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Critical points at prescribed energy level for Schrödinger-Bopp-Podolsky systems

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Abstract. In this note we aim to present an existence result for radial solutions to a Schrödinger-Bopp-Podolsky system in the whole space under the constraint of energy. In particular the problem is depending on a parameter and we ask if, given a priori energy level, for some value of he parameter there is a solution of the system whose energy is the given value. The result is presented in [10] so here just the main ideas are presented, referring the reader to the mentioned paper for all the details and proof.

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1 Introduction

Recently a new kind of elliptic problem appeared in the mathematical literature which considers a variant of the more classical and more studied

Schrödinger-Poisson system. In fact in [3] the authors introduced the following system

$$\begin{cases} -\Delta u + \omega u + \mu\phi u = |u|^{p-2}u & \text{in} \quad \mathbb{R}^3, \\ -\Delta\phi + a^2\Delta^2 u = 4\pi u^2 & \text{in} \quad \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $p \in (2,6)$, $\omega > 0$, and $a \geq 0$. The first equation is a Schödinger type equation for the modulus $u : \mathbb{R}^3 \to \mathbb{R}$ of a stationary wave $\psi(x,t) = u(x)e^{-i\omega t}$ under a potential $\phi : \mathbb{R}^3 \to \mathbb{R}$ generated by the same wave function, as the second equation states: in fact it can be written in the obvious divergence form $-\nabla\Big(\nabla\phi + a^2\nabla\Delta\phi\Big) = 4\pi u^2$. Evidently, unless $a = 0$, ϕ is not the classical electrostatic potential. Indeed this was exactly the main point of the generalized electromagnetic theory developed by Bopp and Podolsky in order to obtain a better description, than the Maxwell electromagnetic theory, of some physical phenomena.

In this paper we do not aim to enter in this physical detail, for which we refer the reader to [3] where a physical and mathematical motivation and approach to the above system is given. We are just interested here in showing existence of solutions for the above system under suitable condition which were never considered before in the literature. For recent results on this kind of physical system we refer the reader to [1, 4, 8, 11, 12].

We first recall that in [3] the existence of a solution has been proved when $p \in (2,3)$, $\omega > 0$, and $a \geq 0$. We look here for radial solutions $u \in H^1_r(\mathbb{R}^3)$ of (1.1). Let $\|\cdot\|$ be given by $\|u\|^2 = \|\nabla u\|_2^2 + \omega\|u\|_2^2$, and

$$\mathcal{D}_r = \left\{\phi \in D^{1,2}_r(\mathbb{R}^3) : \Delta\phi \in L^2_r(\mathbb{R}^3)\right\}.$$

It is known that for every $u \in H^1_r(\mathbb{R}^3)$, there exists a unique $\phi_u \in \mathcal{D}_r$ solving the second equation in (1.1), see [3]. Moreover, critical points of the functional $\Phi_\mu : H^1_r(\mathbb{R}^3) \to \mathbb{R}$ defined by

$$\Phi_\mu(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{\omega}{2}\int_{\mathbb{R}^3} |u|^2 + \frac{\mu}{4}\int_{\mathbb{R}^3} \phi_u u^2 - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p, \tag{1.2}$$

are classical solutions of (1.1).

The main result states that for suitable values c of energy, a priori fixed, there are two family of solutions $(\mu^-_{n,c}, u_{n,c})$ and $(\mu^+_{n,c}, v_{n,c})$ with $n \in \mathbb{N}$. More precisely

Theorem 1.1. *Let $p \in (2,3)$, $\omega > 0$, and $a \geq 0$. There exists $c^* > 0$ such that:*

1. *For any fixed $c < c^*$ there exist infinitely many $(\mu^-_{n,c}, u_{n,c}) \in (0,\infty) \times H^1_r(\mathbb{R}^3) \setminus \{0\}$ such that $\Phi_{\mu^-_{n,c}}(u_{n,c}) = c$ and $\Phi'_{\mu^-_{n,c}}(u_{n,c}) = 0$, i.e., $u_{n,c}$ are weak solutions of (1.1) with $\mu = \mu^-_{n,c}$, having energy c, for every n. Moreover:*
   - *(a) $\{\mu^-_{n,c}\}$ is a non-increasing sequence, $\lim_{n\to+\infty} \mu^-_{n,c} = 0$ and $\|u_{n,c}\| \to +\infty$ as $n \to +\infty$, so $(0, +\infty)$ is a bifurcation point.*
   - *(b) If $c < 0$ and $\mu > \mu^-_{1,c}$ then (1.1) has no radial weak solution having energy c.*

