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Positive solution for a class of system of nonlinear Schrödinger equations with bounded potentials

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Dedicated to Professor Pedro Ubilla
on the occasion of his 60th birthday

Abstract. In this paper, we deal with the following class of system

$$\begin{cases} -\Delta u + V(x)u - [\Delta(u^2)]u = K(x)H_u(u,v) & \text{in} \quad \mathbb{R}^N, \\ -\Delta v + V(x)v - [\Delta(v^2)]v = K(x)H_v(u,v) & \text{in} \quad \mathbb{R}^N, \end{cases}$$

where $N \geq 3$, V and K are bounded continuous nonnegative functions. The nonlinearity $H(u,v)$ is a p-homogeneous function of class C^1 with $4 < p < 22^*$

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where $2^* = 2N/(N-2)$ is critical Sobolev exponent. Using variational techniques in combination with changing variables, penalization method and Moser iteration we prove the existence of positive solutions.

Keywords: Elliptic systems, variational methods, vanishing potential, quasilinear Schrödinger equations, bounded states.

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1 Introduction

In this paper, we are concerned with standing wave solutions of time-dependent quasilinear Schrödinger equations

$$\begin{cases} i\dfrac{\partial\psi}{\partial t} = -\Delta\psi + \widehat{V}(y)\psi - [\Delta(|\psi|^2)]\psi - K(x)\dfrac{\psi}{|\psi|}H_\psi(|\psi|,|\varphi|) & \text{in} \quad \mathbb{R}^N, \\ i\dfrac{\partial\varphi}{\partial t} = -\Delta\varphi + \widehat{V}(y)\varphi - [\Delta(|\varphi|^2)]\varphi - K(x)\dfrac{\varphi}{|\varphi|}H_\varphi(|\psi|,|\varphi|) & \text{in} \quad \mathbb{R}^N, \end{cases}$$

where $V(y) = \widehat{V}(y)+\lambda$, $\psi = e^{i\lambda t}u$ and $\varphi = e^{i\lambda t}v$. This class of systems has been studied recently due to its importance in various areas, for instance,

$$\begin{cases} -\Delta u + V(x)u - [\Delta(u^2)]u = K(x)H_u(u,v) & \text{in} \quad \mathbb{R}^N, \\ -\Delta v + V(x)v - [\Delta(v^2)]v = K(x)H_v(u,v) & \text{in} \quad \mathbb{R}^N, \end{cases} \tag{S}$$

where $N \geq 3$, V and K are bounded continuous nonnegative functions, and, the primitive of nonlinearity, $H : ([0,\infty)\times[0,\infty)) \to \mathbb{R}$ is a $p-$homogeneous function of class C^1 with $4 < p < 22^*$, where 2^* is the critical Sobolev exponent. Such class of systems arise in various branches of mathematical physics and are related to the existence of solitary wave solutions for nonlinear Schrödinger equations and Klein-Gordon equations (for details see for example [7, 22]).

Our study was originated by recent works concerning the study of nonlinear Schrödinger equations by using purely variational approach, see [25]. The semilinear case has also been studied extensively in recent years, see for example [7, 21, 25] and references therein. For quasilinear Schrödinger equations we refer the reader to the recent papers [11, 23] and their references for a discussion on the subject. We found some works involving system, but the most of them used variational methods to study some class of Schrödinger equations with coercive potentials or potential with positive infimum, we indicate for further studies which treat concentration problems, [9, 10, 13, 19], and for studies with constant potentials [5]. Our work has contributed to study some class of systems of quasilinear Schrödinger equations with bounded potentials, we prove the existence of positive solutions for this class of system. In order to apply variational arguments and to overcome the lack of compactness of the associated energy functional some authors have assumed that the potential is coercive and bounded away from zero. Here, in this paper our main purpose is to extend and complement the results in [3, 14] to System (S) with no coercivity condition and with possible vanishing potential. This class of problems treated here has several difficulties. First, there is the usual lack of compactness of the Sobolev embedding, since our domain is a whole space $\mathbb{R}^N$. Second, since we are interested in vanishing and bounded potentials, it is challenging to find an adequate variational framework with an associated functional energy which critical points correspond to weak solutions of system quasilinear (S). Although the approach is similar to that used in the scalar case, for the system case, in addition to the difficulty in handle the coupled terms, the truncation argument differs completely from the scalar case.

In the rest of this paper we will assume that $V, K : \mathbb{R}^N \to \mathbb{R}$ are bounded, non negative and continuous functions satisfying:

(V_0)

$$\lambda_1 := \inf_{(u,v)\in\mathcal{H},\, ||(u,v)||_{\mathcal{L}}=1} ||(u,v)||^2_{\mathcal{H}} > 0,$$

where

$$\mathcal{H} := \left\{ (u,v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)\left(u^2+v^2\right)\,\mathrm{d}x < +\infty \right\}$$

is a Hilbert space when endowed with the inner product

$$\big\langle (u,v),(\phi,\varphi)\big\rangle_{\mathcal{H}} := \int_{\mathbb{R}^N} (\nabla u \nabla \phi + V(x) u\phi + \nabla v \nabla \varphi + V(x) v\varphi)\,\mathrm{d}x,$$

$\forall\ (u,v),(\phi,\varphi) \in \mathcal{H}$. And its correspondent norm

$$||(u,v)||^2_{\mathcal{H}} := \int_{\mathbb{R}^N} (|\ \nabla u\ |^2 + V(x)u^2 + |\ \nabla v\ |^2 + V(x)v^2)\,\mathrm{d}x, \quad \forall\ (u,v) \in \mathcal{H}.$$

We consider $\mathcal{L} := L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ equipped with the norm $||(u,v)||^2_{\mathcal{L}} := ||u||^2_{L^2(\mathbb{R}^N)} + ||v||^2_{L^2(\mathbb{R}^N)}$ (cf. [27]).

Remark 1.1. We can drop the hypothesis (V_0), if the potential V is a constant.

For the potential V and the function K, firstly, we assume that

(V_1) There exist $\lambda > 0\,,\ \ \mu > 0$ and $R > 0$, such that

$$\begin{cases} \exists\, x_o \in B_R(0) := \big\{x \in \mathbb{R}^N : |x| < R\big\} \text{ such that } K(x_o) > 0 \text{ and} \\ 0 < \mu \leq\ K(x) \leq V(x) \leq \lambda \leq \dfrac{2p}{p-2} < k_p = \dfrac{2p}{p-4}, \quad \text{for all} \quad |x| \geq R. \end{cases}$$

We also impose for K, a similar hypothesis used in [3], namely,

(V_2) There exist $\gamma > \mu$ and $R > 0$, such that

$$\sup_{|x| \geq R} \frac{R^{(N-2)}}{|x|^{(N-2)}} K(x) \leq \gamma.$$

Let be $2^* = 2N/(N-2)$ the critical Sobolev exponent. It was firstly proved in an earlier work of J. Liu et al. [23] that $2(2^*) = 4N/(N-2)$ behaves as a critical exponent for the modified Schrödinger equations of the form

$$\Delta u + V(x)u - u\Delta(u^2) = |u|^{p-2}u \quad \text{in} \quad \mathbb{R}^N,$$

in the sense that this equation has no positive solution with $\int u^2 |\nabla u|^2\,\mathrm{d}x < \infty$ in $H^1(\mathbb{R}^N)$ provided that $x \cdot \nabla V(x) \geq 0$ and $p \geq 2(2^*)$ (see also [15, 16] for related problems involving critical growth). In order to state our main result let us introduce the assumptions on the $p-$homogeneous function H that we assume throughout this article:

(H_0) There exists $0 < c_0 \leq \mu^{p/2}$ such that

$$| H_u(u,v) | + | H_v(u,v) | \leq c_0 \, (u^{p-1} + v^{p-1}), \quad \forall \, u, \, v \geq 0.$$

(H_1) $H_u(0,1) = H_v(1,0) = 0.$

(H_2) $H_u(1,0) = H_v(0,1) = 0.$

(H_3) $H_{uv}(u,v) > 0, \quad \forall \, u, v > 0.$

Throughout this paper a positive solution (u,v) of (S) means that $u > 0$ and $v > 0$ in $\mathbb{R}^N$. We now state the main result concerning the existence of solutions of system (S).

Theorem 1.2. *Suppose that* (V_0), $(H_0) - (H_3)$ *are satisfied. Then, there exists* $\gamma^* > 0$ *such that* (S) *has a positive weak solution for any potentials that satisfy* $(V_1) - (V_2)$ *with* $\gamma \leq \gamma^*$.

Remark 1.3. **(a)** A typical example of $p-$homogeneous function H satisfying our hypotheses is given by

$H(u,v) = Q(u,v)^{p/l}$ with $p \geq l$ and $Q(u,v)$ is a l-homogeneous function satisfying

$$Q(u,v) = \sum_{\alpha_i + \beta_i = l} a_i u^{\alpha_i} v^{\beta_i}, \quad u, v \geq 0,$$

where $i \in \mathcal{I}$ $(\#\mathcal{I} < \infty)$, $\alpha_i \geq 2$, $\beta_i \geq 2, N = 3$, $4 \leq l \ \leq p < 22^*$ and $a_i > 0$.

(b) For some $R > 0$ let V, K be bounded, non negative and continuous functions which are constants for all $|x| \leq R$ and such that $0 < \mu \leq K(x) \leq V(x) \leq \lambda < k_p := \frac{2p}{p-4}$ for all $|x| \geq R$. It is easy to see that V, K satisfy assumptions $(V_0) - (V_2)$.

We use variational approach to get a positive solution to the presented system (S). Initially, we modify the system S by penalization techniques to obtain another system (AS) with non linearities which satisfies some properties. But the Euler-Lagrange functional associated to (AS) system, $\hat{I}$, is not well defined in space $\mathcal{H}$. So, it is necessary to make a change of variable. In subsection 3, we use change of variable to get a suitable functional J well defined in an appropriate Hilbert space. To conclude the proof of our main result, in sixth step, we will show that this critical point of J will eventually be a solution of the original system with the help of an uniform $L^\infty-$estimate which will be obtained via Moser iteration scheme.

