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# Stationary Schrödinger equations involving critical growth and vanishing potential

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*Dedicated to Professor Pedro Ubilla*
*on the occasion of his 60th birthday*

**Abstract.** We establish the existence of positive solutions for a class of stationary nonlinear Schrödinger equations involving critical growth in the sense of the Sobolev embedding and potentials, which may decay to zero at infinity. We use min-max techniques combined with an appropriate truncated argument and a priori estimate.

**Keywords:** Vanishing potentials, critical growth.

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## 1 Introduction

We prove the existence of positive solutions for stationary Schrödinger-type equations of the form

$$\begin{cases} -\Delta u + V(x)u = f(u) \quad \text{in} \quad \mathbb{R}^N, \\ u > 0 \quad \text{in} \quad \mathbb{R}^N \quad \text{and} \quad u \in D^{1,2}(\mathbb{R}^N). \end{cases} \tag{1.1}$$

This class of nonlinear elliptic equations in $\mathbb{R}^N$ has been intensively studied in recent years, motivated by a wide variety of problems in mathematics and physics, in particular for the search for standing wave solutions by considering different approaches (see [1, 3, 4, 5, 7, 8]).

In this paper we report a joint work with J.M. do Ó and P. Ubilla which can be seen as a natural completion of recent works [1, 3], where the subcritical case for a certain class of vanishing potentials was studied. We mention that V. Benci and G. Cerami in [6] studied standing wave solutions of the critical problem $-\Delta u + a(x)u = u^{(N+2)/(N-2)}$ in $\mathbb{R}^N$ involving vanishing potential requiring also that $a \in L^{N/2}(\mathbb{R}^N)$. They proved this problem has at least one solution if $\|a\|_{L^{N/2}}$ is sufficiently small. We point out that if $a(x) \approx |x|^{-\theta}$ with $0 < \theta < p-2$ is in the class of potentials satisfying our assumptions, but $a \notin L^{N/2}(\mathbb{R}^N)$ if $\theta \leq 2$, that is, $a(x)$ does not belongs to Benci-Cerami class (see Example 1.2).

We focus our study on the following model problem involving critical growth

$$\begin{cases} -\Delta u + V_\lambda(x)u = |u|^{2^*-2}u + \gamma|u|^{p-2}u & \text{in } \mathbb{R}^N, \\ u > 0 \quad \text{in} \quad \mathbb{R}^3, \quad u \in D^{1,2}(\mathbb{R}^N), \end{cases} \tag{$\mathcal{P}_{\lambda,\gamma}$}$$

depending on $p \in (2, 2^*)$, the potential $V_\lambda(x) = Z(x) + \lambda V(x)$ and the positive real paramenter $\lambda$. Here $N \geq 3$ and $2^* = 2N/(N-2)$ is the critical exponent for the classical Sobolev embedding. This potential $V_\lambda = Z + \lambda V$ appears in some recent works to study a class of nonlinear Schrödinger equations. For instance, [2, 4, 5] and references therein, for the case where the potential is bounded away from zero. In the present paper, the potential $V_\lambda = Z + \lambda V$ may decay to zero at infinity in some direction ($Z$ with compact support, for instance). To state our main results, let us describe in a more precise way the assumptions on the potential $V$:

$$Z(x) \text{ and } V(x) \text{ are continuous and nonnegative functions;} \tag{$V_1$}$$

$$V(x) \equiv 0 \text{ in some ball } B_{r_1}(x_1) \subset \mathbb{R}^N; \tag{$V_2$}$$

$$\liminf_{|x|\to\infty} |x|^{p-2}V(x) > 0; \tag{$V_3$}$$

Our first result for equation ($\mathcal{P}_{\lambda,\gamma}$) is the following.