2. *For any fixed $c \in (0, c^*)$ there exist infinitely many $(\mu^+_{n,c}, v_{n,c}) \in \mathbb{R} \times H^1_r(\mathbb{R}^3) \setminus \{0\}$ such that $\Phi_{\mu^+_{n,c}}(v_{n,c}) = c$ and $\Phi'_{\mu^+_{n,c}}(v_{n,c}) = 0$, i.e., $v_{n,c}$ are weak solutions of (1.1) with $\mu = \mu^+_{n,c}$, having energy c, for every n. Moreover:*
   - *(a) $\{\mu^+_{n,c}\}$ is a non-increasing sequence, $\lim_{n\to+\infty} \mu^+_{n,c} = -\infty$ and $v_{n,c} \rightharpoonup 0$ as $n \to +\infty$.*
   - *(b) $\mu^+_{n,c} < \mu^-_{n,c}$ for every n.*

Moreover the maps $c \mapsto \mu^\pm_{n,c}$ enjoy the following behavior:

Theorem 1.2. *Under the conditions of Theorem 1.1 the following properties hold for every fixed $n \in \mathbb{N}$ (see Figure 1.1):*

1. *The map $c \mapsto \mu^-_{n,c}$ is continuous and non-decreasing in $(-\infty, c^*)$, and $\lim_{c\to-\infty} \mu^-_{n,c} = 0$.*

2. *The map* $c \mapsto \mu^+_{n,c}$ *is continuous and non-decreasing in* $(0, c^*)$, *and* $\lim_{c \to 0^+} \mu^+_{n,c} = -\infty$.

3. *For every* $\mu \in (0, \mu^-_{n,0})$ *the problem* (1.1) *has at least* n *pairs of radial solutions with negative energy.*

4. *For every* $\mu < \mu^+_{n,0}$ *the problem* (1.1) *has at least* n *pairs of radial solutions with positive energy.*

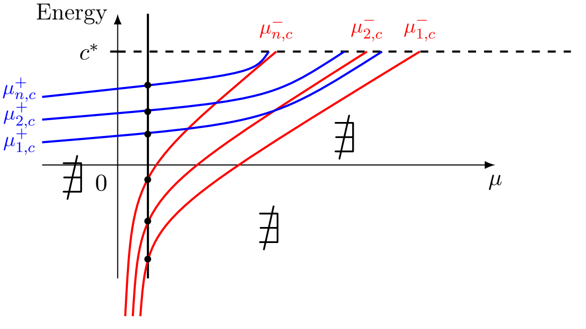

Figure 1.1: Energy curves for (1.1)

In the case $a = 0$ this result should be compared with [2, Theorem 3.1], where the authors show the existence of multiple solutions with positive and negative energy for small values of the parameter μ. Moreover when $a > 0$ Theorem 1.2 improves [3, Theorem 1.1] where the existence of a nontrivial solution at a positive energy level is proved for small values of μ.

The structure of the paper is the following. In the next Section we give the idea of the general theory which permits to have existence of solutions for problems where the constraint is the prescribed energy. In the subsequent Section, we show that it is possible to applies these ideas to the functional defined in (1.2).

2 General theory for abstract functionals

Many problems in nonlinear pdes can be formulated in terms of variational equations, namely

$$\Phi'(u) = 0, \tag{2.1}$$

where Φ' is the Fréchet derivative of a certain functional Φ defined on a suitable function space X. Φ is called energy functional and the previous equation is the related Euler-Lagrange equation.

In this context the Critical Point Theory has a major role since it gives conditions to guarantee that a functional has critical points, then giving solutions of a certain equation.