2 Variational Setting

2.1 Penalization method: The auxiliary system

Since here we are interested in the existence of positive solutions of (S) in the sense that each coordinate is a positive function, we redefine the non linearity $H(t,s)$ as $H(t,s)=0$ if $t\leq 0$ or $s\leq 0$. Using the definition of the weighted Sobolev space $\mathcal{H}$ and the Sobolev embedding theorem, the following embedding are continuous by condition (V_0):

$$\mathcal{H} \hookrightarrow H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)\hookrightarrow L^q(\mathbb{R}^N)\times L^q(\mathbb{R}^N),\ 2\leq q\leq 2^*,\ N\geq 3.$$

We are looking for solutions of (S) defined in $\mathbb{R}^N$. We observe that formally (S) is the Euler-Lagrange equation associated to the energy functional

$$\begin{aligned} I(u,v) &= \frac{1}{2}\int_{\mathbb{R}^N}\left[\left(1+2|u|^2\right)|\nabla u|^2+\left(1+2|v|^2\right)|\nabla v|^2\right]\,\mathrm{d}x \\ &\quad +\frac{1}{2}\int_{\mathbb{R}^N}V(x)\left(|u|^2+|v|^2\right)\,\mathrm{d}x-\int_{\mathbb{R}^N}K(x)H(u,v)\,\mathrm{d}x, \end{aligned}$$

but the functional I is not well defined in $H^1(\mathbb{R}^N)$ because of the term $|u|^2|\nabla u|^2$. Moreover, we have the following difficulties: lack of compactness because our equation (S) is defined in whole $\mathbb{R}^N$, the pontential V is bounded and can vanish in $B_R(0)$. So we will make some modifications which are appropriated to obtain a new class of problems where we are able to apply the mountain–pass argument (cf. [4, 24]). For that, first we will consider a reformulation of the problem following a penalization argument used in Alves [2], for the scalar case see an argument introduced by del Pino and Felmer [12]. Let us denote by χ_Λ the characteristic function of the set $\Lambda \subset \mathbb{R}^N$. For that, we formulate our problem in the weighted Sobolev space $\mathcal{H}$ and we introduce an auxiliary system modifying the gradient $(H_u(u,v), H_v(u,v))$ for a C^1 gradient, for which we can guarantee that Cerami sequences for the associated functional, J, of the auxiliary system are bounded and that J has a critical point in $\mathcal{H}$.

Remark 2.1. 1. Using condition (H_0), we have

$$|pH(u,v)| \leq 3c_0\left(u^p + v^p\right), \quad \forall\, u,v \geq 0. \tag{2.1}$$

Moreover, if $4 < p < 2^*$ we get $H_u(u,v),\ H_v(u,v) \in L^{p/(p-1)}(\mathbb{R}^N)$ for all $u, v \in L^p(\mathbb{R}^N)$.

2. By assumptions (H_1)–(H_2), we deduce that $H_u(u,v) = H_v(u,v) = 0$ if either $u = 0$ or $v = 0$. In addition, by the homogeneity of H,

$$pH(u,v) = uH_u(u,v) + vH_v(u,v), \quad \text{for all} \quad u,v \geq 0,$$

and thus, $H(u,0) = H(0,v) = 0, \quad \text{for all} \quad u,v \geq 0.$

3. From the hypotheses $(H_2) - (H_3)$, we see that $H_u(u,v), H_v(u,v)$ are non negative functions for $u,v \geq 0$, and thus thanks to the homogeneity property, the same holds for $H(u,v)$.

4. By assumption (V_1), $\mathcal{Z} = \left\{x \in \mathbb{R}^N : V(x) = 0\right\}$ is a compact set. We recall that $\mathcal{Z}$ can be empty set.

Let R given in (V_1), $\Lambda = B_R(0) \subset \mathbb{R}^N$ and $k = k_p = 2p/(p-4) > 2$. Let $a > 0$ be a real constant which will be chosen appropriately. Let us consider the cutoff function

$$\begin{cases} \eta : \mathbb{R} \to \mathbb{R} \quad \text{decreasing and } C^1, \\ \eta(t) = 1, \quad \forall t \in (-\infty, a], \\ \eta(t) = 0, \quad \forall t \in [5a, +\infty), \\ |\eta'(t)| \leq 1/5a, \quad \forall t \in \mathbb{R}. \end{cases} \tag{2.2}$$

Now, consider $\widehat{H} : \left(\mathbb{R}^N \setminus \Lambda\right) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by

$$\widehat{H}(x,t,s) := \begin{array}{r} \eta(\left|\left(K(x)\right)^{1/2}(t,s)\right|\Big) H(t,s) \\ + \left(1 - \eta\left(\left|(K(x))^{1/2}(t,s)\right|\right)\right) A(x) \dfrac{K(x)}{2k}\left(t^2+s^2\right), \end{array}$$

where

$$A(x) := \max\left\{\frac{2kH(t,s)}{K(x)(t^2+s^2)} : (s,t) \in \mathbb{R}^2 \text{ and } a \leq \sqrt{K(x)(t^2+s^2)} \leq 5a\right\}.$$

Thus, from (1), $A(x) \to 0$ uniformly as $a \to 0$. Note that $\widehat{H}$ is well defined, nonnegative and is of class C^1 (cf. item 3, Remark 2.1). We can now define $G : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by

$$G(x,t,s) := \chi_\Lambda(x) H(t,s) + (1 - \chi_\Lambda(x))\, \widehat{H}(x,t,s)$$

where χ_Λ denotes the characteristic function of the set Λ. Thus, G is a p-homogeneous function on Λ and $G(x,t,s) \geq 0$, $\forall (x,t,s) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}$. Moreover, for fixed $x \in \mathbb{R}^N$ the function $(t,s) \mapsto G(x,t,s)$ is of class C^1 and for each fixed $(t,s) \in \mathbb{R}^2$, the function $x \mapsto G(x,s,t)$ is Lebesgue measurable in $\mathbb{R}^N$.

Lemma 2.2. *The function G satisfies the following properties:*

$$pG(x,u,v) = uG_u(x,u,v) + vG_v(x,u,v), \quad \forall\, x \in \Lambda, \tag{2.3}$$

Moreover, for $k = k_p = 2p/(p-4)$, we can choose the constant a sufficiently small such that

$$2G(x,u,v) \leq uG_u(x,u,v) + vG_v(x,u,v) \leq \frac{K(x)}{k}\left(u^2+v^2\right),\ \forall\, x \in \mathbb{R}^N \setminus \Lambda. \tag{2.4}$$

Proof. Using the definition of G, we have that $G(x,u,v) = H(u,v)$ for all $x \in \Lambda$, and consequently (2.3) holds.

Observe that $G(x,u,v) = \widehat{H}(x,u,v)$ for all $x \in \mathbb{R}^N \setminus \Lambda$, besides that from the definition of the function $\widehat{H}$, we have for all $x \in \mathbb{R}^N \setminus \Lambda$

$$\widehat{H}_u = \eta\left(\left|K^{1/2}(u,v)\right|\right) H_u(u,v) + \frac{K}{k} uA(x)\left(1-\eta\left(\left|K^{1/2}(u,v)\right|\right)\right)$$

$$+\eta^{'}\left(\left|K^{1/2}(u,v)\right|\right)\left(K^{1/2}u(u^2+v^2)^{-1/2}\right)[H(u,v)]$$
$$+\eta^{'}\left(\left|K^{1/2}(u,v)\right|\right)\left(K^{1/2}u(u^2+v^2)^{-1/2}\right)\left[-A(x)\left(\frac{K(x)}{2k}(u^2+v^2)\right)\right] \tag{2.5}$$

and

$$\widehat{H}_v = \eta\left(\left|K^{1/2}(u,v)\right|\right) H_v(u,v) + \frac{K}{k} vA(x)\left(1-\eta\left(\left|K^{1/2}(u,v)\right|\right)\right)$$

$$+\eta^{'}\left(\left|K^{1/2}(u,v)\right|\right)\left(K^{1/2}v(u^2+v^2)^{-1/2}\right)[H(u,v)]$$
$$+\eta^{'}\left(\left|K^{1/2}(u,v)\right|\right)\left(K^{1/2}v(u^2+v^2)^{-1/2}\right)\left[-A(x)\left(\frac{K}{2k}(u^2+v^2)\right)\right] \tag{2.6}$$

They imply that

$$\begin{aligned} u\widehat{H}_u + v\widehat{H}_v = {} & p\eta\left(\left|K^{1/2}(u,v)\right|\right) H(u,v) \\ & + A(x)\left[\frac{K}{k}(u^2+v^2)\right]\left(1-\eta\left(\left|K^{1/2}(u,v)\right|\right)\right) \\ & + \eta^{'}\left(\left|K^{1/2}(u,v)\right|\right)\left(K^{1/2}(u^2+v^2)^{1/2}\right)[H(u,v)] \\ & + \eta^{'}\left(\left|K^{1/2}(u,v)\right|\right)\left(K^{1/2}(u^2+v^2)^{1/2}\right)\left[-A(x)\left(\frac{K}{2k}(u^2+v^2)\right)\right]. \end{aligned} \tag{2.7}$$

Therefore,

$$\begin{aligned} u\widehat{H}_u + v\widehat{H}_v \geq {} & p\eta\left(\left|K^{1/2}(u,v)\right|\right) H(u,v) \\ & + A(x)\left[\frac{K}{k}(u^2+v^2)\right]\left(1-\eta\left(\left|K^{1/2}(u,v)\right|\right)\right) \\ & \geq 2\widehat{H}(x,u,v) \quad \text{for all} \quad x \in \mathbb{R}^N \setminus \Lambda. \end{aligned}$$

Observe that

$$\frac{pH(u,v)}{\frac{K}{2k}(u^2+v^2)} \leq 3c_0 \frac{|u|^p+|v|^p}{\frac{K}{2k}(u^2+v^2)} \leq \frac{5^p 6kc_0 a^{p-2}}{\mu^{p/2}} \leq Ca^{p-2}.$$

Where $(u,v) \in \mathbb{R}^2$ such that $a \leq \sqrt{K(x)(u^2+v^2)} \leq 5a$ and $C > 0$ is a real constant. Since $p > 4$ then, we can conclude that taking a sufficiently small, we get

$$\frac{pH(u,v)}{\frac{K(x)}{2k}(u^2+v^2)} \leq 1.$$

Beside that we recall that $A(x) \geq 0$ and $A(x) \to 0$ uniformly as $a \to 0$ for all $x \in \mathbb{R}^N \setminus \Lambda$. From (2.7)

$$\frac{u\widehat{H}_u + v\widehat{H}_v}{\frac{K}{2k}(u^2+v^2)} = \eta\left(\left|K^{1/2}(u,v)\right|\right) \frac{pH(u,v)}{\frac{K}{2k}(u^2+v^2)} + 2A(x)\left(1-\eta\left(\left|K^{1/2}(u,v)\right|\right)\right)$$

$$+\eta'\left(\left|(K^{1/2}(u,v)\right|\right)\left(K^{1/2}(u^2+v^2)^{1/2}\right)\left[\frac{H(u,v)}{\frac{K}{2k}(u^2+v^2)} - A(x)\right].$$

