2 & 0 \\ 0 & \eta m & 0 & -k_1\tau^2 \end{pmatrix}$$

and the principal form of the operator is given by

$$\begin{aligned} Q(\xi, \eta, \tau) &= \det(\Lambda((\xi, \eta, \tau))) \\ &= (\rho_1\xi^2 - k\tau^2)(\rho_2\xi^2 - b\tau^2)(\rho_1\xi^2 - k_0\tau^2)(-k_1\tau^2). \end{aligned}$$

The line $\Pi = \{(x,t) \in \mathbb{R} \times \mathbb{R} : \tau r + \xi t = C\}$ is characteristic with respect to homogeneous system (1.1) if and only if

$$\tau^2 = 0$$
$$\rho_1\xi^2 - k\tau^2 = 0, \quad \tau = \pm\sqrt{\frac{\rho_1}{k}}\xi$$
$$\rho_2\xi^2 - b\tau^2 = 0, \quad \tau = \pm\sqrt{\frac{\rho_2}{b}}\xi$$
$$\rho_1\xi^2 - k_0\tau^2 = 0, \quad \tau = \pm\sqrt{\frac{\rho_1}{k_0}}\xi.$$

In consequence, the characteristic lines of the system are

$$\begin{cases} t = C, \\ t \pm \sqrt{\frac{k}{\rho_1}} r = C, \\ t \pm \sqrt{\frac{b}{\rho_2}} r = C, \\ t \pm \sqrt{\frac{k_0}{\rho_1}} r = C. \end{cases} \tag{2.5}$$

By Holmgren's uniqueness theorem (see [3]) $(\varphi, \psi, w, \theta) = (0, 0, 0, 0)$ in $(0, L) \times (0, T)$. □

Corollary 2.3. *Suppose that* $(\varphi, \psi, w) = (0, 0, 0)$ *in* $(l_1, l_2) \times (0, T)$*. Then, there exists some* $C \in \mathbb{R}$ *such that* $(\varphi, \psi, w, \theta) = (0, 0, 0, C)$ *in* $(0, L) \times (0, T)$.

Proof. In fact, if $(\varphi, \psi, w) = (0, 0, 0)$ in $(l_1, l_2) \times (0, T)$, from the homogeneous system $(1.1)_2$, $\theta_x = 0$ in $(l_1, l_2) \times (0, T)$ and $\theta_t = 0$ in $(l_1, l_2) \times (0, T)$. Then we have that $\theta = C$ in $(l_1, l_2) \times (0, T)$, where C is a constant that does not depend on x neither t. From Lemma 2.2 we deduce that $(\varphi, \psi, w, \theta) = (0, 0, 0, C)$ in $(0, L) \times (0, T)$. □

Proposition 2.4. *Suppose that* $T > 2\alpha R$*. If the solution of the homogeneous system (1.1),* $(\varphi, \psi, w, \theta)$*, is such that* $(\varphi, \psi, w) = (0, 0, 0)$ *in* $(l_1, l_2) \times (0, T)$*, then* $(\varphi, \psi, w, \theta) = (0, 0, 0, 0)$ *in* $(0, L) \times (0, T)$*. Here* $R := \max\{l_1, L - l_2\}$ *and* $\alpha := \max\{1, \frac{\rho_1}{k}, \frac{\rho_2}{b}, \frac{\rho_1}{k_0}\}$.

Proof. In fact, from Corollary 2.3 we have that $(\varphi, \psi, w, \theta) = (0, 0, 0, C)$ in $(0, L) \times (0, T)$ and since $\theta(0, .) = \theta(L, .) = 0$ in $(0, T)$, then $C = 0$. Therefore, $(\varphi, \psi, w, \theta) = (0, 0, 0, 0)$ in $(0, L) \times (0, T)$. □

Proposition 2.5. *Suppose that* $T > 2\alpha R$*. If* (u, v, z, p) *is a solution of problem (2.1) such that* $(u, v, z) = (0, 0, 0)$ *in* $(l_1, l_2) \times (0, T)$*, then* $(u, v, z, p) = (0, 0, 0, 0)$ *in* $(0, L) \times (0, T)$.