In general these equations are often coupled to some additional constraint on u (e.g. a *sign constraint* $u > 0$ or a *mass constraint* $\|u\| = m$) and a huge bibliography is available on this subject.

We revise in this note how prove the existence of solutions for (2.1) under the constraint, namely, the level (or energy) constraint $\Phi(u) = c$, with $c \in \mathbb{R}$. Of course for a single functional Φ is quite improbable that given a priori value c it is a critical value. For this reason we deal here with a family of functionals Φ_μ with $\mu \in \mathbb{R}$: we ask if, given $c \in \mathbb{R}$, at least one functional of the family has a critical point to the level c. It is clear then that a solution is a pair: the critical point u and the parameter μ which selects the functional that has u as critical point at the given level c.

In general this method applies for a family of functionals of type

$$\Phi_\mu := I_1 - \mu I_2,$$

where $\mu \in \mathbb{R}$, $I_1, I_2 \in C^1(X)$, and X is Banach space which is assumed to be infinite-dimensional, uniformly convex, and equipped with $\|\cdot\| \in C^1(X \setminus \{0\})$.

To this aim, consider the abstract problem

$$\Phi'_\mu(u) = 0, \quad \Phi_\mu(u) = c \in \mathbb{R}, \tag{2.2}$$

in the unknowns $(\mu, u) \in \mathbb{R}\times X\setminus\{0\}$. The general strategy is to analyse the *nonlinear generalized Rayleigh quotient*, introduced by Y. Ilyasov [6]. As a byproduct, we obtain te structure of the *solution set* of (2.2), considered with respect to μ and c:

$$\mathcal{S} := \{(\mu, c) \in \mathbb{R}^2 : \Phi_\mu \text{ has a critical point at the level } c\}.$$

Let us note that some preliminary results on (2.2) can be found in [7] and [9], where the case $c = 0$ has been treated.

We shall see that there exist infinitely many pairs $(\mu_{n,c}, \pm u_{n,c})$ solving (2.2). This result, which is proved via the Ljusternik-Schnirelman theory, will be established not only for a single value of c, but for c lying in an open interval $\mathcal{I} \subset \mathbb{R}$. Thus it makes sense to study the behaviour of $\mu_{n,c}$ and $u_{n,c}$ with respect to $c \in \mathcal{I}$. In many cases the values $\mu_{n,c}$ depend continuously on c, so that letting c vary we shall obtain a family of *energy curves* $\{(\mu_{n,c}, c); c \in \mathcal{I}\}_{n\in\mathbb{N}}$. The properties of these *energy curves* will give also informations about the existence of solutions for the unconstrained problem $\Phi'_\mu(u) = 0$ (without the restriction on the energy) and then will give bifurcation and multiplicity results for the problem.

Assume that $I_2(u) \neq 0$ for every $u \in X \setminus \{0\}$ (which is the case also in many elliptic problems), and observe that

$$\Phi_\mu(u) = c \quad \Longleftrightarrow \quad \mu = \mu(c, u) := \frac{I_1(u) - c}{I_2(u)}. \tag{2.3}$$

A simple computation gives

$$\frac{\partial \mu}{\partial u}(c, u) = \frac{\Phi'_{\mu(c,u)}(u)}{I_2(u)}, \quad \forall u \in X \setminus \{0\}.$$

Here $\frac{\partial \mu}{\partial u}(c, u)$ denotes the Fréchet derivative of the functional $u \mapsto \mu(c, u)$. Then we have:

$$\Phi'_\mu(u) = 0, \quad \Phi_\mu(u) = c \quad \Longleftrightarrow \quad \mu = \mu(c, u), \quad \frac{\partial \mu}{\partial u}(c, u) = 0,$$

i.e. solving (2.2) is reduced to find critical points (and critical values) of the functional $u \mapsto \mu(c, u)$. If we set $\mathcal{K}(c)$ the set of critical values of $\mu(c, \cdot)$, we get a sufficient and necessary condition for the solvability of (2.2):