Therefore,

$$\frac{u\widehat{H}_u + v\widehat{H}_v}{\frac{K}{2k}(u^2+v^2)} \leq 2.$$

That is,

$$u\widehat{H}_u + v\widehat{H}_v \leq \frac{K}{k}(u^2+v^2) \quad \text{for all} \quad x \in \mathbb{R}^N \setminus \Lambda.$$

□

We can now introduce the auxiliary system

$$\begin{cases} -\Delta u + V(x)u - [\Delta(u^2)]u = K(x)G_u(x,u,v) & \text{in} \quad \mathbb{R}^N, \\ -\Delta v + V(x)v - [\Delta(v^2)]v = K(x)G_v(x,u,v) & \text{in} \quad \mathbb{R}^N. \end{cases} \tag{AS}$$

The Euler-Lagrange functional associated with (AS), $\widehat{I}$ is given by

$$\widehat{I}(u,v) = \frac{1}{2}\int_{\mathbb{R}^N} \left[\left(1+2|u|^2\right)|\nabla u|^2 + \left(1+2|v|^2\right)|\nabla v|^2\right] \mathrm{d}x$$
$$+\frac{1}{2}\int_{\mathbb{R}^N} V(x)\left(|u|^2+|v|^2\right) \mathrm{d}x - \int_{\mathbb{R}^N} K(x)G(x,u,v)\,\mathrm{d}x,$$

As before, since the functional $\widehat{I}$ is not well defined in the space $\mathcal{H}$ we have to perform a suitable change of variable to get a new problem, which the associated functional is well defined in a new class of the function space defined in the next section.

2.2 Changing the variable

From the variational point of view, the second difficulty that we have to deal with is to find an appropriate variational setting in order to apply minimax methods to study the existence of nontrivial solutions of (AS). However, it should be pointed out that we may not apply directly such methods since the natural associated functional $\widehat{I}$ is not well defined in the usual Sobolev spaces. To overcome this difficulty, we follow the idea developed by Liu, Wang and Wang in [23], that is, we make the change of variables $w = f^{-1}(u), z = f^{-1}(v)$ where f is defined by

$$f^{'}(t) = \frac{1}{(1+2f^2(t))^{1/2}} \quad \text{on} \quad [0,\infty), \quad f(t) = -f(-t) \quad \text{on} \quad (-\infty, 0].$$

Proposition 2.3. *Basic properties of the change of variable $f(t)$ are listed below:*

(1) f *is a uniquely defined* C^∞ *function and invertible.*

(2) $|f'(t)| \leq 1$ *for all* $t \in \mathbb{R}$.

(3) $|f(t)| \leq |t|$ *for all* $t \in \mathbb{R}$.

(4) $f(t)/t \to 1$ *as* $t \to 0$.

(5) $f(t)/\sqrt{t} \to 2^{1/4}$ *as* $t \to +\infty$.

(6) $f(t)/2 \leq tf'(t) \leq f(t)$ *for all* $t \geq 0$.

(7) $|f(t)| \leq 2^{1/4}|t|^{1/2}$ *for all* $t \in \mathbb{R}$.

(8) the function $f^2(t)$ *is strictly convex.*

(9) there exists a positive constant C such that

$$|f(t)| \geq \begin{cases} C|t|, & |t| \leq 1 \\ C|t|^{1/2}, & |t| \geq 1. \end{cases}$$

(10) there exist positive constants C_1 and C_2 satisfying

$$|t| \leq C_1|f(t)| + C_2|f(t)|^2 \quad \textit{for all} \quad t \in \mathbb{R}.$$

(11) $|f(t)f'(t)| \leq 1/\sqrt{2}$ for all $t \in \mathbb{R}$.

(12) $f^2(\lambda s) \leq \lambda^2 f^2(s)$, for all $s \in \mathbb{R}$ and $\lambda \geq 1$.

(13) The function $f(t)f'(t)t^{-1}$ is decreasing for $t > 0$.

(14) The function $f^3(t)f'(t)t^{-1}$ is increasing for $t > 0$.

Proof. The proof of items (1)–(11) and (13)–(14) can be seen in [17, Proposition 2.2, Corollary 2.3] (see also [11, 23]). For item (12) one can see [18, Lemma 2.1]. □

Thus, after this change we obtain the following functional

$$J(w,z) := \widehat{I}(f(w), f(z)) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla w|^2 + V(x)f^2(w)\right) \mathrm{d}x$$
$$\frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla z|^2 + V(x)f^2(z)\right) \mathrm{d}x - \int_{\mathbb{R}^N} G(x, f(w), f(z))\, \mathrm{d}x,$$

which is well defined on $\mathcal{H}$. Moreover, nontrivial critical points of J correspond precisely to the positive weak solutions of the system (AS). As in [26] (see also [18]) we see that if (w,z) is a weak solution for (MS) then $u = f(w), v = f(z)$ is a weak solution for (AS). Our goal here is to prove the existence of a critical point (w,z) for J, associated with the system,

$$\begin{cases} -\Delta w = f'(w)\left[G_u(x, f(w), f(z)) - V(x)f(w)\right] & \text{in} \quad \mathbb{R}^N, \\ -\Delta z = f'(z)\left[G_v(x, f(w), f(z)) - V(x)f(z)\right] & \text{in} \quad \mathbb{R}^N. \end{cases} \tag{MS}$$

Remark 2.4. 1. Using Remark (2.1) and the property (7), Proposition (2.3), we have

$$|pG(x2,f(w),f(z))| \leq 3c_0\left(|f(w)|^p + |f(z)|^p\right) \leq 2^{p/4}3c_0\left(|w|^{p/2} + |z|^{p/2}\right),$$

for all $f(w), f(z) \geq 0$, $x \in \Lambda$.

Moreover, $G_u(x, f(w), f(z)),\ G_v(x, f(w), f(z)) \in L^{p/(p-1)}(\mathbb{R}^N)$ for all $(w, z) \in \mathcal{H}$, $x \in \Lambda$.

3 Compactness results

3.1 Mountain-pass geometry

Proposition 3.1. *Suppose that* $(H_0) - (H_3)$ *and* $(V_0) - (V_1)$ *hold. The Euler-Lagrange functional* J *associated with* (MS) *satisfies the following conditions:*

1. J *is well defined and continuous in* $\mathcal{H}$.
2. J *is Gateaux-differentiable in* $\mathcal{H}$ *and its derivative is given by*

$$J'(w,z)(\phi,\varphi) = \int_{\mathbb{R}^N}\left(\nabla w \nabla \phi + V(x)f(w)f'(w)\phi\right)\mathrm{d}x$$
$$+ \int_{\mathbb{R}^N}\left(\nabla z \nabla \varphi + V(x)f(z)f'(z)\varphi\right)\mathrm{d}x - \int_{\mathbb{R}^N} G_u(x, f(w), f(z))f'(w)\phi\,\mathrm{d}x$$
$$- \int_{\mathbb{R}^N} G_v(x, f(w), f(z))f'(z)\phi\,\mathrm{d}x, \ \textit{for all } (w,z), (\phi,\varphi) \ \in \mathcal{H}.$$

3. *For* $(w, z) \in \mathcal{H}$, $J'(w, z) \in \mathcal{H}'$ *and if* $(w_n, z_n) \to (w, z)$ *in* $\mathcal{H}$ *then* $J'(w_n, z_n) \to J'(w, z)$ *in the weak-∗topology of* $\mathcal{H}'$, *that is, for each* $(\phi, \varphi) \in \mathcal{H}$ *we have*

$$\langle J'(w_n, z_n), (\phi, \varphi)\rangle \to \langle J'(w, z), (\phi, \varphi)\rangle.$$

Proof. The proof is essentially the same as in [17, Proposition 2.5].

It is standard to prove that J satisfies the mountain pass geometry, we include the proof for ready reference. See [24,25] for more details. For this next result we consider the functional $Q : \mathcal{H} \to \mathbb{R}$ defined by

$$Q(w,z) = \int_{\mathbb{R}^N} \left(|\nabla w|^2 + V(x) f^2(w) + |\nabla z|^2 + V(x) f^2(z)\right) \mathrm{d}x \qquad (3.1)$$

and the set

$$S(\rho) := \{(w,z) \in \mathcal{H} : \; Q(w,z) = \rho^2\}.$$

Remark 3.2.

From the item (3) Proposition (2.3), the functional $Q(w,z) \in C^1$ and $Q(w,z) \leq ||(w,z)||^2_{\mathcal{H}}$.
There is a real constant $\beta > 0$ such that $\beta ||(w,z)||^2_{\mathcal{H}} \leq Q(w,z) + (Q(w,z))^{2^*/2}$.

For further details to consult [1].

Lemma 3.3. *(Mountain–pass geometry) Suppose that* $(H_0) - (H_3)$ *and* $(V_0) - (V_1)$ *hold. The functional J has the Mountain Pass geometry, that is, J satisfies*

1. *Exists ρ, $\alpha > 0$, such that $J(w,z) \geq \alpha$ if $(w,z) \in S(\rho)$,*
2. *For any $(w,z) \in \mathcal{H}$, $w, z > 0$ with compact support on Λ, there is $0 < \theta < 1$ such that $J(t^\theta w, t^\theta z) \to -\infty$ as $t \to +\infty$.*

Proof. From Remark 2.4 we have that

$$|pG(x, f(w), f(z))| \leq \;\; 3c_0 \left(|f(w)|^p + |f(z)|^p\right) \text{ for all } \quad x \in \Lambda.$$

By the Sobolev imbedding

$$\begin{aligned}\int_\Lambda K(x) G(x, f(w), f(z))\,\mathrm{d}x &= \int_\Lambda K(x) H(f(w), f(z))\,\mathrm{d}x \\ &\leq C_0 \int_{\mathbb{R}^N} \left(|f(w)|^p + |f(z)|^p\right) \mathrm{d}x \\ &\leq C_1 \int_{\mathbb{R}^N} \left(|w|^{p/2} + |z|^{p/2}\right) \mathrm{d}x \leq C_2 Q(w,z)^{p/4},\end{aligned}$$

where C_0, C_1, C_2 are positive constants. Using above estimate, we have that choose $\rho > 0$ small enough such that $D = \left(\frac{1}{2} - \frac{\lambda}{2k} - C_2 \rho^{(p-4)/2}\right) > 0$.