Proof. Making a time reversal in (2.1), $u(x, t) = \widetilde{u}(x, T - t)$, $v(x, t) = \widetilde{v}(x, T - t)$, $z(x, t) = \widetilde{z}(x, T - t)$, $p(x, t) = \widetilde{p}(x, T - t)$, differentiating $(2.1)_4$

with respect to t and letting $\widetilde{p}_t = \widetilde{\eta}$ we deduce that

$$\begin{cases} \rho_1 \widetilde{u}_{tt} - k(\widetilde{u}_x + \widetilde{v} + l\widetilde{z})_x - k_0 l[\widetilde{z}_x - l\widetilde{u}] = 0, & in \quad (0,L)\times(0,T) \\ \rho_2 \widetilde{v}_{tt} - b\widetilde{v}_{xx} + k(\widetilde{u}_x + \widetilde{v} + l\widetilde{z}) + m\widetilde{\eta}_x = 0, & in \quad (0,L)\times(0,T) \\ \rho_1 \widetilde{z}_{tt} - k_0[\widetilde{z}_x - l\widetilde{u}]_x + kl(\widetilde{u}_x + \widetilde{v} + l\widetilde{z}) = 0, & in \quad (0,L)\times(0,T) \\ \eta_t - k_1\eta_{xx} + \gamma v_{xt} = 0, & in \quad (0,L)\times(0,T). \end{cases} \tag{2.6}$$

As $(u,v,z) = (0,0,0)$ in $(l_1,l_2)\times(0,T)$, then by the definition of $(\widetilde{u},\widetilde{v},\widetilde{z})$ we have that $(\widetilde{u},\widetilde{v},\widetilde{z}) = (0,0,0)$ in $(l_1,l_2)\times(0,T)$ and, by Proposition 2.4, we deduce that $(\widetilde{u},\widetilde{v},\widetilde{z},\widetilde{\eta} = \widetilde{p}_t) = (0,0,0,0)$ in $(0,L)\times(0,T)$. Therefore, $\widetilde{p} = \widetilde{p}(x)$ and by (2.1) we have $p_{xx} = 0$ in $(0,L)\times(0,T)$. So, $p = Cx$ and by the fact $p(0,.) = p(L,.) = 0$ in $(0,T)$ we have $C = 0$, and therefore $p = 0$ in $(0,L)\times(0,T)$. □

2.2 Observability inequality and internal control

Consider the problem

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi + lw)_x - k_0 l[w_x - l\varphi] = 0, & in \quad (0,L)\times(0,T) \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + lw) + \frac{m\gamma}{k_1}P\psi_t = 0, & in \quad (0,L)\times(0,T) \\ \rho_1 w_{tt} - k_0[w_x - l\varphi]_x + kl(\varphi_x + \psi + lw) = 0, & in \quad (0,L)\times(0,T) \\ \varphi(0,t) = \varphi(L,t) = \psi(0,t) = \phi(L,t) = w(0,t) = w(L,t) = 0, \\ t \in (0,T) \\ \varphi(.,0) = \varphi_0, \quad \varphi_t(.,0) = \varphi_1, & in \quad (0,L) \\ \psi(.,0) = \psi_0, \quad \psi_t(.,0) = \psi_1, & in \quad (0,L) \\ w(.,0) = w_0, \quad w_t(.,0) = w_1, & in \quad (0,L), \end{cases} \tag{2.7}$$

with $P\psi_t = \psi_t - \dfrac{1}{L}\displaystyle\int_0^L \psi_t dx.$

Theorem 2.6 (Observability inequality). *For $T > 2\alpha R$, there exists a positive constant, $C > 0$, such that the weak solution of (2.7) satisfies*

$$\begin{aligned} \|\{\varphi_0,\psi_0,\omega_0\}\|^2_{[H_0^1(0,L)]^3} + \|\{\varphi_1,\psi_1,\omega_1\}\|^2_{[L^2(0,L)]^3} \\ \leq C\int_0^T\int_{l_1}^{l_2}(\varphi_t^2 + \psi_t^2 + \omega_t^2)dxdt. \end{aligned} \tag{2.8}$$

Proof. See [4], pages 121-137, or [11]. □

Proposition 2.7. *For $T > 2\alpha R$ and for every bounded set $B \subset L^2(0,L)$ there exists $\delta = \delta(B) > 0$ such that,*

$$\delta \leq \int_0^T \int_{l_1}^{l_2} (|u|^2 + |v|^2 + |z|^2)\; dxdt \tag{2.9}$$

for any solution of (2.1) with initial data satisfying

$$\|((\rho_1 u_1, \rho_2 v_1 + m p_{0x}, \rho_1 z_1), (u_0, v_0, z_0))\|_{(H^{-1}(0,L))^3 \times (L^2(0,L))^3} \geq 1,$$

and $p_0 \in B$.