Theorem 2.1. *For a given $c \in \mathbb{R}$ the problem (2.2) has a solution (μ, u) if, and only if, $\mu \in \mathcal{K}(c)$ and u is the associated critical point. In particular, if $u \mapsto \mu(c, u)$ has a ground state (or least energy) level $GS(c)$ then (2.2) has no solution for $\mu < GS(c)$.*

To find critical points (and critical levels) of $\mu(c, \cdot)$, we consider the fibering map associated to the functional $u \mapsto \mu(c, u)$, namely, the real-valued function $\psi_{c,u}$ given by

$$\psi_{c,u}(t) := \mu(c, tu) = \frac{I_1(tu) - c}{I_2(tu)}, \quad t > 0 \tag{2.4}$$

for any fixed $(c, u) \in \mathbb{R} \times X \setminus \{0\}$. We assume that

(A) There exists an open set $\mathcal{I} \subset \mathbb{R}$ such that:

1. the map $(c, u, t) \mapsto \psi'_{c,u}(t)$ belongs to $C^1(\mathcal{I} \times X \setminus \{0\} \times (0, \infty))$;
2. for every $(c, u) \in \mathcal{I} \times X \setminus \{0\}$ the map $\psi_{c,u}$ has exactly one local minimizer $t^+(c, u) > 0$ of Morse type or for every $(c, u) \in \mathcal{I} \times X \setminus \{0\}$ the map $\psi_{c,u}$ has exactly one local maximizer $t^-(c, u) > 0$ of Morse type.

It should be noted that both possibilities in (2) can occur, as we shall see. For simplicity, assume for the moment that the first one occurs (the other case is similar), and set $t(c, u) := t^+(c, u)$. We introduce the reduced functional $\Lambda \in C^1(\mathcal{I} \times X \setminus \{0\})$ given by

$$\Lambda(c, u) := \psi_{c,u}(t(c, u)) = \mu(c, t(c, u)u). \tag{2.5}$$

It is easy to see that, for any $c \in \mathcal{I}$ the functional $u \mapsto \Lambda(c, u)$ is 0-homogeneous, so we can restrict to find its critical point on $\mathcal{I} \times S$, where S is the unit sphere in X.

If we assume that I_1, I_2 are even, also $\mu(c, \cdot)$, $t(c, \cdot)$ and $\Lambda(c, \cdot)$ are even. In order to apply the Ljusternick-Schnirelmann theory, let us recall that given a nonempty symmetric and closed set $F \subset S$, the Krasnoselskii genus of F is given by

$$\gamma(F) := \inf\{n \in \mathbb{N} : \exists h : F \to \mathbb{R}^n \setminus \{0\} \text{ odd and continuous}\}.$$

For every $n \in \mathbb{N}$ we set

$$\mathcal{F}_n := \{F \subset S : F \text{ is compact, symmetric, and } \gamma(F) \geq n\}. \tag{2.6}$$

We shall assume the following condition, which contains two alternatives in accordance with the behavior of $u \mapsto \widetilde{\Lambda}(c,u)$:

(B) The functional $u \mapsto \widetilde{\Lambda}(c,u)$ is bounded from below (respect. from above) and satisfies the Palais-Smale condition at the level $\mu_{n,c} := \inf\limits_{F \in \mathcal{F}_n} \sup\limits_{u \in F} \Lambda(c,u)$ (respect. at the level $\mu_{n,c} := \sup\limits_{F \in \mathcal{F}_n} \inf\limits_{u \in F} \Lambda(c,u)$) for every $n \in \mathbb{N}$.

Recall, by definition, that the functional $u \mapsto \widetilde{\Lambda}(c,u)$ satisfies the Palais-Smale condition at the level μ if any sequence $\{u_k\} \subset S$ such that $\widetilde{\Lambda}(c,u_k) \to \mu$ and $\frac{\partial \widetilde{\Lambda}}{\partial u}(c,u_k) \to 0$ has a convergent subsequence.

The main result is the following.