Let $\alpha = D\rho^2$. Then exists ρ, $\alpha > 0$ such that $J(u,v) \geq \alpha$, for all $(u,v) \in \partial B(0,\rho)$.

$$\begin{aligned}
J(w,z) &= \frac{1}{2}Q(w,z) - \int_{\Lambda} K(x)G(x,f(w),f(z))\,\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^N\setminus\Lambda} K(x)G(x,f(w),f(z))\,\mathrm{d}x \\
&\geq \frac{1}{2}Q(w,z) - C_2 Q(w,z)^{p/4} - \int_{\mathbb{R}^N\setminus\Lambda} \lambda\frac{V(x)}{2k}\left(f^2(w)+f^2(z)\right)\,\mathrm{d}x \\
&\geq \left(\left(\frac{1}{2}-\frac{\lambda}{2k}\right)\rho^2 - C_2\rho^{p/2}\right) = \left(\frac{1}{2}-\frac{\lambda}{2k} - C_2\rho^{(p-4)/2}\right)\rho^2.
\end{aligned}$$

Now, we will prove (2). Consider $\theta > 0$ such that $1/p < \theta < 1/2$. From the definition of G, $G(x,f(w),f(z)) = H(f(w),f(z))$ for $x \in \Lambda$. So,

$$pG(x,f(w),f(z)) = f(w)G_u(x,f(w),f(z)) + f(z)G_v(x,f(w),f(z))$$

where $x \in \Lambda$.

Therefore we have

$$\begin{aligned}
\frac{d}{dt}\left\{G(x,t^\theta f(w),t^\theta f(z))\right\} &= p\theta\frac{1}{t}G(x,t^\theta f(w),t^\theta f(z)) \\
&\geq \frac{1}{t}G(x,t^\theta f(w),t^\theta f(z)). \qquad (3.2)
\end{aligned}$$

Observe that $p\theta > 1$ and from $(H_2)-(H_3)$ and the definition of G, we have $G(x,f(w),f(z)) = H(f(w),f(z)) > 0$ where $x \in \Lambda$ (cf. Remark 2.1 (3)).

The inequality (3.2) implies that $G(x,t^\theta f(w),t^\theta f(z)) \geq t\tilde{Q}(x,f(w),f(z))$ for some function $\tilde{Q}$. Then for $t > 1$, we have

$$\begin{aligned}
J(t^\theta w,t^\theta z) \leq \frac{1}{2}t^{2\theta}\int_{\Lambda}\left(|\nabla w|^2 + V(x)f^2(w) + |\nabla z|^2 + V(x)f^2(z)\right)\,\mathrm{d}x \\
- \int_{\Lambda} tK(x)\tilde{Q}(x,f(w),f(z))\,\mathrm{d}x.
\end{aligned}$$

Since $2\theta < 1$, we conclude that for any $(w,z),(\phi,\varphi) \in \mathcal{H}\setminus(0,0)$ fixed, $w,z > 0$ with compact support on Λ, we have

$$J(t^\theta f(w),t^\theta f(z)) \to -\infty \quad \text{as} \quad t \to +\infty.$$

This completes the proof. □

3.2 The Cerami condition

In order to apply critical point theory to prove the existence of weak solutions of (AS), we first need to study some compactness property of functional J.

Lemma 3.4. *Suppose that* $(H_0)-(H_3)$ *and* $(V_0)-(V_1)$ *hold. Then, any Cerami sequence for* J *is bounded in* $\mathcal{H}$.

Proof. Let $(w_n, z_n) \subset \mathcal{H}$ be a Cerami sequence for J, that is,

$$|J(w_n, z_n)| \leq c \quad \text{and}\, (1 + ||(w_n, z_n)||)\, ||J'(w_n, z_n)|| \to 0 \quad \text{as } n \to \infty \text{ in } \quad \mathcal{H}'.$$

Therefore

$$\begin{aligned}\langle J'(w_n, z_n), (\psi, \phi)\rangle = \\ &\int_{\mathbb{R}^N} \Big(\nabla w_n \nabla \psi + V(x) f(w_n) f'(w_n)\psi\Big)\, \mathrm{d}x \\ &+ \int_{\mathbb{R}^N} \Big(\nabla z_n \nabla \phi + V(x) f(z_n) f'(z_n)\phi\Big)\, \mathrm{d}x \\ &- \int_{\mathbb{R}^N} K(x) \left(G_u(f(w_n), f(z_n)) f'(w_n)\psi\right)\, \mathrm{d}x \\ &- \int_{\mathbb{R}^N} K(x) \left(G_v(f(w_n), f(z_n)) f'(z_n)\phi\right)\, \mathrm{d}x \\ &\text{for all } (w, z), (\psi, \phi) \in \mathcal{H}.\end{aligned}$$

We can note that $\left(\frac{f(w_n)}{f'(w_n)}, \frac{f(z_n)}{f'(z_n)}\right) \in \mathcal{H}$ because from the property item (6), Proposition (2.3), and the estimates below

$$\left|\nabla\left(\frac{f(w_n)}{f'(w_n)}\right)\right| = \left(1 + \frac{2f^2(w_n)}{1 + 2f^2(w_n)}\right)|\nabla w_n|\,;$$
$$\left|\nabla\left(\frac{f(z_n)}{f'(z_n)}\right)\right| = \left(1 + \frac{2f^2(z_n)}{1 + 2f^2(z_n)}\right)|\nabla z_n|.$$

Moreover, $\left\|\left(\frac{f(w_n)}{f'(w_n)}, \frac{f(z_n)}{f'(z_n)}\right)\right\| \leq 2||(w_n, z_n)||$. Therefore, we have

$$\left\langle J'(w_n, z_n), \left(\frac{f(w_n)}{f'(w_n)}, \frac{f(z_n)}{f'(z_n)}\right)\right\rangle$$
$$= \int_{\mathbb{R}^N} K(x) \left(G_u(f(w_n), f(z_n)) f(w_n) + G_v(f(w_n), f(z_n)) f(z_n)\right) \mathrm{d}x + o_n(1)$$

Let the function Q defined in (3.1). Thus,

$$Q(w_n, z_n) + \int_{\mathbb{R}^N} \frac{2f^2(w_n)}{1+2f^2(w_n)} |\nabla w_n|^2 \, \mathrm{d}x + \int_{\mathbb{R}^N} \frac{2f^2(z_n)}{1+2f^2(z_n)} |\nabla z_n|^2 \, \mathrm{d}x$$
$$= \int_{\mathbb{R}^N} K(x) \left(G_u(f(w_n), f(z_n)) f(w_n) + G_v(f(w_n), f(z_n)) f(z_n) \right) \mathrm{d}x + o_n(1)$$

and

$$\frac{1}{2} Q(w_n, z_n) = \int_{\mathbb{R}^N} K(x) G(x, f(w_n), f(z_n)) \, \mathrm{d}x + O_n(1).$$

Consider $\Omega = \mathbb{R}^N \setminus \Lambda$. Recalling that $k = k_p = 2p/(p-4)$, we have

$$\begin{aligned}
\left(\frac{1}{2} - \frac{2}{p}\right) Q(w_n, z_n) \le & \int_\Lambda K(x) \left(G(x, f(w_n), f(z_n)) \right) \mathrm{d}x \\
& + \int_\Lambda K(x) \left(-\frac{1}{p} \left[f(w_n) G_u(x, f(w_n), f(z_n)) \right] \right) \mathrm{d}x \\
& + \int_\Lambda K(x) \left(-\frac{1}{p} \left[f(z_n) G_v(x, f(w_n), f(z_n)) \right] \right) \mathrm{d}x \\
& + \int_\Omega K(x) \left(G(x, f(w_n), f(z_n)) \right) \mathrm{d}x \\
& + \int_\Omega K(x) \left(-\frac{1}{p} \left[f(w_n) G_u(x, f(w_n), f(z_n)) \right] \right) \mathrm{d}x \\
& + \int_\Omega K(x) \left(-\frac{1}{p} \left[f(z_n) G_v(x, f(w_n), f(z_n)) \right] \right) \mathrm{d}x \\
& + O_n(1) + o_n(1).
\end{aligned}$$

Using (2.3), (2.4), the condition (V_1), we deduced that

$$\begin{aligned}
\left(\frac{1}{2} - \frac{2}{p}\right) Q(w_n, z_n) \le & \frac{\lambda(p-2)}{2pk} \int_\Omega \left(V(x) f^2(w_n) + V(x) f^2(z_n) \right) \mathrm{d}x \\
& + O_n(1) + o_n(1),
\end{aligned} \tag{3.3}$$

which implies that

$$\left(\frac{1}{k} - \frac{\lambda(p-2)}{2pk} \right) Q(w_n, z_n) \le O_n(1) + o_n(1).$$

Using that $\lambda < 2p/(p-2)$ and remark (3.2), we have (w_n, z_n) is bounded in $\mathcal{H}$. □

Lemma 3.5. *Suppose that* (H_0)–(H_3) *and* (V_0)–(V_1) *hold. Let* $(w_n, z_n) \rightharpoonup (w, z)$. *Then*

1.

$$\begin{aligned} \lim_{n\to\infty} \int_{\mathbb{R}^N} V(x) f^2(w_n)\,\mathrm{d}x &= \int_{\mathbb{R}^N} V(x) f^2(w)\,\mathrm{d}x, \\ \lim_{n\to\infty} \int_{\mathbb{R}^N} V(x) f^2(z_n)\,\mathrm{d}x &= \int_{\mathbb{R}^N} V(x) f^2(z)\,\mathrm{d}x. \end{aligned} \tag{3.4}$$

2. Moreover, if $(w_n, z_n) \subset \mathcal{H}$ *be an arbitrary Cerami sequence of* J*. Then,* (w, z) $(w, z) \in \mathcal{H}$ *is a critical point for the functional* J*.*