**Theorem 1.1.** *Suppose that* ($V_1$)–($V_3$) *are satisfied and* $2 < p < 2^*$. *Then, there exists* $\gamma^* > 0$ *such that for any* $\gamma \geq \gamma^*$ *there exists* $\lambda^* = \lambda^*(\gamma) > 0$ *such that* ($\mathcal{P}_{\lambda,\gamma}$) *possesses a positive solution for all* $\lambda \geq \lambda^*$.

Let us give some examples which illustrate the above result.

**Example 1.2.** Given $C > 0$, $0 < \theta < p-2$ and $R_o > 0$, we can check that any continuous and nonnegative function $V : \mathbb{R}^3 \to \mathbb{R}$ such that $V(x) = C/|x|^\theta$ for all $|x| \geq R_o$ verifies ($V_3$).

**Remark 1.3.** One can see that under our assumptions, the natural functional of ($\mathcal{P}_{\lambda,\gamma}$) is not well defined. To face this difficulty, we propose a suitable modification on the nonlinearity $f_\gamma(s) := |u|^{2^*-2}u + \gamma|u|^{p-2}u$ such that the energy functional associated to the modified problem has compactness and allow us to prove the existence of a ground state solution by using the min-max techniques. Next, by choosing a sufficiently large $\gamma$, we verify that the solution to the auxiliary problem is indeed a solution to our original problem ($\mathcal{P}_{\lambda,\gamma}$).

Using a similar approach as in Theorem 1.1, with some minor modifications, a more general result for the following problem can be proved.

$$\begin{cases} -\Delta u + W_\lambda(x)u = |u|^{2^*-2}u + \gamma|u|^{p-2}u \quad \text{in } \mathbb{R}^N, \\ u > 0 \quad \text{in} \quad \mathbb{R}^3, u \in D^{1,2}(\mathbb{R}^N), \end{cases} \tag{$\mathcal{Q}_{\lambda,\gamma}$}$$

where $W_\lambda$ verifies the following hypotheses:

$$\inf_{z\in\mathbb{R}^N} \int_{B_1(z)} W_\lambda(x)\, dx < 1. \tag{$V_4$}$$

$$\text{There exists } R_o > 0 \text{ and } C > 0 \text{ such that } \inf_{|x|\geq R_o} W_\lambda(x)|x|^{p-2} > C\lambda. \tag{$V_5$}$$

**Theorem 1.4.** *Suppose that* ($V_4$)–($V_5$) *are satisfied and* $2 < p < 2^*$*. Then, there exist* $\gamma^* > 0$ *such that for all* $\gamma \geq \gamma^*$ *there is a* $\lambda^* = \lambda^*(\gamma) > 0$ *such that* ($\mathcal{Q}_{\lambda,\gamma}$) *possesses a positive solution for all* $\lambda \geq \lambda^*$*.*

**Example 1.5.** As an example of a class of potentials which satisfies conditions ($V_4$)–($V_5$) is given by $W_\lambda(x) = \lambda^2/(\lambda|x|^\theta + 1)$ where $0 < \theta < p - 2$ for $|x - z| \geq r_1$ and $W_\lambda$ bounded in $|x - z| \leq r_o$ uniformly in $\lambda > 0$. Notice that $W_\lambda$ does not verifies ($V_1$)–($V_3$).

Notation: Let us introduce the following notations:

- $C$, $\tilde{C}$, $C_1$, $C_2$,... denote positive constants (possibly different).
- $B_R(x_0)$ denotes the open ball centered at $x_0$ and radius $R > 0$.
- The norms in $L^p(\mathbb{R}^N)$ and $L^\infty(\mathbb{R}^N)$ will be denoted respectively by $\|\cdot\|_p$ and $\|\cdot\|_\infty$.

Outline: In the Section 2, we consider some auxiliary functionals and we obtain estimates for their mountain pass levels. Section 3 is devoted to a study of a $L^{2^*}$-estimate on the solutions of some auxiliary problem and its $L^\infty$-estimate is done in the Section 4. We conclude the proof of Theorem 1.1 in the Section 5.