Theorem 2.2. *Suppose that (A) holds and let $c \in \mathcal{I}$.*

1. *Assume that $t(c,u)$ is the only critical point of $\psi_{c,u}$, for every $u \in X \setminus \{0\}$. If $u \mapsto \Lambda(c,u)$ is bounded from below (respect. above) and $\mu < \mu_{1,c} = \inf\limits_{u \in X\setminus\{0\}} \Lambda(c,u)$ (respect. $\mu > \mu_{1,c} = \sup\limits_{u \in X\setminus\{0\}} \Lambda(c,u)$) then there exists no $u \in X \setminus \{0\}$ such that*

$$\Phi'_\mu(u) = 0 \qquad \text{and} \qquad \Phi_\mu(u) = c.$$

2. *If (B) holds then there exist infinitely many $u_{n,c} \in X \setminus \{0\}$ such that*

$$\Phi'_{\mu_{n,c}}(\pm u_{n,c}) = 0 \quad \text{and} \quad \Phi_{\mu_{n,c}}(\pm u_{n,c}) = c \quad \forall n \in \mathbb{N}.$$

If, in addition, $\Lambda(c, w_n) \to +\infty$ *whenever* $w_n \rightharpoonup 0$ *in* X*, then* $\mu_{n,c} \to +\infty$ *as* $n \to +\infty$*.*

Remark 2.3. From the definition of $\mu_{n,c}$ we see that if (B) holds with $u \mapsto \Lambda(c, u)$ bounded from below (respect. above) then $\{\mu_{n,c}\}$ is nondecreasing (respect. nonincreasing). Moreover, in the second case we also have $\mu_{n,c} = -\inf_{F \in \mathcal{F}_n} \sup_{u \in F}(-\Lambda(c, u))$, as well as $\mu_{n,c} = \left(\inf_{F \in \mathcal{F}_n} \sup_{u \in F}(\Lambda(c, u))^{-1}\right)^{-1}$ if, in addition, $\Lambda(c, u)$ is positive on S. These characterizations will provide us at least two possible behaviors for $\{\mu_{n,c}\}$ as $n \to +\infty$: $\mu_{n,c} \to 0$ or $\mu_{n,c} \to -\infty$, cf. Theorem 1.1 below.

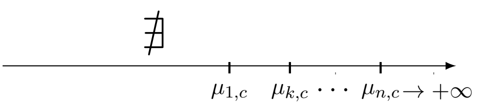

Figure 2.1: The sequence $\{\mu_{n,c}\}$ provided by Theorem 2.2.

2.1 Energy curves

Then the study of $\mu_{n,c}$ with respect to c is given. The following conditions on Λ shall provide the continuity of $c \mapsto \mu_{n,c}$, which gives rise to the family of *energy curves* $\{(\mu_{n,c}, c); c \in \mathcal{I}\}_{n \in \mathbb{N}}$:

(C) For any $u \in S$ the map $c \mapsto \Lambda(c, u)$ is decreasing (respect. increasing) in $\mathcal{I}$. In addition, Λ is bounded from above (respect. below) in any compact set $K \subset \mathcal{I} \times S$. Finally, if Λ is bounded in $[a, b] \times S_0 \subset \mathcal{I} \times S$ then $\frac{\partial \Lambda}{\partial c}$ is also bounded and away from zero therein.

Conditions to verify (C) are given in [9]

The following generalized Palais-Smale condition shall be required as well:

(D) If $c_n \to c \in \mathcal{I}$ and $\{u_n\} \subset S$ are such that $\{\Lambda(c_n, u_n)\}$ is bounded and $\frac{\partial \widetilde{\Lambda}}{\partial u}(c_n, u_n) \to 0$, then $\{u_n\}$ has a convergent subsequence.

The next result combined with the asymptotics of the maps $c \mapsto \mu_{n,c}$ provide us with some informations on the structure of the set

$$\mathcal{S} := \{(\mu, c) \in \mathbb{R}^2 : \Phi_\mu \text{ has a critical point at the level } c\}.$$

Theorem 2.4. *Assume (A), (B),(C) and (D). Then for every $n \in \mathbb{N}$ the map $c \mapsto \mu_{n,c}$ is decreasing (respect. increasing) and locally Lipschitz continuous in $\mathcal{I}$.*

Remark 2.5. The statement of Theorem 2.4 has to be understood in the sense that

- the map $c \mapsto \mu_{n,c}$ is decreasing if, in (B), the functional $u \mapsto \widetilde{\Lambda}(c, u)$ is bounded below and, in (C), the map $c \mapsto \Lambda(c, u)$ is decreasing.
- the map $c \mapsto \mu_{n,c}$ is increasing if, in (B), the functional $u \mapsto \widetilde{\Lambda}(c, u)$ is bounded above and, in (C), the map $c \mapsto \Lambda(c, u)$ is increasing.