Proof. From Lemma 3.4, up to a subsequence, we can assume that there exists $(w, z) \in \mathcal{H}$ such that Let be $R > 0$ given in (V_1). For each $\varepsilon > 0$, let $r > R$ be such that

$$4\left(1 - \frac{\lambda}{k}\right)^{-1} \omega_N^{1/N} C \left(\int_{r\le |x| \le 2r} \max\left(|w(x)|, |z(x)|\right)^{2^*}\,\mathrm{d}x\right)^{1/2^*} < \frac{\varepsilon}{8},$$

where $C \geq \; ||(w_n, z_n)||_{\mathcal{H}}$ is a positive constant and ω_N is the volume of the unitary ball in $\mathbb{R}^N$. Let $\xi = \xi_r \in C_0^\infty(\mathbb{R}^N, [0, 1])$, be a function verifying $supp\,\xi \subseteq B_r^c(0)$, $\xi \equiv 1$ in $B_{2r}^c(0)$, $0 < \xi < 1$ if $r < |x| < 2r$ and

$$|\xi'(x)| \leq \frac{1}{r}, \quad \text{for all}\, x \in \mathbb{R}^N.$$

Since (w_n, z_n) is bounded in H, $|\xi(x)| \leq 1$ and using the property (6), Proposition (2.3),so we get that the sequence $(\xi\,(f(w_n)/f'(w_n), f(z_n)/f'(z_n)))$ is also bounded, and hence $J'(u_n, v_n) \cdot (\xi\,(f(w_n)/f'(w_n), f(z_n)/f'(z_n))) = o_n(1)$, that is,

$$\begin{aligned} &\int_{\mathbb{R}^N} \left(\nabla w_n \nabla \left(\frac{\xi f(w_n)}{f'(w_n)}\right) + \nabla z_n \nabla \left(\frac{\xi f(z_n)}{f'(z_n)}\right)\right) \mathrm{d}x \\ &+ \int_{\mathbb{R}^N} \left[V(x) f(w_n) f'(w_n) \left(\frac{\xi f(w_n)}{f'(w_n)}\right) + V(x) f(z_n) f'(z_n) \left(\frac{\xi f(z_n)}{f'(z_n)}\right)\right] \mathrm{d}x = \\ &\int_{\mathbb{R}^N} K(x) \left(f'(w_n) \left(\frac{\xi f(w_n)}{f'(w_n)}\right) G_u(x, f(w_n), f(z_n))\right) \mathrm{d}x \\ &+ \int_{\mathbb{R}^N} K(x) \left(f'(z_n) \left(\frac{\xi f(z_n)}{f'(z_n)}\right) G_v(x, f(w_n), f(z_n))\right) \mathrm{d}x \\ &+ o_n(1). \end{aligned}$$

Since $\xi \equiv 0$ in $B_r(0)$ and $\Lambda = B_R(0)$, the last equality combined with the property (2.4) and condition (V_1) yields

$$\int_{|x|\geq r} \left(\left(1 + \frac{2f^2(w_n)}{1+2f^2(w_n)}\right) |\nabla w_n|^2 + \left(1 + \frac{2f^2(z_n)}{1+2f^2(z_n)}\right) |\nabla z_n|^2 \right) \mathrm{d}x$$
$$+ \int_{|x|\geq r} \left(V(x) f^2(w_n) + V(x) f^2(z_n)\right) \xi \, \mathrm{d}x \leq$$
$$\leq \frac{\lambda}{k} \int_{|x|\geq r} \left(V(x) f^2(w_n) + V(x) f^2(z_n)\right) \xi \, \mathrm{d}x - \int_{|x|\geq r} \left(\frac{f(w_n)}{f'(w_n)}\right) \nabla w_n \nabla \xi \, \mathrm{d}x$$
$$- \int_{|x|\geq r} \left(\frac{f(z_n)}{f'(z_n)}\right) \nabla z_n \nabla \xi \, \mathrm{d}x + o_n(1).$$

and so, using the Proposition (2.3), property (6), we have

$$\begin{aligned} \left(1 - \frac{\lambda}{k}\right) \int_{|x|\geq 2r} & \left(|\nabla w_n|^2 + V(x) f^2(w_n) + |\nabla z_n|^2 + V(x) f^2(z_n)\right) \xi \, \mathrm{d}x \\ \leq \quad \frac{2}{r} \int_{r\leq|x|\leq 2r} & \left(|w_n||\nabla w_n| + |z_n||\nabla z_n|\right) \mathrm{d}x + o_n(1). \end{aligned} \tag{3.5}$$

Here we have used assumption (V_1) to guarantee that $\lambda < k$. By Holder's inequality,

$$\begin{cases} \displaystyle\int_{r\leq|x|\leq 2r} |w_n||\nabla w_n| \, \mathrm{d}x \leq & ||\nabla w_n||_{L^2(\mathbb{R}^N)} \left(\displaystyle\int_{r\leq|x|\leq 2r} |w_n|^2 \, \mathrm{d}x\right)^{1/2} \\ \displaystyle\int_{r\leq|x|\leq 2r} |z_n||\nabla z_n| \, \mathrm{d}x \leq & ||\nabla z_n||_{L^2(\mathbb{R}^N)} \left(\displaystyle\int_{r\leq|x|\leq 2r} |z_n|^2 \, \mathrm{d}x\right)^{1/2} \end{cases}.$$

Due to the Rellich-Kondrachov Compactness Theorem, we have that $(w_n, z_n) \to (w, z)$ as $n \to \infty$ in $L^2(B_{2r} \setminus B_r)$ and using that (w_n, z_n) is bounded, it follows that

$$\begin{cases} \displaystyle\limsup_n \int_{r\leq|x|\leq 2r} |w_n||\nabla w_n| \, \mathrm{d}x \leq C \left(\int_{r\leq|x|\leq 2r} |w|^2 \, \mathrm{d}x\right)^{1/2} \\ \\ \displaystyle\limsup_n \int_{r\leq|x|\leq 2r} |z_n||\nabla z_n| \, \mathrm{d}x \leq C \left(\int_{r\leq|x|\leq 2r} |z|^2 \, \mathrm{d}x\right)^{1/2} \end{cases}. \tag{3.6}$$

On the other hand, using again Holder's inequality

$$\begin{cases} \left(\int_{r\le|x|\le 2r} |w|^2 \,\mathrm{d}x\right)^{1/2} \le \left(\int_{r\le|x|\le 2r} |w|^{2^*} \,\mathrm{d}x\right)^{1/2^*} |B_{2r}\setminus B_r|^{1/N} \\ \left(\int_{r\le|x|\le 2r} |z|^2 \,\mathrm{d}x\right)^{1/2} \le \left(\int_{r\le|x|\le 2r} |z|^{2^*} \,\mathrm{d}x\right)^{1/2^*} |B_{2r}\setminus B_r|^{1/N}. \end{cases} \tag{3.7}$$

Recalling that $|B_{2r}\setminus B_r| \le |B_{2r}| = \omega_N 2^N r^N$, from (3.6) and (3.7)

$$\begin{cases} \limsup_n \int_{r\le|x|\le 2r} |w_n||\nabla w_n| \,\mathrm{d}x \le 2\omega_N^{1/N} rC \left(\int_{r\le|x|\le 2r} |w|^{2^*} \,\mathrm{d}x\right)^{1/2^*} \\ \limsup_n \int_{r\le|x|\le 2r} |z_n||\nabla z_n| \,\mathrm{d}x \le 2\omega_N^{1/N} rC \left(\int_{r\le|x|\le 2r} |z|^{2^*} \,\mathrm{d}x\right)^{1/2^*}. \end{cases} \tag{3.8}$$

By choosing $r > 0$ in (3.5), (3.5) and (3.8), it implies that

$\limsup_n \int_{|x|\ge 2r}(|\nabla w_n|^2 + V(x)f^2(w_n) + |\nabla z_n|^2 + V(x)f^2(z_n)) \,\mathrm{d}x < \dfrac{\varepsilon}{4}$.

Therefore,

$$\begin{aligned} &\limsup_n \int_{|x|\ge 2r} K(x)\left(f(w_n)G_u(x, f(w_n), f(z_n)))\right) \,\mathrm{d}x \\ &+ \limsup_n \int_{|x|\ge 2r} K(x)\left(f(z_n)G_v(x, f(w_n), f(z_n))\right) \,\mathrm{d}x \\ &\le \frac{\varepsilon}{4}. \end{aligned} \tag{3.9}$$

Observe that $G_u(x, f(w_n), f(z_n)), G_v(x, f(w_n), f(z_n)) \in L^{p/(p-1)}(\mathbb{R}^N)$ for $x \in B_{2r}(0)$. Now, using the Sobolev compact embedding and Dominated Convergence Theorem, they lead to

$$\begin{aligned} &\lim_{n\to\infty} \int_{|x|\le 2r} K(x)\left(f(w_n)G_u(x, f(w_n), f(z_n)) + f(z_n)G_v(x, f(w_n), f(z_n))\right) \,\mathrm{d}x = \\ &\int_{|x|\le 2r} K(x)\left(f(w)G_u(x, f(w), f(z)) + f(z)G_v(x, f(w), f(z))\right) \,\mathrm{d}x. \end{aligned} \tag{3.10}$$

and

$$\lim_{n\to\infty}\int_{|x|\le 2r} V(x)f^2(w_n)\,\mathrm{d}x = \int_{|x|\le 2r} V(x)f^2(w)\,\mathrm{d}x.$$
$$\lim_{n\to\infty}\int_{|x|\le 2r} V(x)f^2(z_n)\,\mathrm{d}x = \int_{|x|\le 2r} V(x)f^2(z)\,\mathrm{d}x. \tag{3.11}$$

Hence, we can conclude that , as $n\to\infty$

$$\int_{\mathbb{R}^N} K(x)G_u(x,f(w_n),f(z_n))f(w_n)\,\mathrm{d}x \to \int_{\mathbb{R}^N} K(x)G_u(x,f(w_n),f(z_n))f(w)\,\mathrm{d}x,$$
$$\int_{\mathbb{R}^N} K(x)G_v(x,f(w_n),f(z_n))f(z_n)\,\mathrm{d}x \to \int_{\mathbb{R}^N} K(x)G_v(x,f(w_n),f(z_n))f(z)\,\mathrm{d}x, \tag{3.12}$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} V(x)f^2(w_n)\,\mathrm{d}x = \int_{\mathbb{R}^N} V(x)f^2(w)\,\mathrm{d}x.$$
$$\lim_{n\to\infty}\int_{\mathbb{R}^N} V(x)f^2(z_n)\,\mathrm{d}x = \int_{\mathbb{R}^N} V(x)f^2(z)\,\mathrm{d}x. \tag{3.13}$$

From (3.12) we have that

$$\begin{aligned}
&\lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)\left(f(w_n)G_u(x,f(w_n),f(z_n))\right)\,\mathrm{d}x+\\
&\lim_{n\to\infty}\int_{\mathbb{R}^N} K(x)\left(f(z_n)G_v(x,f(w_n),f(z_n))\right)\,\mathrm{d}x=\\
&\int_{\mathbb{R}^N} K(x)\left(f(w)G_u(x,f(w),f(z))+f(z)G_v(x,f(w),f(z))\right)\,\mathrm{d}x.
\end{aligned} \tag{3.14}$$

Finally, Let's prove that $(w,z)\in\mathcal{H}$ is a critical point for the functional J. In fact,

$$\langle J'(w_n,z_n),(\psi,\phi)\rangle \to \langle J'(w,z),(\psi,\phi)\rangle \quad \text{for all}\,(w,z),$$
$$\text{and }(\psi,\phi)\ \in C_0^\infty(\mathbb{R}^N).$$

The results follows from (3.13) and (3.14). □

Hereafter, we denote by B the ball in $\mathbb{R}^N$ with center 0 and radius $R/2$, that is, $B = B_{R/2}(0)$, the set $\Lambda = B_R(0)$ and by $J_0 : H_0^1(B) \times H_0^1(B) \to \mathbb{R}$ the functional

$$J_0(w,z) = \frac{1}{2}\int_B \left(|\nabla w|^2 + \max_B (V(x),1)\, f^2(w) \right) \mathrm{d}x$$
$$+ \frac{1}{2}\int_B \left(|\nabla z|^2 + \max_B (V(x),1)\, f^2(z) \right) \mathrm{d}x - \int_B K(x) H(f(w), f(v))\, \mathrm{d}x.$$

Moreover, we denote by d the mountain level associated with J_0, that is,

$$d = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J_0(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C\left([0,1], H_0^1(B) \times H_0^1(B)\right) : \gamma(0) = 0 \quad \text{and} \quad \gamma(1) = e\},$$

with $e \in H_0^1(B) \times H_0^1(B) \setminus \{(0,0)\}$ verifying $J_0(e) < 0$.