The previous proposition is equivalent to

Proposition 2.8. *For $T > 2\alpha R$ and for every bounded set B of $L^2(0,L)$ there exists $\delta = \delta(B) > 0$ such that,*

$$\delta \leq \int_0^T \int_{l_1}^{l_2} (|U_t|^2 + |V_t|^2 + |Z_t|^2)\; dx\; dt \tag{2.10}$$

for $\{U, V, Z, p\}$ solution of (2.3) with initial data satisfying

$$\|\{\chi_1, \chi_2, \chi_3\}, \{u_0, v_0, z_0\}\|_{H_0^1(0,L)^3 \times L^2(0,L)^3} \geq 1, \quad p_0 \in B. \tag{2.11}$$

In the proof of this proposition, we will use the decoupled system associated with system (2.3)

$$\left\{\begin{array}{l}
\rho_1 \widetilde{U}_{tt} - k(\widetilde{U}_x + \widetilde{V} + l\widetilde{Z})_x - k_0 l[\widetilde{Z}_x - l\widetilde{U}] = 0, \quad in \quad (0,L) \times (0,T) \\
\rho_2 \widetilde{V}_{tt} - b\widetilde{V}_{xx} + k(\widetilde{U}_x + \widetilde{V} + l\widetilde{Z}) - \frac{m\gamma}{k_1} P\widetilde{V}_t = 0, \quad in \quad (0,L) \times (0,T) \\
\rho_1 \widetilde{Z}_{tt} - k_0[\widetilde{Z}_x - l\widetilde{U}]_x + kl(\widetilde{U}_x + \widetilde{V} + l\widetilde{Z}) = 0, \quad in \quad (0,L) \times (0,T) \\
-\widetilde{p}_t - k_1 \widetilde{p}_{xx} - \gamma \widetilde{v}_{xt} = 0, \quad in \quad (0,L) \times (0,T) \\
\widetilde{U}(0,t) = \widetilde{U}(L,t) = \widetilde{V}(0,t) = \widetilde{V}(L,t) = \widetilde{Z}(0,t) = \widetilde{Z}(L,t) \\
= \widetilde{p}(0,t) = \widetilde{p}(L,t) = 0, \quad t \in (0,T) \\
\widetilde{U}(.,T) = \chi_1, \quad \widetilde{U}_t(.,T) = u_0, \quad in \quad (0,L) \\
\widetilde{V}(.,T) = \chi_2, \quad \widetilde{V}_t(.,T) = v_0, \quad in \quad (0,L) \\
\widetilde{Z}(.,T) = \chi_3, \quad \widetilde{Z}_t(.,T) = z_0, \quad in \quad (0,L) \\
\widetilde{p}(.,T) = p_0, \quad in \quad (0,L).
\end{array}\right. \tag{2.12}$$

Proof. Making a change in the time variable in (2.12) we obtain a system equivalent to (2.6) in the first three equations, and by (2.6) we obtain

$$\begin{aligned}\|\{\chi_1,\chi_2,\chi_3\}\|_{H^{-1}(0,L)^3} + \|\{u_0,v_0,z_0\}\|_{L^2(0,L)^3} \\ \le \int_0^T \int_{l_1}^{l_2} (|\widetilde{U}_t|^2 + |\widetilde{V}_t|^2 + |\widetilde{Z}_t|^2)\ dxdt.\end{aligned} \tag{2.13}$$