## 2 Preliminaries

We start observing that from ($V_1$), we can introduce the natural Hilbert space

$$E = \left\{ v \in \mathcal{D}^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V_\lambda(x) v^2 \, dx < \infty \right\}$$

endowed with the scalar product and norm given, respectively, by

$$\langle u, v \rangle_\lambda = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + V_\lambda(x) uv) \, dx, \quad \|u\|_\lambda^2 = \int_{\mathbb{R}^N} \left( |\nabla u|^2 + V_\lambda(x) u^2 \right) dx.$$

An initial difficulty that appears to attach variational problems like ($\mathcal{P}_{\lambda,\gamma}$) in the case that the potential converges to zero at infinity is that, in general, we do not have the embedding "$E \hookrightarrow L^p(\mathbb{R}^N)$" for $2 \le p < 2^*$ and the Euler-Lagrange functional associated to $(\mathcal{P}_{\lambda,\gamma})$ is not well defined in $E$. For this reason, we will consider an auxiliary problem defined in bounded domains.

From $(V_2)$, without loss of generality, we suppose that $V(x) = 0$ for all $x \in B_1(0)$. Now let us consider the energy functional $I_0 : H^1_o(B_1(0)) \to \mathbb{R}$ defined by

$$I_0(u) = \frac{1}{2}\|\nabla u\|^2_{L^2(B_1(0))} + \frac{1}{2}\int_{B_1(0)} Z(x)u^2\, dx - \frac{\gamma}{p}\int_{B_1(0)} |u|^p\, dx.$$

It is clear that $I_0$ is well defined, belongs to class $C^1$ and does not depend on $\lambda$. Moreover, under our assumptions one can verify that $I_0$ has the mountain pass geometry and thus it is well defined the mountain pass level,

$$d_\gamma = \inf_{v \in H^1_o(B_1(0))} \max_{t>0} I_0(tv).$$

Next, we have a crucial upper bound estimate on this min-max level $d_\gamma$.

**Proposition 2.1.** *There exist constants $C_p > 0$ and $\gamma_o > 0$ such that for all $\gamma \ge \gamma_o$ it holds*

$$0 < d_\gamma \le \frac{C_p}{\gamma^{\frac{2}{p-2}}}.$$

*In particular $d_\gamma \to 0$ as $\gamma \to +\infty$.*

*Proof.* Let $v_o \in C_0^\infty(\mathbb{R}^3)$ such that $0 \le v_o \le 1$ and define

$$a := \|\nabla v_o\|^2_{L^2(B_1(0))} + \int_{B_1(0)} Z(x)v_o^2\, dx, \quad b := \|v_o\|^p_{L^p(B_1(0))}.$$

Let us estimate $\max_{t>0} I_0(tv_o)$. It is clear that the function $h(t) := I_0(tv_o)$ has a unique critical point which is a global maximum point. Indeed, $h'(t) = 0$ is equivalent to $a = \gamma b t^{p-2}$. Thus, there is a unique $t > 0$ such that $h'(t) = 0$. We also have for $\gamma > 0$ sufficiently large,

$$2h(1) = a - \frac{2\gamma b}{p} < 0.$$

If $t_o$ is the critical point of $h(t)$, it is easy to check that $h(t)$ is increasing in $(0, t_o)$ and it is decreasing in $(t_o, \infty)$. Then, since $h(1) < 0$, we must have $t_o < 1$ such that

$$\max_{t>0} h(t) = h(t_o),$$

which implies

$$h(t_o) = \frac{a\, t_o^2}{2} - \frac{\gamma b t_o^p}{p} = \left(\frac{1}{2} - \frac{1}{p}\right) \gamma b t_o^p = \frac{C_p}{\gamma^{\frac{2}{p-2}}}$$

where,

$$C_p := \frac{a^{\frac{p}{p-2}}}{b^{\frac{2}{p-2}}} \left(\frac{1}{2} - \frac{1}{p}\right),$$