Remark 3.6. We observe that $J(w,z) \leq J_0(w,z)$ for all w, $z \in H_0^1(B)$. In particular we have $J(e) \leq J_0(e) < 0$. We denote by m_J the mountain pass level associated with J, that is,

$$m_J = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C\left([0,1], H_0^1(B) \times H_0^1(B)\right) : \gamma(0) = 0 \quad \text{and} \quad \gamma(1) = e\}.$$

It is easily seen that $m_J \leq d$.

In order to prove the existence of a nontrivial critical point for J we will use the following version of the Mountain Pass theorem, which is a consequence of the Ekeland Variational Principle as developed in [28] (see also [6], [20]).

Proposition 3.7. *Le E be a Banach space and $\Phi \in C(E, \mathbb{R})$, Gateaux-differentiable for all $v \in E$ with G-derivative Φ' continuous from the norm topology of E to the weak $*$ topology of E'. Suppose also that Φ satisfies Cerami condition and $\Phi(0) = 0$. Let S be a closed subset of E which disconnects (archwise) E. Let $v_0 = 0$ and $v_1 \in E$ be points belonging to distinct connected components of $E \backslash S$. Suppose that*

$$\inf_S \Phi \geq \alpha > 0 \quad \text{and} \quad \Phi(v_1) \leq 0.$$

Then, Φ possesses a critical value c which can be characterized as

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \Phi(\gamma(t)) \geq \alpha,$$

where

$$\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0 \quad \text{and} \quad \gamma(1) = v_1\}.$$

Lemma 3.8. *Let be $(w, z) \in \mathcal{H}$ a critical point of the functional J at the minimax level m_J. Then (w, z) satisfies the estimate*

$$\| (w,z) \|_{\mathcal{H}}^2 \leq \beta^{-1} \left(\frac{1}{k} - \frac{\lambda(p-2)}{2pk} \right)^{-1} d + \left(\beta^{-1} \left(\frac{1}{k} - \frac{\lambda(p-2)}{2pk} \right)^{-1} \right)^{2^*/2} d^{2^*/2}.$$

Proof. It is enough to combine remarks (3.2) and (3.3) with definition of d and the fact that $m_J \leq d$. □

Lemma 3.9. *If $\{(w,z)\} \in \mathcal{H}$ is a point critical of the functional J, then $w, z \geq 0$.*

Proof. Observe that

$$\begin{aligned}
&\left\langle J'(w,z), \left(\frac{f(w)}{f'(w)}, \frac{f(z)}{f'(z)}\right)\right\rangle \\
&= Q(w,z) + \int_{\mathbb{R}^N} \frac{2f^2(w)}{1+2f^2(w)}|\nabla w|^2 \,\mathrm{d}x + \int_{\mathbb{R}^N} \frac{2f^2(z)}{1+2f^2(z)}|\nabla z|^2 \,\mathrm{d}x \\
&- \int_{\mathbb{R}^N} K(x)\left(G_u(f(w),f(z))f(w) + G_v(f(w),f(z))f(z)\right) \mathrm{d}x \\
&= 0.
\end{aligned}$$

Let the function Q defined in (3.1). Thus,

$$\begin{aligned}
&Q(w,z) + \int_{\mathbb{R}^N} \frac{2f^2(w)}{1+2f^2(w)}|\nabla w|^2 \,\mathrm{d}x + \int_{\mathbb{R}^N} \frac{2f^2(z)}{1+2f^2(z)}|\nabla z|^2 \,\mathrm{d}x \\
&= \int_{\mathbb{R}^N} K(x)\left(G_u(f(w),f(z))f(w) + G_v(f(w),f(z))f(z)\right) \mathrm{d}x.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&Q(w^-,z^-) + \int_{\mathbb{R}^N} \frac{2f^2(w^-)}{1+2f^2(w^-)}|\nabla w^-|^2 \,\mathrm{d}x + \int_{\mathbb{R}^N} \frac{2f^2(z^-)}{1+2f^2(z^-)}|\nabla z^-|^2 \,\mathrm{d}x \\
&= \int_{\mathbb{R}^N} K(x)\big(G_u(f(w^-),f(z^-))f(w^-) + G_v(f(w^-),f(z^-))f(z^-)\big) \,\mathrm{d}x.
\end{aligned}$$

Now, from the definition of the non linearities H, G and properties of the function f, we have $H_u(f(w^-), f(z^-) = H_v(f(w^-), f(z^-) = 0$ and $G_u(f(w^-), f(z^-)) = G_v(f(w^-), f(z^-)) = 0$ and the inequality (2.4), we have

$$\begin{aligned}
Q(w^-,z^-) \;\leq\; & \int_{\Lambda} K(x)H_u(f(w^-),f(z^-))f(w^-)\,\mathrm{d}x \\
& + \int_{\Lambda} K(x)H_v(f(w^-),f(z^-))f(z^-)\,\mathrm{d}x \\
& + \int_{\mathbb{R}^N\setminus\Lambda} K(x)G_u(f(w^-),f(z^-))f(w^-)\,\mathrm{d}x \\
& + \int_{\mathbb{R}^N\setminus\Lambda} K(x)G_v(f(w^-),f(z^-))f(z^-)\,\mathrm{d}x \\
\leq\; & 0.
\end{aligned}$$

Recall that $||(w,z)||^2_{\mathcal{H}} \leq \beta^{-1}Q(w,z) + \left(\beta^{-1}Q(w,z)\right)^{2^*/2}$. Therefore, we conclude $||(u^-,v^-)||_{\mathcal{H}} = 0$. □

We have proved up this moment the following result:

Proposition 3.10. *There is a critical point $(\varphi,\phi) \in \mathcal{H}$ with φ,ϕ non negative functions associated to the functional*

$$J(w,z) \;=\; \frac{1}{2}Q(w,z) - \int_{\mathbb{R}^N} K(x)G(x,f(w),f(z))\,\mathrm{d}x, \quad (w,z) \in \mathcal{H}.$$

at the critical level

$$m_J = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t))$$

where

$$\Gamma = \{\gamma \in C\left([0,1],\mathcal{H}\right) : \gamma(0) = 0 \;\text{ and }\; \gamma(1) = e\},$$

with $e \in H^1_0(B) \times H^1_0(B) \setminus \{(0,0)\}$ *verifying,* $J_0(e) < 0$.

Proof. It follows from Proposition 3.7. □

4 Finding a positive solution

In this section, we show some qualitatives properties of (AS) systems solutions which are necessary to we get positive solutions by Maximum Principle. The next proposition establishes an important estimate involving the $L^\infty(\mathbb{R}^N)$ norm for solutions (w,z) of the system (AS). Here, we used the Moser iteration scheme which was adapted to our problem from the classical paper [8]).

Proposition 4.1. *Let $Y \in L^q(\mathbb{R}^N)$, $2q > N$, and $(w,z) \in \mathcal{H} \subset H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ be a weak solution of the problem*

$$\begin{cases} -\Delta w = f'(w)\left[G_u(x,f(w),f(z)) - a(x)f(w)\right] \quad \text{in} \quad \mathbb{R}^N, \\ -\Delta z = f'(z)\left[G_v(x,f(w),f(z)) - b(x)f(z)\right] \quad \text{in} \quad \mathbb{R}^N, \end{cases} \tag{4.1}$$

where $G_u, G_v : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *are a continuous functions verifying*

$$|G_u(x, f(w), f(z))| + |G_v(x, f(w), f(z))| \quad \leq \quad Y(x)\left(|w| + |z|\right), a.e.$$

for all w, $z \geq 0$, *and* a, b *are nonnegative continuous functions in* $\mathbb{R}^N$. *Then exists a constant* $M = M(N, q, ||Y||_{L^q(\mathbb{R}^N)}) > 0$ *such that*

$$\begin{aligned} ||(w, z)||_\infty &= \max\left(||w||_\infty, ||z||_\infty\right) \\ &\leq M|| \max\left(|w(x)|, |z(x)|\right) ||_{L^{2^*}(\mathbb{R}^N)}. \end{aligned} \tag{4.2}$$

Proof. For each $m \in \mathbb{N}$ and $\beta > 1$, let us consider the subsets of $\mathbb{R}^N$,

$$A_m \quad = \quad \{x \in \mathbb{R}^N : |z|^{\beta-1} \leq m\} \quad \text{and} \quad B_m = \mathbb{R}^N \setminus A_m,$$

and the function

$$z_m = \begin{cases} z|z|^{2(\beta-1)} & \text{in} \quad A_m, \\ m^2 z & \text{in} \quad B_m. \end{cases}$$

Since $z|z|^{2(\beta-1)} = m^2 z$ on ∂A_m, using standard properties of Sobolev spaces we can conclude that $z_m \in H^1(\mathbb{R}^N)$. From $\int_{\mathbb{R}^N} V(x) z_m^2(x)\, \mathrm{d}x < \infty$, we have $z_m \in \mathcal{H}$, and an easy calculation yields that

$$\nabla z_m = \begin{cases} (2\beta - 1)|z|^{2(\beta-1)} \nabla z & \text{in} \quad A_m \\ m^2 \nabla z & \text{in} \quad B_m \end{cases} \tag{4.3}$$

So,

$$\int_{\mathbb{R}^N} \nabla z \nabla z_m \, \mathrm{d}x = (2\beta - 1) \int_{A_m} |z|^{2(\beta-1)} |\nabla z|^2 \, \mathrm{d}x + m^2 \int_{B_m} |\nabla z|^2 \, \mathrm{d}x. \tag{4.4}$$