Decomposing $(U,V,Z,p) = (\widetilde{U},\widetilde{V},\widetilde{Z},\widetilde{p}) + (\varphi,\psi,w,\theta)$, we have

$$\left\{\begin{array}{l} \rho_1\varphi_{tt} - k(\varphi_x+\psi+lw)_x - k_0 l[w_x - l\varphi] = 0, \quad in \quad (0,L)\times(0,T) \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x+\psi+lw) = -\frac{m\gamma}{k_1}P\widetilde{V}_t - mp_x, \ in\ (0,L)\times(0,T) \\ \rho_1 w_{tt} - k_0[w_x - l\varphi]_x + kl(\varphi_x+\psi+lw) =, \quad in \quad (0,L)\times(0,T) \\ \theta_t - k_1\theta_{xx} + m\psi_{xt} = 0, \quad in \quad (0,L)\times(0,T) \\ \varphi(0,t) = \varphi(L,t) = \psi(0,t) = \psi(L,t) = w(0,t) = w(L,t) \\ = \theta(0,t) = \theta(L,t) = 0, \quad t\in(0,T) \\ \varphi(.,T) = \varphi_0, \quad \varphi_t(.,T) = \varphi_1, \quad in \quad (0,L) \\ \psi(.,T) = \psi_0, \quad \psi_t(.,T) = \psi_1, \quad in \quad (0,L) \\ w(.,T) = w_0, \quad w_t(.,T) = w_1, \quad in \quad (0,L) \\ \theta(.,T) = \theta_0, \quad in \quad (0,L), \end{array}\right. \tag{2.14}$$

and from (2.13),

$$\begin{aligned}\|\{\chi_1,\chi_2,\chi_3\}\|_{(H^{-1}(0,L))^3} + \|\{u_0,v_0,z_0\}\|_{(L^2(0,L))^3} \\ \le C\int_0^T \int_{l_1}^{l_2} (|U_t|^2 + |V_t|^2 + |Z_t|^2)\ dx\ dt \\ + \int_0^T \int_{l_1}^{l_2} (|\varphi_t|^2 + |\psi_t|^2 + |w_t|^2)\ dx\ dt.\end{aligned} \tag{2.15}$$

Suppose that (2.10) is false. Then there exists a bounded set $B \subset L^2(0,L)$ and a sequence of initial data $(\chi_1^j,\chi_2^j,\chi_3^j,u_0^j,v_0^j,z_0^j,p_0^j)$, with $p_0^j \in B$, satisfying (2.11) and such that

$$\int_0^T \int_{l_1}^{l_2} (|U_t^j|^2 + |V_t^j|^2 + |Z_t^j|^2)\ dx\ dt \to 0, \quad j\to\infty. \tag{2.16}$$

From (2.15) and (2.16) and since

$\|\{\chi_1,\chi_2,\chi_3\},\{u_0,v_0,z_0\}\|_{H_0^1(0,L)^3\times L^2(0,L)^3}\geq 1$, we deduce that

$$\liminf_{j\to\infty}[\int_0^T\int_{l_1}^{l_2}(|\varphi_t|^2+|\psi_t|^2+|w_t|^2)\;dx\;dt]>0. \tag{2.17}$$

We introduce the normalized data

$$(\widehat{\chi}_1^j,\widehat{\chi}_2^j,\widehat{\chi}_3^j,u_0^j,v_0^j,z_0^j,p_0^j)=\frac{(\widehat{\chi}_1^j,\widehat{\chi}_2^j,\widehat{\chi}_3^j,u_0^j,v_0^j,z_0^j,p_0^j)}{\|\varphi_t^j,\psi_t^j,w_t^j\|_{L^2(l_1,l_2;(0,T))}} \tag{2.18}$$

and $(\widehat{U}^j,\widehat{V}^j,\widehat{Z}^j,\widehat{p}_j)$, $(\widehat{\varphi}^j,\widehat{\psi}^j,\widehat{w}^j,\widehat{\theta}^j)$, solutions of (2.3) and (2.14) respectively. Then,

$$\begin{aligned}&\int_0^T\int_{l_1}^{l_2}|\widehat{\varphi}^j|+|\widehat{\psi}^j|+|\widehat{w}^j|\;dx\;dt=1,\quad\forall j\geq 1,\text{and}\\&\int_0^T\int_{l_1}^{l_2}|\widehat{U}^j|+|\widehat{V}^j|+|\widehat{Z}^j|\;dx\;dt\to 0.\end{aligned} \tag{2.19}$$

From (2.15) we deduce that

$$\|\{\widehat{\chi}_1^j,\widehat{\chi}_2^j,\widehat{\chi}_3^j\}\|_{H^{-1}(0,L)}+\|\{\widehat{u}_0^j,\widehat{v}_0^j,\widehat{z}_0^j\}\|_{L^2(0,L)}\leq C.$$