which completes our proof. □

From Proposition 2.1, we can choose $\gamma^* > 0$ such that for all $\gamma \geq \gamma^*$ it holds

$$d_\gamma < \min\left\{1, \frac{p-2}{2p}, \frac{p-2}{2p} S^{\frac{N}{N-1}}\right\}, \tag{2.1}$$

where $S$ is the best constant for the Sobolev embedding $\mathcal{D}^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$, that is,

$$S = \inf\left\{\|\nabla v\|_2^2 : v \in \mathcal{D}^{1,2}(\mathbb{R}^N),\ \|v\|_{2^*} = 1\right\}.$$

## 3 Auxiliary problem

We begin this section by recalling that since we deal with a class of potentials that may decay to zero at infinity, the variational method cannot be applied because the natural Euler-Lagrange functional associated with Problem $(\mathcal{P}_{\lambda,\gamma})$ is not well defined on the space $E$. To overcome this difficulty, we are going to modify the critical nonlinearity $f_\gamma(s) := |u|^{2^*-2}u + \gamma |u|^{p-2}u$ as follows: choose $R \geq 1$ and define

$$g(x,s) = \begin{cases} f_\gamma(s), & \text{if} \quad x \in B_R \qquad \text{or} \quad f_\gamma(s) \leq \dfrac{V_\lambda(x)}{p} s, \\[2mm] \dfrac{V_\lambda(x)}{p} s, & \text{if} \quad x \notin B_R \qquad \text{and} \quad f_\gamma(s) > \dfrac{V_\lambda(x)}{p} s. \end{cases}$$

Let us consider the auxiliar problem

$$-\Delta u+V_\lambda(x)u=g(x,u),\ \text{ in } \mathbb{R}^N. \tag{$\mathcal{AP}$}$$

It is easy to check that $g(x,s)$ is a Carathéodory function and its primitive

$$G(x,s)=\int_0^s g(x,\tau)\,d\tau$$

is such that

$$G(x,s)=F_\gamma(s) \quad \text{ if } \quad x\in B_R \quad \text{ or } \quad f_\gamma(s)\le \frac{V_\lambda(x)}{p}s,$$

where

$$F_\gamma(s)=\int_0^s f_\gamma(\tau)\,d\tau=\frac{|s|^{2^*}}{2^*}+\frac{\gamma|s|^p}{p}.$$

Moreover, since $f(s)/s$ is increasing for $s>0$ and decreasing if $s<0$, one can see that

$$sg(x,s)\le |s|^{2^*}+\gamma|s|^p, \qquad \text{for all } s\in\mathbb{R}; \tag{$g_1$}$$

$$sg(x,s)-pG(x,s)\ge\left[\frac{1}{p}-\frac{1}{2}\right]V_\lambda(x)s^2, \qquad \text{for all } s\in\mathbb{R}; \tag{$g_2$}$$

$$sg(x,s)\le\frac{V_\lambda(x)}{p}s^2, \qquad \text{for all } s\in\mathbb{R} \text{ and } x\in B_R^c; \tag{$g_3$}$$

Using standard arguments, from condition ($g_3$), the corresponding energy functional $J:E\to\mathbb{R}$ is given by

$$J(u)=\frac{1}{2}\|u\|_\lambda^2\,dx-\int_{\mathbb{R}^N}G(x,u)\,dx,$$

is well defined and of class $C^1$ with

$$J'(u)v=\int_{\mathbb{R}^N}(\nabla u\nabla v+\int_{\mathbb{R}^N}V_\lambda(x)uv)\,dx-\int_{\mathbb{R}^N}g(x,u)v\,dx \text{ for all } u,v\in E.$$