Taking z_m as a test function in (4.1) we have,

$$\int_{\mathbb{R}^N} \left(\nabla z \nabla z_m + b(x) f(z) f'(z) z_m\right) \, \mathrm{d}x \quad = \quad \int_{\mathbb{R}^N} G_v(x, f(w), f(z)) f'(z) z_m \, \mathrm{d}x.$$

Considering

$$\omega_m = \begin{cases} z|z|^{\beta-1} & \text{in } A_m \\ mz & \text{in } B_m = \mathbb{R}^N \setminus A_m. \end{cases}$$

by a similar argument it follows that

$$\nabla\omega_m = \begin{cases} \beta|z|^{\beta-1}\nabla z & \text{in } A_m \\ m\nabla z & \text{in } B_m \end{cases} \tag{4.5}$$

and $\omega_m \in \mathcal{H}$. Observe that

$$\int_{\mathbb{R}^N} |\nabla\omega_m|^2 \,\mathrm{d}x = \beta^2 \int_{A_m} |z|^{2(\beta-1)}|\nabla z|^2 \,\mathrm{d}x + m^2 \int_{B_m} |\nabla z|^2 \,\mathrm{d}x. \tag{4.6}$$

From (4.4) and (4.6)

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla\omega_m|^2 \,\mathrm{d}x \;&-\; \int_{\mathbb{R}^N} \nabla z\nabla z_m \,\mathrm{d}x = \\ &= \int_{A_m} \left(\beta^2 - 2\beta + 1\right) |z|^{2(\beta-1)}|\nabla z|^2 \,\mathrm{d}x. \end{aligned}$$

Using (4.4), we get the inequality

$$\Big(2\beta - 1\Big) \int_{A_m} |z|^{2(\beta-1)}|\nabla z|^2 \,\mathrm{d}x \le \int_{\mathbb{R}^N} \left(\nabla z\nabla z_m + b(x)f(z)f'(z)z_m\right) \,\mathrm{d}x,$$

which leads to

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla\omega_m|^2 \,\mathrm{d}x \le \\ &\left[\frac{\left(\beta^2 - 2\beta + 1\right)}{(2\beta - 1)} + 1\right] \int_{\mathbb{R}^N} \left(\nabla z\nabla z_m + b(x)b(x)f(z)f'(z)z_m\right) \,\mathrm{d}x. \end{aligned}$$

Using (4.1), it follows that

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla\omega_m|^2 \,\mathrm{d}x \le \;& \frac{\beta^2}{(2\beta - 1)} \int_{\mathbb{R}^N} G_v(x, f(w)f(z))f'(z)z_m \,\mathrm{d}x. \\ \le \;& \beta^2 \int_{\mathbb{R}^N} G_v(x, f(w)f(z))f'(z)z_m \,\mathrm{d}x. \end{aligned}$$

Let S be the best constant which verifies

$$||z||^2_{L^{2^*}(\mathbb{R}^N)} \le S \int_{\mathbb{R}^N} |\nabla z|^2 \,\mathrm{d}x \quad \text{for all} \quad z \in H^1(\mathbb{R}^N).$$

Using the definition of v_m it follows that $|z_m| \le |z|^{2\beta-1}$. Then,

$$|G_v(x, f(w)f(z))f'(z)z_m| \le 2Y(x)\max\left(|w(x)|, |z(x)|\right)^{2\beta}$$

and therefore we have

$$\left[\int_{A_m} |\omega_m|^{2^*}\right]^{(N-2)/N} \leq S\beta^2 \int_{\mathbb{R}^N} 2Y(x) \max\left(|w(x)|, |z(x)|\right)^{2\beta} \, \mathrm{d}x.$$

If $1/q_1 + 1/q = 1$, from Holder's inequality,

$$\left[\int_{A_m} |\omega_m|^{2^*}\right]^{(N-2)/N} \leq 2S\beta^2 ||Y||_{L^q(\mathbb{R}^N)} \left[\int_{\mathbb{R}^N} |\max\left(|w(x)|, |z(x)|\right)|^{2\beta q_1} \, \mathrm{d}x\right]^{1/q_1}.$$

Since $|\omega_m| \leq |z|^\beta$ in $\mathbb{R}^N$ and $|\omega_m| = |z|^\beta$ in A_m, it follows that

$$\left[\int_{A_m} \left(|z|^\beta\right)^{2^*}\right]^{(N-2)/N} \leq 2S\beta^2 ||Y||_{L^q(\mathbb{R}^N)} \left[\int_{\mathbb{R}^N} |\max\left(|w(x)|, |z(x)|\right)|^{2\beta q_1} \, \mathrm{d}x\right]^{1/q_1}.$$

Taking the limit as $m \to \infty$ and using Monotone Convergence Theorem, we obtain

$$||z||_{2^*\beta}^{2\beta} \leq 2S\beta^2 ||Y||_{L^q(\mathbb{R}^N)} ||\max\left(|w(x)|, |z(x)|\right)||_{2\beta q_1}^{2\beta}.$$

and

$$||z||_{2^*\beta} \leq \beta^{1/\beta} \left(2S||Y||_{L^q(\mathbb{R}^N)}\right)^{1/(2\beta)} ||\max\left(|w(x)|, |z(x)|\right)||_{2\beta q_1}. \tag{4.7}$$

Since $N/(N-2) > q_1$, set $\sigma = N/(q_1(N-2))$. When $\beta = \sigma$ in (4.7), we have: $2q_1\beta = 2^*$ and

$$||z||_{2^*\sigma} \leq \sigma^{1/\sigma} \left(2S||Y||_{L^q(\mathbb{R}^N)}\right)^{1/(2\sigma)} ||\max\left(|w(x)|, |z(x)|\right)||_{2^*}. \tag{4.8}$$

When $\beta = \sigma^2$ in (4.7) we get $2q_1\beta = 2^*\sigma$ and

$$||z||_{2^*\sigma^2} \leq \sigma^{2/\sigma^2} \left(2S||Y||_{L^q(\mathbb{R}^N)}\right)^{1/(2\sigma^2)} ||\max\left(|w(x)|, |z(x)|\right)||_{2^*\sigma}. \tag{4.9}$$

The inequalities (4.8) and (4.9) imply that

$$||z||_{2^*\sigma^2} \leq \sigma^{\frac{1}{\sigma}+\frac{2}{\sigma^2}} \left(2S||Y||_{L^q(\mathbb{R}^N)}\right)^{\frac{1}{2}\left(\frac{1}{\sigma}+\frac{1}{\sigma^2}\right)} ||\max\left(|w(x)|,|z(x)|\right)||_{2^*}. \quad (4.10)$$

An iteration argument, replacing β by σ^j in (4.7), lead us to

$$||z||_{2^*\sigma^j} \leq \sigma^{\sum_{l=1}^{j}\frac{l}{\sigma^l}} \left(2S||Y||_{L^q(\mathbb{R}^N)}\right)^{\frac{1}{2}\left(\sum_{l=1}^{j}\frac{1}{\sigma^l}\right)} ||\max\left(|w(x)|,|z(x)|\right)||_{2^*}. \quad (4.11)$$

Once that

$$\begin{aligned} \sum_{j=1}^{\infty}\frac{j}{\sigma^j} &= \frac{\sigma}{(\sigma-1)^2} \\ \frac{1}{2}\sum_{j=1}^{\infty}\frac{1}{\sigma^j} &= \frac{1}{2}\frac{1}{(\sigma-1)}, \end{aligned}$$

it follows from (4.11)

$$||z||_i \leq \sigma^{\sigma/(\sigma-1)^2}\left(2S||Y||_{L^q(\mathbb{R}^N)}\right)^{1/(2(\sigma-1))} ||\max\left(|w(x)|,|z(x)|\right)||_{2^*},$$

for all $i \geq 2^*$. Since

$$||z||_\infty = \lim_{i\to\infty}||z||_i,$$

we get

$$||z||_\infty \leq M||\max\left(|w(x)|,|z(x)|\right)||_{2^*},$$

where

$$M = \max\left(1, \sigma^{\sigma/(\sigma-1)^2}\left(2S||Y||_q\right)^{1/(2(\sigma-1))}\right) \quad \text{with} \quad \sigma = \frac{N(q-1)}{q(N-2)}.$$

By a similar argument to that used above, we can also prove that

$$||w||_\infty \leq M||\max\left(|w(x)|,|z(x)|\right)||_{2^*}$$

which complete the proof of Proposition 4.1. □

Lemma 4.2. *If* $(w,z) \in \mathcal{H}$ *is a critical point of the functional* J, *then* $w,z > 0$.

Proof. If $(w,z) \in \mathcal{H}$ is a critical point of J, from Lemma 3.9 we have $w,z \geq 0$. By Proposition 4.1 we obtain $w,z \in L^\infty(\mathbb{R}^N)$, and therefore the result follows with help of the Maximum Principle. □

Lemma 4.3. *For any* $R > 1$, *any bound state solution* (w,z) *with* $w,z > 0$ *of* (AS) *satisfies*

$$||(w,z)||_\infty \quad \leq \quad \tilde{M}\delta^{1/2}.$$

where

$$\delta := S\left[\beta^{-1}\left(\frac{1}{k} - \frac{\lambda(p-2)}{2pk}\right)^{-1} d + \left(\beta^{-1}\left(\frac{1}{k} - \frac{\lambda(p-2)}{2pk}\right)^{-1}\right)^{2^*/2} d^{2^*/2}\right].$$

Proof. If (w,z) is a bound state solution of (MS) with $w > 0$ e $z > 0$, then it satisfies the following system

$$\begin{cases} -\Delta w = f'(w)\left[G_u(x,f(w),f(z)) - b(x)f(w)\right] & \text{in} \quad \mathbb{R}^N, \\ -\Delta z = f'(z)\left[G_v(x,f(w),f(z)) - b(x)f(z)\right] & \text{in} \quad \mathbb{R}^N, \end{cases}$$

where $G_u,\ G_v : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $b : \mathbb{R}^N \to \mathbb{R}$ are given by

$$\begin{aligned} G_u(x,f(w),f(z)) =\ & \chi_\Lambda(x)K(x)H_u(f(w),f(z)) \\ & + (1-\chi_\Lambda(x))\,\eta\left(\left|K(x)^{1/2}(f(w),f(z))\right|\right)K(x)H_u(f(w),f(z)) \\ & + (1-\chi_\Lambda(x))\,\eta'\left(\left|K(x)^{1/2}(f(w),f(z))\right|\right)\frac{f(w)}{(f(w)^2+f(z)^2)^{1/2}}B_1(w,z), \\ G_v(x,f(w),f(z)) =\ & \chi_\Lambda(x)K(x)H_v(f(w),f(z)) \\ & + (1-\chi_\Lambda(x))\,\eta\left(\left|K(x)^{1/2}(f(w),f(z))\right|\right)K(x)^{1/2}H_v(f(w),f(z)) \\ & + (1-\chi_\Lambda(x))\,\eta'\left(\left|K(x)^{1/2}(f(w),f(z))\right|\right)\frac{f(z)}{(f(w)^2+f(z)^2)^{1/2}}B_1(w,z), \\ b(x) =\ & V(x) - (1-\chi_\Lambda(x))\,\frac{1}{k}K(x)A(x)\left(1-\eta\left(\left|K(x)^{1/2}(f(w),f(z))\right|\right)\right) \\ & + (1-\chi_\Lambda(x))\,\eta'\left(\left|K(x)^{1/2}(f(w),f(z))\right|\right)\frac{K(x)^{1/2}}{(f(w)^2+f(z)^2)^{1/2}}B_2(w,z). \end{aligned}$$

where χ_Λ is the characteristic function of the set Λ, η was defined in (2.2) and the functions B_1, B_2 are defined by $B_1(w,z) := K(x)^{1/2}H(f(w), f(z))$, and $B_2(w,z) := \left[A(x)K(x)\frac{1}{2k}\left(f(w)^2 + f(z)^2\right)\right]$. Observe that it is easy to see that G_u and G_v are continuous functions.