On the other hand, by (2.17) and taking into account that $p_0^j\in B$ we know that $\widehat{p}_0^j$ remains in a bounded set $\widehat{B}\subset L^2(0,L)$. By extracting a subsequence we deduce that

$$\begin{aligned}&((\widehat{\chi}_1^j,\widehat{\chi}_2^j,\widehat{\chi}_3^j),(\widehat{u}_0^j,\widehat{v}_0^j,\widehat{z}_0^j))\rightharpoonup((\widehat{\chi}_1,\widehat{\chi}_2,\widehat{\chi}_3),(\widehat{u}_0,\widehat{v}_0,\widehat{z}_0))\\&in\quad H_0^1(0,L)^3\times L^2(0,L)^3,\\&\widehat{p}_0^j\to\widehat{p}_0\quad in\quad L^2(0,L),\end{aligned} \tag{2.20}$$

and

$$(\widehat{\varphi}_t^j,\widehat{\psi}_t^j,\widehat{w}_t^j)\rightharpoonup(\widehat{\varphi}_t,\widehat{\psi}_t,\widehat{w}_t)\quad in\quad L^2((0,L)\times(0,T))^3, \tag{2.21}$$

$$(\widehat{U}_t^j,\widehat{V}_t^j,\widehat{Z}_t^j)\rightharpoonup(\widehat{U}_t,\widehat{V}_t,\widehat{Z}_t)\quad in\quad L^2((0,L)\times(0,T))^3, \tag{2.22}$$

where $(\widehat{u},\widehat{v},\widehat{z},\widehat{p})$, $(\widehat{U},\widehat{V},\widehat{Z},\widehat{p})$, $(\widehat{\varphi},\widehat{\psi},\widehat{w},\widehat{\theta})$ are solution of (2.1), (2.3) and (2.14) respectively ($\widehat{U}_t=\widehat{u},\widehat{V}_t=\widehat{v},\widehat{Z}_t=\widehat{z}$).

On the other hand (Theorem 2.1) $(\widehat{\varphi}_t^j, \widehat{\psi}_t^j, \widehat{w}_t^j)$ is compact in $C([0,T]; L^2(0,L))^3$ and therefore

$$(\widehat{\varphi}_t^j, \widehat{\psi}_t^j, \widehat{w}_t^j) \to (\widehat{\varphi}_t, \widehat{\psi}_t, \widehat{w}_t) \quad in \quad L^2((0,L)\times(0,T))^3. \tag{2.23}$$

From (2.19) and (2.22) we deduce that

$$\begin{array}{lll} \widehat{U}_t = \widehat{u} & in & (l_1,l_2)\times(0,T) \\ \widehat{V}_t = \widehat{v} & in & (l_1,l_2)\times(0,T) \\ \widehat{Z}_t = \widehat{z} & in & (l_1,l_2)\times(0,T), \end{array} \tag{2.24}$$

and from (2.24) and Proposition 2.5 we have

$$\widehat{u}_0 \equiv 0, \widehat{v}_0 \equiv 0, \widehat{z}_0 \equiv 0, \widehat{p}_0 \equiv 0, \widehat{u}_1 \equiv 0, \widehat{v}_1 \equiv 0, \widehat{z}_1 \equiv 0, \tag{2.25}$$

which implies that

$$(\widehat{\varphi}, \widehat{\psi}, \widehat{w}) = (0,0,0). \tag{2.26}$$

From (2.19) and (2.23) we have

$$\|(\widehat{\varphi}_t, \widehat{\psi}_t, \widehat{w}_t)\| = 1, \tag{2.27}$$

which contradicts (2.25) and (2.26). □

Given $(\Phi_0, \Phi_1, \Psi_0, \Psi_1, W_0, W_1, \eta_0) \in \mathcal{H} = H_0^1(0,L)\times L^2(0,L)\times H_0^1(0,L)\times L^2(0,L)\times H_0^1(0,L)\times L^2(0,L)\times L^2(0,L)$ and $\widetilde{H} = L^2(0,L)\times H^{-1}(0,L)\times L^2(0,L)\times H^{-1}(0,L)\times L^2(0,L)\times H^{-1}(0,L)\times L^2(0,L)$ we introduce the functional $J : \widetilde{H} \to \mathbb{R}$ defined as follows:

$$\begin{aligned} &J(u_0,u_1,v_0,v_1,z_0,z_1,p_0) = \frac{1}{2}\int_0^T\int_{l_1}^{l_2} (|u|^2+|v|^2+|z|^2)\,dx\,dt \\ &-\rho_1\int_0^L \Phi_1 u_0 dx - \rho_2\int_0^L \Psi_1 v_0 dx - \rho_1\int_0^L W_1 z_0 dx + \rho_1\langle \Phi_0, u_1\rangle \\ &+\rho_2\langle \Psi_0, v_1\rangle + \rho_1\langle W_0, z_1\rangle - \int_0^L (\eta_0 + m\Psi_x)p_0\,dx + \varepsilon\|p_0\|_{L^2(0,L)}, \end{aligned} \tag{2.28}$$

where $\langle,\rangle$ denotes the duality product between $H_0^1(0,L)$ and $H^{-1}(0,L)$ and (u,v,z,p) the solution of (2.1).

Lemma 2.9. *Suppose that $T > 2\alpha R$, then*

$$\liminf_{\|(u_0,\rho_1 u_1,v_0,\rho_2 v_1+mp_{0x},z_0,\rho_1 z_1,p_0)\|_{\widetilde{H}}\to\infty} \frac{J(u_0,u_1,v_0,v_1,z_0,z_1,p_0)}{\|(u_0,\rho_1 u_1,v_0,\rho_2 v_1+mp_{0x},z_0,\rho_1 z_1,p_0)\|_{\widetilde{H}}} \geq \varepsilon. \tag{2.29}$$

Proof. Consider a sequence of initial data $(u_0^j, u_1^j, v_0^j, v_1^j, z_0^j, z_1^j, p_0^j)$ in $\widetilde{H}$ such that

$$N_j = \|(u_0^j, \rho_1 u_1^j, v_0^j, \rho_2 v_1^j + mp_{0x}^j, z_0^j, \rho_1 z_1^j, p_0^j)\|_{\widetilde{H}} \to \infty, \quad j \to \infty.$$

We introduce the normalized sequence of initial data

$$(\widehat{u}_0^j, \widehat{u}_1^j, \widehat{v}_0^j, \widehat{v}_1^j, \widehat{z}_0^j, \widehat{z}_1^j, \widehat{p}_0^j) = \frac{(u_0^j, u_1^j, v_0^j, v_1^j, z_0^j, z_1^j, p_0^j)}{N_j}$$

and the corresponding solutions of (2.1), $(\widehat{u}^j, \widehat{v}^j, \widehat{z}^j, \widehat{p}^j) = \dfrac{(u^j, v^j, z^j, p^j)}{N_j}$. We have

$$\begin{aligned}
&\frac{J(u_0,u_1,v_0,v_1,z_0,z_1,p_0)}{N_j} = \frac{N_j}{2}\int_0^T\int_{l_1}^{l_2}(|\widehat{u}^j|^2+|\widehat{v}^j|^2+|\widehat{z}^j|^2)\,dx\;dt\\
&-\rho_1\int_0^L \Phi_1\widehat{u}_0^j dx - \rho_2\int_0^L \Psi_1\widehat{v}_0^j dx - \rho_1\int_0^L W_1\widehat{z}_0^j dx + \rho_1\langle\Phi_0,\widehat{u}_1^j\rangle +\\
&\rho_2\langle\Psi_0,\widehat{v}_1^j\rangle + \rho_1\langle W_0,\widehat{z}_1^j\rangle - \int_0^L(\eta_0+m\Psi_x)\widehat{p}_0^j\;dx + \varepsilon\|\widehat{p}_0^j\|_{L^2(0,L)},
\end{aligned} \tag{2.30}$$

and we distinguish the following two cases:

$$i)\ \liminf_{j\to\infty}\int_0^T\int_{l_1}^{l_2}(|\widehat{u}^j|^2+|\widehat{v}^j|^2+|\widehat{z}^j|^2)\,dx\;dt > 0, \tag{2.31}$$

or there exists a sequence such that

$$ii)\int_0^T\int_{l_1}^{l_2}(|\widehat{u}^j|^2+|\widehat{v}^j|^2+|\widehat{z}^j|^2)\,dx\;dt \to 0, \ \ j\to\infty. \tag{2.32}$$