Remark 2.6. Let us assume that $t(c, u)$ is the only critical point of $\psi_{c,u}$, for every $u \in X \setminus \{0\}$. Then the solution $u_{1,c}$ of (2.2) with $\mu = \mu_{1,c}$ is the ground state solution of the problem $\Phi'_\mu(u) = 0$. In other words, whenever achieved, the ground state level of Φ_μ is the value c such that $\mu = \mu_{1,c}$.

The next results are the core of our approach. Throughout this section we assume that (A) holds. In addition, $t(c, u)$ can be either $t^+(c, u)$ or $t^-(c, u)$, i.e. a nondegenerate minimizer or a nondegenerate maximizer of $\psi_{c,u}$.

Lemma 2.7. *The following statements hold.*

1. *The map* $(c,u)\mapsto t(c,u)$ *belongs to* $C^1(\mathcal{I}\times X\setminus\{0\})$ *and*

$$\frac{\partial t}{\partial c}(c,u)=\frac{-I_2'(t(c,u)u)u}{I_2(t(c,u)u)^2\psi''_{c,u}(t(c,u))}\qquad \forall (c,u)\in\mathcal{I}\times X\setminus\{0\}. \quad (2.7)$$

   In particular, for every $u\in X\setminus\{0\}$, *the map* $c\mapsto t(c,u)$ *is increasing (respect. decreasing) if* $\psi''_{c,u}(t(c,u))I_2'(t(c,u)u)u<0$ *(respect.* >0*);*

2. $\Lambda\in C^1(\mathcal{I}\times X\setminus\{0\})$, *and for any* $v\in X$

$$\frac{\partial\Lambda}{\partial u}(c,u)v=\frac{\Phi'_{\Lambda(c,u)}(t(c,u)u)t(c,u)v}{I_2(t(c,u)u)},\qquad \forall (c,u)\in\mathcal{I}\times X\setminus\{0\}. \quad (2.8)$$

   In particular, $\frac{\partial\Lambda}{\partial u}(c,u)u=0$ *for any* $(c,u)\in\mathcal{I}\times X\setminus\{0\}$. *Furthermore,*

$$\frac{\partial\Lambda}{\partial c}(c,u)=-\frac{1}{I_2(t(c,u)u)}, \quad (2.9)$$

   so that for every $u\in X\setminus\{0\}$ *the map* $c\mapsto\Lambda(c,u)$ *is decreasing (respect. increasing) if* $I_2(t(c,u)u)>0$ *(respect.* <0*).*

3. *For every* $c>0$ *the maps* $u\mapsto t(c,u),\Lambda(c,u)$ *are* (-1)*-homogeneous and* 0*-homogeneous, respectively.*

Proof. The proof is straightforward: it is based on the Implicit Function Theorem and suitable computations. All the details are given in [9]. □

From (2.8) and the definition of $\Lambda(c,u)$ we derive the following result:

Corollary 2.8. *If* $\frac{\partial\Lambda}{\partial u}(c,u)=0$ *then*

$$\Phi'_{\Lambda(c,u)}(t(c,u)u)=0$$

and

$$\Phi_{\Lambda(c,u)}(t(c,u)u)=c.$$

Since $\|\cdot\|\in C^1(X\setminus\{0\})$, the tangent space to S at u is given by

$$\mathcal{T}_u(S)=\{v\in X: i'(u)v=0\}$$

where $i(u) = \frac{1}{2}\|u\|^2$, and then $X = \mathcal{T}_u(S) \oplus \mathbb{R}u$.

The interesting fact now is that the unit sphere S is a *natural constraint* for $\Lambda(x, \cdot)$.