From our assumptions, one can see that $J$ fulfills the mountain pass geometry, and then the min-max level

$$c_{\lambda,\gamma}=\inf_{v\in E}\max_{t>0}J(tv)$$

is well defined, and satisfies $0 < c_{\lambda,\gamma} \leq d_\gamma$ due to $J(v) \leq I_0(v)$ for all $v \in H_0^1(B_1(0))$. We can use the Ekeland Variational Principle [11] to produces a Palais-Smale sequence $(u_n) \subset E$ at the ninimax level $c_{\lambda,\gamma}$, that is,

$$J(u_n) \to c_{\lambda,\gamma} \text{ and } J'(u_n) \to 0. \tag{3.1}$$

**Lemma 3.1.** *The sequence $(u_n)$ is bounded in $E$ and $\|\nabla u_n\|_2 \leq 1$ for large $n$.*

*Proof.* Indeed, using (3.1) for $n$ big enough, we have

$$c_{\lambda,\gamma} + 1 + \|u_n\|_\lambda \geq J(u_n) - \frac{1}{p} J'(u_n)u_n.$$

From ($g_2$), it is easy to check that

$$\begin{aligned} d_\gamma + 1 + \|u_n\|_\lambda \geq \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\mathbb{R}^N} [|\nabla u_n|^2 + V_\lambda(x) u_n^2] dx \\ + \int_{\mathbb{R}^N} \left[\frac{1}{p} g(x, u_n)u_n - G(x, u_n)\right] \, dx \\ \geq \left(\frac{1}{2} - \frac{1}{p}\right)\left(1 - \frac{1}{p}\right) \int_{\mathbb{R}^N} V_\lambda(x) u_n^2 \, dx + \left(\frac{1}{2} - \frac{1}{p}\right) \|\nabla u_n\|_2^2. \end{aligned}$$

This last inequality show that $(u_n)$ is bounded in $E$. Besides, using (2.1), for all $n$ large enough, we have

$$\|\nabla u_n\|_2^2 \leq (d_\gamma + o_n(1)) \frac{p-2}{2p} \leq 1, \tag{3.2}$$

which completes the proof. □

**Lemma 3.2.** *Up to a subsequence, we have that $(u_n)$ converges in $L^{2^*}(\mathbb{R}^N)$.*

*Proof.* We may suppose that $u_n \rightharpoonup u$ weakly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, $|\nabla u_n|^2$ and $|u_n|^{2^*}$ converge tightly to $\mu$ and $\nu$, where $\mu$ and $\nu$ are bounded nonnegative measures on $\mathbb{R}^3$. Moreover, $u_n \to u$ in $L^r_{loc}(\mathbb{R}^N)$, for all $2 \leq r < 2^*$. Then, in view of Lions concentration compactnes principle (see [14, Lemma I.1], page 158), we have

1. there exists a sequence $(\nu_j)_{j\in\mathbb{N}}$ in $\mathbb{R}_+$, $(x_j)_{j\in\mathbb{N}}$ in $\mathbb{R}^N$ such that
$$\nu = |u|^{2^*} + \sum_{j=1}^{\infty} \nu_j \delta_{x_j};$$

2. besides, we have
$$\mu \geq |\nabla u|^2 + S \sum_{j=1}^{\infty} \nu_j^{\frac{1}{N}} \delta_{x_j}.$$

Let $\phi \in C_o^\infty(\mathbb{R}^N, [0,1])$ such that $\phi(x) = 1$, if $|x| \leq 1/2$ and $\phi(x) = 0$ if $|x| \geq 1$. For each $\varepsilon \in (0,1)$ let us consider
$$\phi_\varepsilon(x) = \phi\left(\frac{x - x_j}{\varepsilon}\right).$$

Notice that if $2 \leq r < 2^*$,
$$\lim_n \int_{\mathbb{R}^N} \phi_\varepsilon |u_n|^r \, dx = \int_{\mathbb{R}^3} \phi_\varepsilon |u|^r \, dx := B_{\varepsilon,u,r}$$

and for each fixed $u \in E$, we have $\operatorname{supp}(\phi_\varepsilon) \subset B(0,1)$ and $|\phi_\varepsilon |u|^r| \leq |u|^r$. Thus by Lebesgue's dominated convergence theorem,
$$\lim_{\varepsilon\to 0} B_{\varepsilon,u,r} = 0.$$