Taking a sufficiently small, we can conclude that $b(x)$ is a non negative function,

$$\int_{\mathbb{R}^N} b(x)(f^2(w) + f^2(z))\,\mathrm{d}x < +\infty.$$

Besides that, $V(x)$ is a bounded function and using the Remark 2.1, we have the following inequalities

$$\begin{aligned}
&|G_u(x, f(w), f(z))| \leq \ |\chi_\Lambda(x)K(x)H_u(f(w), f(z))| \\
&+ \left|(1-\chi_\Lambda(x))\,\eta\left(\left|K(x)^{1/2}(f(w), f(z))\right|\right)K(x)H_u(f(w), f(z))\right| \\
&+ \left|(1-\chi_\Lambda(x))\,\eta'\left(\left|K(x)^{1/2}(f(w), f(z))\right|\right)\frac{f(w)}{(f(w)^2 + f(z)^2)^{1/2}}B_1(w,z)\right| \\
&\leq C_1\left(|f(w)|^{(p-2)} + |f(z)|^{(p-2)/2}\right)(|f(w)| + |f(z)|) \\
&+ \frac{C_2}{a^2}\left(|f(w)|^p + |f(z)|^p\right)|f(w)| \\
&\leq C_3\left(|w|^{(p-2)/2} + |z|^{(p-2)/2}\right)(|w| + |z|)\,.
\end{aligned}$$

and

$$\begin{aligned}
&G_v(x, f(w), f(z)) \leq \ \chi_\Lambda(x)K(x)H_v(f(w), f(z)) \\
&+ (1-\chi_\Lambda(x))\,\eta\left(\left|K(x)^{1/2}(f(w), f(z))\right|\right)K(x)^{1/2}H_v(f(w), f(z)) \\
&+ (1-\chi_\Lambda(x))\,\eta'\left(\left|K(x)^{1/2}(f(w), f(z))\right|\right)\frac{f(z)}{(f(w)^2 + f(z)^2)^{1/2}}B_1(w,z) \\
&\leq C_1\left(|f(w)|^{(p-2)} + |f(z)|^{(p-2)}\right)(|f(w)| + |f(z)|) \\
&+ \frac{C_2}{a^2}\left(|f(w)|^p + |f(z)|^p\right)|f(z)| \\
&\leq C_3\left(|w|^{(p-2)/2} + |z|^{(p-2)/2}\right)(|w| + |z|)\,,
\end{aligned}$$

where C_1, C_2 are real positive constants such that $C_1 = 2c_0 \sup_\Lambda K(x)$,

$C_2 = 30c_0/(p\mu)$ and $C_3 = \max(C_1, C_2)$. Therefore,

$$|G_u(x, f(w), f(z))| + |G_v(x, f(w), f(z))| \quad \leq \quad |Y(x)|\,(|w| + |z|) \quad \text{in} \quad \mathbb{R}^N.$$

With Y given by

$$Y(x) = \quad C\,(\max(w(x), z(x)))^{(p-2)/2}.$$

where C is a positive constant such that $C = 2C_3$. A direct computation shows that $Y \in L^q(\mathbb{R}^N)$ for $q = \dfrac{22^*}{(p-2)} > N/2$. From the Sobolev inequality and the Lemma 3.8, we get

$$||Y||_q \leq C \parallel (w, z) \parallel_{2^*}^{(p-2)/2} \leq C \left(S^{1/2} \parallel (w, z) \parallel_{\mathcal{H}} \right)^{(p-2)/2} \leq C\delta^{(p-2)/4},$$

Now, from the proof of Proposition 4.1, we get

$$\begin{aligned} M &= \max\left(1, \sigma^{\sigma/(\sigma-1)^2} \left(2S||Y||_q\right)^{1/(2(\sigma-1))}\right) \\ &\leq \max\left(1, \sigma^{\sigma/(\sigma-1)^2} \left(2SC\delta^{(p-2)/4}\right)^{1/(2(\sigma-1))}\right) := \tilde{M}. \end{aligned}$$

Observe that $\tilde{M}$ does not depend on R, w or z. □

Lemma 4.4. *Let $(w, z) \in \mathcal{H}$ a bound state solution for the system (AS) with w, z positive functions and the potential V is non negative bounded continuous functions satisfying $(H_0) - (H_3)$. Then w and z satisfy*

$$|\max(w(x), z(x))| \leq \frac{R^{N-2}||(w, z)||_\infty}{|x|^{N-2}} \leq \frac{R^{N-2}\tilde{M}\delta^{1/2}}{|x|^{N-2}}$$

for all $x \in \mathbb{R}^N \setminus B_R(0)$.

Proof. Consider the harmonic function $\psi : \left(\mathbb{R}^N \setminus B_R(0)\right) \longrightarrow \mathbb{R}$ such that

$$\psi(x) \quad = \quad \frac{R^{N-2}||(w, z)||_\infty}{|x|^{N-2}}.$$

Observe that the function ψ satisfies

$$-\Delta\psi \;=\; 0.$$

Therefore

$$-\Delta\psi + \left(1-\frac{\lambda}{k}\right)V(x)f(\psi)f'(\psi) \geq \;0.$$

Now, take as a test function

$$\phi = \begin{cases} (w-\psi)^+ & \text{if} \quad x\in\mathbb{R}^N\setminus B_R(0), \\ 0 & \text{if} \quad x\in B_R(0), \end{cases}$$

and

$$\varphi = \begin{cases} (z-\psi)^+ & \text{if} \quad x\in\mathbb{R}^N\setminus B_R(0), \\ 0 & \text{if} \quad x\in B_R(0). \end{cases}$$

Observe that

$$w\leq\psi \,\mathrm{in}\, \partial B_R \quad \text{and} \quad z\leq\psi\,\mathrm{in}\,\partial B_R(0),$$

then we can conclude that $\phi,\, \varphi\in H^1(\mathbb{R}^N)$. Besides that

$$J'(w,z)(\phi,\varphi) - \int_{\mathbb{R}^N\setminus B_R(0)} \frac{\lambda}{k}\big(V(x)f(w)f'(w)\phi + V(x)f(z)f'(z)\varphi\big) \leq 0.$$

Consider $\Omega := \{x\in\mathbb{R}^N\setminus B_R(0) : \min\{w,z\} > \psi\}$. Hence, combining these estimates

$$\begin{aligned} 0 &\geq \int_\Omega (\nabla w - \nabla\psi)\nabla\phi + \left(1-\frac{\lambda}{k}\right)V(x)(f(w)f'(w) - f(\psi)f'(\psi))\phi\,\mathrm{d}x \\ &+ \int_\Omega (\nabla z - \nabla\psi)\nabla\varphi + \left(1-\frac{\lambda}{k}\right)V(x)(f(z)f'(z) - f(\psi)f'(\psi))\varphi\,\mathrm{d}x \\ &\geq \int_\Omega \left(1-\frac{\lambda}{k}\right)\big[V(x)(f(w)f'(w) - f(\psi)f'(\psi))\phi\big]\,\mathrm{d}x \\ &+ \int_\Omega \left(1-\frac{\lambda}{k}\right)\big[V(x)(f(z)f'(z) - f(\psi)f'(\psi))\varphi\big]\,\mathrm{d}x. \end{aligned}$$

Using that $f^2(t)$ is strictly convex, we have that ff' is a increasing function and so

$$\begin{aligned} &V(x)(f(w)f'(w) - f(\psi)f'(\psi))\phi \geq 0 \quad \text{and} \\ &V(x)(f(z)f'(z) - f(\psi)f'(\psi))\varphi \geq 0 \quad \text{in } \Omega. \end{aligned}$$

Thus, the set Ω is empty and the proof is complete. □

5 Proof of Theorem 1.2 completed

Proof. From Lemmas 3.4 and 3.5, problem (AS) has a bound state solution $(w,z) \in \mathcal{H}^1(\mathbb{R}^N) \times \mathcal{H}^1(\mathbb{R}^N)$ with w, z positive functions. Therefore, it is enough to show that (w,z) satisfies the inequality $|K(x)^{1/2}(f(w), f(z))| \leq a$. Using the hyphotesis (V_2),

$$K(x)\left(|f(w)|^2 + |f(z)|^2\right) \leq K(x)\left(|w| + |z|\right) \leq \ 2K(x)\frac{R^{(N-2)}\tilde{M}\delta^{1/2}}{|x|^{(N-2)}}$$
$$\leq \ \gamma 2\tilde{M}\delta^{1/2}.$$

for all $|x| \geq R$. Consider $\gamma^* = a^2/\left(2\tilde{M}\delta^{1/2}\right)$. Observe that the constant $\tilde{M}$ not depend on R, w or z, see Lemma 4.3. So, for all γ with $0 < \gamma \leq \gamma^*$, we have that $G_u(x, f(w), f(z)) = H_u(f(w), f(z))$ and $G_v(x, f(w), f(z)) = H_v(f(w), f(z))$, because $\eta(|K(x)^{1/2}(f(w), f(z))|) \equiv 1$ for all $|x| \geq R$. Hence, we complete the proof. $\square$

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