Proposition 2.9. *Let $u \in S$. Then*

$$\frac{\partial \widetilde{\Lambda}}{\partial u}(c, u) = 0 \quad \textit{if and only if} \quad \frac{\partial \Lambda}{\partial u}(c, u) = 0.$$

Proof. Let $u \in S$ be such that $\frac{\partial \widetilde{\Lambda}}{\partial u}(c, u) = 0$. From Lemma 2.7 (2) we know that $\frac{\partial \Lambda}{\partial u}(c, u)u = 0$. Note that $\frac{\partial \widetilde{\Lambda}}{\partial u}(c, u) = \frac{\partial \Lambda}{\partial u}(c, u)_{|\mathcal{I} \times \mathcal{T}_u(S)}$. If $w \in X$, then $w = v + su$ for some $v \in \mathcal{T}_u(S)$ and $s \in \mathbb{R}$, which implies that $\frac{\partial \Lambda}{\partial u}(c, u)w = \frac{\partial \widetilde{\Lambda}}{\partial u}(c, u)v + \frac{\partial \Lambda}{\partial u}(c, u)tu = 0$. Since the converse is obvious, the proof is complete. □

2.2 Proofs of Theorem 2.2 and 2.4

Let us give know the proof of the main theorems, which is a consequence of our general hypotheses.

2.2.1 Proof of Theorem 2.2

(1) Take $u \in X \setminus \{0\}$ such that $\Phi'_\mu(u) = 0$ and $\Phi_\mu(u) = c$ so that $\mu = \frac{I_1(u)-c}{I_2(u)} = \psi_{c,u}(1)$.

Equation (2.4) give that $\psi'_{c,u}(1) = \frac{\Phi'_{\psi_{c,u}(1)}(u)u}{I_2(u)} = \frac{\Phi'_\mu(u)u}{I_2(u)} = 0$, i.e. $t(c, u) = 1$. Thus $\mu = \Lambda(c, u) \geq \mu_{1,c}$ (respect. $\leq \mu_{1,c}$) as soon as $u \mapsto \Lambda(c, u)$ is bounded from below (respect. above).

(2) We consider the case where (B) holds with $u \mapsto \widetilde{\Lambda}(c, u)$ bounded from below. By the Ljusternick-Schnirelman theorem (see e.g. [5, Corollary 4.17] or [13]) there exist infinitely many $u_{n,c} \in S$ such that

$$\frac{\partial \Lambda}{\partial u}(c, \pm u_{n,c}) = 0 \quad \text{and} \quad \Lambda(c, \pm u_{n,c}) = \mu_{n,c} \quad \forall n \in \mathbb{N}$$

(we used here Proposition 2.9). From Corollary 2.8 and the fact that $u \mapsto t(c, u)$ is even, the sequence $v_n := t(c, u_{n,c})u_{n,c}$ satisfies

$$\Phi_{\mu_{n,c}}(v_{n,c}) = c \quad \text{and} \quad \Phi'_{\mu_{n,c}}(v_{n,c}) = 0 \quad \forall n \in \mathbb{N}.$$

Now, if (B) holds with $u \mapsto \widetilde{\Lambda}(c, u)$ bounded from above, then we deal with the functional $u \mapsto -\widetilde{\Lambda}(c, u)$, which is bounded from below. The details can be seen in [10].

Finally, the second assertion follows from [9, Lemma A.1]. □

2.2.2 Proof of Theorem 2.4

Let us fix n. Of course, being $c \mapsto \Lambda(c, u)$ decreasing in $\mathcal{I}$, it is clear that $c \mapsto \mu_{n,c}$ is nonincreasing in $\mathcal{I}$.