In the first case we have $\liminf_{j\to\infty} \dfrac{J(u_0,u_1,v_0,v_1,z_0,z_1,p_0)}{N_j} = \infty.$

In case ii), we have that $(\widehat{u}_0^j, \widehat{u}_1^j, \widehat{v}_0^j, \widehat{v}_1^j, \widehat{z}_0^j, \widehat{z}_1^j, \widehat{p}_0^j)$ is bounded in $\widetilde{H}$, so we can extract a subsequence such that

$$(\widehat{u}_0^j, \widehat{u}_1^j, \widehat{v}_0^j, \widehat{v}_1^j, \widehat{z}_0^j, \widehat{z}_1^j, \widehat{p}_0^j) \rightharpoonup (\widehat{u}_0, \widehat{u}_1, \widehat{v}_0, \widehat{v}_1, \widehat{z}_0, \widehat{z}_1, \widehat{p}_0) \quad in \quad \widetilde{H}. \tag{2.33}$$

We denote $(\widehat{u}, \widehat{v}, \widehat{z}, \widehat{p})$ the solution from (2.1). From (2.32) we get $\widehat{u} = \widehat{v} = \widehat{z} = 0$ in $(l_1, l_2) \times (0, T)$. By Proposition 2.5 it follows that $(\widehat{u}_0, \widehat{u}_1, \widehat{v}_0, \widehat{v}_1, \widehat{z}_0, \widehat{z}_1, \widehat{p}_0) \equiv (0,0,0,0,0,0,0)$. So,

$$(\widehat{u}_0^j, \widehat{u}_1^j, \widehat{v}_0^j, \widehat{v}_1^j, \widehat{z}_0^j, \widehat{z}_1^j, \widehat{p}_0^j) \rightharpoonup (0,0,0,0,0,0,0) \quad in \quad \widetilde{H}. \tag{2.34}$$

From (2.34) we deduce that

$$\begin{aligned} &\liminf_{j\to\infty} \frac{J(u_0, u_1, v_0, v_1, z_0, z_1, p_0)}{N_j} \\ &= \liminf_{j\to\infty} \Big(\frac{N_j}{2} \int_0^T \int_{l_1}^{l_2} (|\widehat{u}^j|^2 + |\widehat{v}^j|^2 + |\widehat{z}^j|^2)\, dx\, dt + \|\widehat{p}_0^j\|_{L^2(0,L)}\Big). \end{aligned} \tag{2.35}$$

If

$$\liminf_{j\to\infty} \|\widehat{p}_0^j\|_{L^2(0,L)} \geq 1 \tag{2.36}$$

then (2.29) is immediate. Suppose that $\liminf_{j\to\infty} \|\widehat{p}_0^j\|_{L^2(0,L)} < 1$. So, since $\|(\widehat{u}_0^j, \rho_1 \widehat{u}_1^j, \widehat{v}_0^j, \rho_2 \widehat{v}_1^j + m\widehat{p}_{0x}^j, \widehat{z}_0^j, \rho_1 \widehat{z}_1^j, \widehat{p}_0^j)\|_{\widetilde{H}} = 1$, we deduce

$$\liminf_{j\to\infty} \|(\widehat{u}_0^j, \rho_1 \widehat{u}_1^j, \widehat{v}_0^j, \rho_2 \widehat{v}_1^j + m\widehat{p}_{0x}^j, \widehat{z}_0^j, \rho 1 \widehat{z}_1^j\| > 0. \tag{2.37}$$

From (2.37) and since $\widehat{p}_0^j$ bounded in $L^2(0, L)$, by (2.7) we have that

$$\liminf_{j\to\infty} \int_0^T \int_{l_1}^{l_2} (|\widehat{u}^j|^2 + |\widehat{v}^j|^2 + |\widehat{z}^j|^2)\, dx\, dt > 0, \tag{2.38}$$

which contradicts (2.32).

Thus, we necessarily have (2.36) and so (2.32), that is, the functional J is coercive. □

The functional J is lower semicontinuous, because it is continuous and also convex (for this, it suffices to observe that $\int_0^T \int_{l_1}^{l_2} |u|^2 + |v|^2 + |z|^2\, dx\, dt$

is strictly convex, because $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=x^2$ is strictly convex and the other terms of J are convex).

The second Gateaux derivative of J is

$$\begin{aligned}
&\lim_{\lambda\to\infty}\tfrac{J((u_0,u_1,v_0,v_1,z_0,z_1,p_0)+\lambda(U_0,U_1,V_0,V_1,Z_0,Z_1,P_0))-J(u_0,u_1,v_0,v_1,z_0,z_1,p_0)}{\lambda}\\
&=\int_0^T\int_{l_1}^{l_2}(uU+vV+zZ)\;dx\;dt\\
&-\rho_1\int_0^L\Phi_1U_0dx-\rho_2\int_0^L\Psi_1V_0dx-\rho_1\int_0^L W_1Z_0dx\\
&+\rho_1\langle\Phi_0,U_1\rangle+\rho_2\langle\Psi_0,V_1\rangle+\rho_1\langle W_0,Z_1\rangle\\
&-\int_0^L(\eta_0+m\Psi_{0x})P_0\;dx+\varepsilon\frac{\int_0^L p_0P_0\;dx}{(\int_0^L|p_0|^2)^{\frac12}}.
\end{aligned}\tag{2.39}$$

From the coercivity Lemma 2.9, the lower semicontinuity and the fact that J is strictly convex it has a single minimum $(\widehat{u}_0,\widehat{u}_1,\widehat{v}_0,\widehat{v}_1,\widehat{z}_0,\widehat{z}_1,\widehat{p}_0)$ in $\widetilde{H}$. For the minimum of the functional J,

$$\begin{aligned}
&\Big|\int_0^T\int_{l_1}^{l_2}(\widehat{u}U+\widehat{v}V+\widehat{z}Z)\;dx\;dt\\
&-\rho_1\int_0^L\Phi_1U_0dx-\rho_2\int_0^L\Psi_1V_0dx-\rho_1\int_0^L W_1Z_0dx\\
&+\rho_1\langle\Phi_0,U_1\rangle+\rho_2\langle\Psi_0,V_1\rangle+\rho_1\langle W_0,Z_1\rangle\\
&-\int_0^L(\eta_0+m\Psi_{0x})P_0\;dx\Big|\le\|P_0\|_{L^2(0,L)},
\end{aligned}\tag{2.40}$$

for all $(U_0,U_1,V_0,V_1,Z_0,Z_1,P_0)\in\widetilde{H}$ and $\widehat{u},\widehat{v},\widehat{z},\widehat{p}$ solution of (2.1) with data $(\widehat{u}_0,\widehat{u}_1,\widehat{v}_0,\widehat{v}_1,\widehat{z}_0,\widehat{z}_1,\widehat{p}_0)$ and (U,V,Z,P) solution of (2.3) with data $(U_0,U_1,V_0,V_1,Z_0,Z_1,P_0)$.

Note that the solution of (1.1) with null initial data and $f_1=\widehat{u}$, $f_2=\widehat{v}$, $f_3=\widehat{z}$ satisfies

$$\begin{aligned}
&\int_0^T\int_{l_1}^{l_2}\widehat{u}U+\widehat{v}V+\widehat{z}Z\;dx\;dt=\\
&\rho_1\int_0^L\varphi_t(T)U_0\;dx+\rho_2\int_0^L\psi_t(T)V_0\;dx+\rho_1\int_0^L w_t(T)Z_0\;dx\\
&-\rho_1\langle\varphi(T),U_1\rangle-\rho_2\langle\psi(T),V_1\rangle-\rho_1\langle w(T),Z_1\rangle\\
&+\int_0^L(\theta(T)+m\psi_x(T))P_0\;dx.
\end{aligned}\tag{2.41}$$

Then, from (2.40), (2.41) and taking $P_0 = 0$ we obtain

$$\begin{aligned} \varphi(T) &= \Phi_0, \quad \varphi_t(T) = \Phi_1, \\ \psi(T) &= \Psi_0, \quad \psi_t(T) = \Psi_1, \\ w(T) &= W_0, \quad w_t(T) = W_1. \end{aligned} \tag{2.42}$$

From (2.40) – (2.42) we obtain

$$|\int_0^L (\theta(T) - \eta_0)P_0| \leq \varepsilon \|P_0\|_{L^2(0,L)}$$

and this is equivalent to $|\theta(T) - \eta_0|_{L^2(0,L)} \leq \varepsilon$.

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