Fixed an interval $[a, b] \subset \mathcal{I}$, we show that there exist $T > 0$ such that for all $c \in [a, b]$ there exists a Palais-Smale sequence $\{u_{k,c}\}$ of the functional $u \mapsto \tilde{\Lambda}(c, u)$ at the level $\mu_{n,c}$ satisfying

$$\{u_{j,c}\} \subset S_{a,T} := \{u \in S : \Lambda(a, u) \leq T\}.$$

If it were not true, we could find a sequence $\{c_k\} \subset [a, b]$ such that for any k there exists a Palais-Smale sequence $\{u_{j,c_k}\}$ of the functional $u \mapsto \tilde{\Lambda}(c_k, u)$ at the level μ_{n,c_k} and $\{u_{j,c_k}\} \not\subset S_{a,T}$. Thus we can extract a sequence $\{u_k\} \subset S$ such that $\Lambda(c_k, u_k)$ is bounded, $\frac{\partial \tilde{\Lambda}}{\partial u}(c_k, u_k) \to 0$ and $\Lambda(a, u_k) \to +\infty$. Since $c_k \to c$ we infer by (D) that $\{u_k\}$ is compact and hence the set $K = \{(a, u_k) : k \in \mathbb{N}\}$ is compact too, and this contradicts (C). Therefore

$$\mu_{n,c} = \inf_{F \in \widetilde{\mathcal{F}}_n} \sup_{u \in F} \Lambda(c, u),$$

where $\widetilde{\mathcal{F}}_n = \{F \in \mathcal{F}_n : F \subset S_{a,T}\}$. Now, by the mean value theorem we have $|\Lambda(c_2, u) - \Lambda(c_1, u)| = \frac{\partial \Lambda}{\partial c}(\theta, u)|c_2 - c_1|$, where $\theta := \theta(u) \in (\min\{c_1, c_2\}, \max\{c_1, c_2\})$. Since $c \mapsto \Lambda(c, u)$ is decreasing, we infer that Λ is bounded in $[a, b] \times S_{a,T}$, and by (C) there exist $M, m > 0$ such that $m|c_2 - c_1| \leq |\Lambda(c_2, u) - \Lambda(c_1, u)| \leq M|c_2 - c_1|$ for $c_1, c_2 \in [a, b]$ and $u \in S_T$. It follows that $m|c_2 - c_1| \leq |\mu_{n,c_1} - \mu_{n,c_2}| \leq M|c_2 - c_1|$ for $c_1, c_2 \in [a, b]$, so that $c \mapsto \mu_{n,c}$ is decreasing and locally Lipschitz continuous. In a similar way we can prove the result whenever $c \mapsto \Lambda(c,u)$ is increasing in $\mathcal{I}$. $\square$

3 Application to the Schrödinger-Bopp-Podolsky functional

The previous general theory can be applied to the functional Φ_μ given in (1.2) for the Schrödinger-Bopp-Podolsky system. In fact we can write

$$\Phi_\mu(u) = I_1(u) - \mu I_2(u), \quad u \in H_r^1(\mathbb{R}^3), \ \mu \in \mathbb{R}$$

with

$$I_1(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 + \frac{\omega}{2}\int_{\mathbb{R}^3} |u|^2 - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p,$$

and

$$I_2(u) = \frac{\mu}{4}\int_{\mathbb{R}^3} \phi_u u^2.$$

Straightforward computations show that all the assumptions (A), (B), (C) and (D) of the previous section are satisfied. The details can be found in the paper [10].

We are not going to show the details, since are quite technical. Actually in [10] three class of functional of type $I_1 - \mu I_2$ with different assumptions on I_1 and I_2 are treated in order to satisfy the general conditions (A), (B), (C) and (D). Here we just say that the results in Theorem 1.1 and Theorem 1.2 are obtained by combining the general result of the abstract Theorem 2.2 and Theorem 2.4 and some straightforward computations which are typical of the Schrödinger-Bopp-Podolsky system. In fact what happens for the Schrödinger-Bopp-Podolsky system is that we can apply the general theory with two intervals $\mathcal{I}_1 = (-\infty, 0)$ and $\mathcal{I}_2 = (0, c^*)$.

In fact assumption (A) can be used with

- $\mathcal{I}_1 = (-\infty, 0)$, where a unique nondegenerate maximum, $t^-(c,u)$, for $\psi_{c,u}$ exists, and

- $\mathcal{I}_2 = (0, c^*)$ where a unique nondegenerate minimum , $t^+(c, u)$, for $\psi_{c,u}$ exists.

and then obtain the two families of solutions stated in Theorem 1.1.

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