From $(g_1)$, we get
$$\left|\int_{\mathbb{R}^N} (u_n\phi_\varepsilon) g(x,u_n) \, dx\right| \leq \gamma \int_{\mathbb{R}^N} \phi_\varepsilon |u_n|^{q+1} \, dx + \int_{\mathbb{R}^N} \phi_\varepsilon u_n^{2^*} \, dx$$

and consequently
$$\limsup_n \left|\int_{\mathbb{R}^N} (u_n\phi_\varepsilon) g(x,u_n) \, dx\right| \leq C\left(B_{\varepsilon,u,q+1} + \int_{\mathbb{R}^N} \varphi_\varepsilon \, d\nu\right).$$

By using a Hölder indquality we obtain
$$\left|\int_{\mathbb{R}^N} u_n \nabla\phi_\varepsilon \nabla u_n \, dx\right| \leq \varepsilon^{-1}\left(\int_{|x-x_j|\leq 2\varepsilon} u_n^2 \, dx\right)^{\frac{1}{2}} \left(\int_{|x-x_j|\leq 2\varepsilon} |\nabla u_n|^2 \, dx\right)^{\frac{1}{2}}.$$

As $(u_n)$ is bounded in $E$, we have

$$\left|\int_{\mathbb{R}^N} u_n \nabla\phi_\varepsilon \nabla u_n \, dx\right| \le C \left(\int_{|x-x_j|\le 2\varepsilon} u_n^2 \, dx\right)^{\frac{1}{2}}, \text{ for all } n, \ \varepsilon$$

and

$$\limsup_n \left|\int_{\mathbb{R}^N} u_n \nabla\phi_\varepsilon \nabla u_n \, dx\right| \le C \left(\int_{|x-x_j|\le 2\varepsilon} u^2 \, dx\right)^{\frac{1}{2}}, \text{ for all } \varepsilon,$$

which shows that

$$\lim_{\varepsilon\to 0}\left(\limsup_n \left|\int_{\mathbb{R}^N} u_n \nabla\phi_\varepsilon \nabla u_n dx\right|\right) = 0.$$

Now, we can see that

$$\begin{aligned} o_n(1) = J'(u_n)(u_n\phi_\varepsilon) &= \int_{\mathbb{R}^N} \nabla u_n \nabla(u_n\phi_\varepsilon) \, dx + \int_{\mathbb{R}^N} V(x) u_n (u_n\phi_\varepsilon) \, dx \\ &\qquad - \int_{\mathbb{R}^N} g(x,u_n)(u_n\phi_\varepsilon) \, dx \\ &= \int_{\mathbb{R}^N} |\nabla u_n|^2 \phi_\varepsilon \, dx + \int_{\mathbb{R}^N} V_\lambda(x) u_n^2 \varphi_\varepsilon \, dx + \int_{\mathbb{R}^N} u_n \nabla\phi_\varepsilon \nabla u_n \, dx \\ &\qquad - \int_{\mathbb{R}^N} g(x,u_n).u_n\phi_\varepsilon \, dx, \end{aligned}$$

or,

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla u_n|^2 \phi_\varepsilon \, dx + \int_{\mathbb{R}^3} V_\lambda(x) u_n^2 \phi_\varepsilon \, dx &= - \int_{\mathbb{R}^N} u_n \nabla\phi_\varepsilon \nabla u_n \, dx \\ &\qquad + \int_{\mathbb{R}^N} g(x,u_n) u_n.\phi_\varepsilon \, dx + o_n(1). \end{aligned}$$

Passing to the limit as $n \to \infty$, we have

$$\left|\int \phi_\varepsilon d\mu + B_{\varepsilon,u,2} - \int \phi_\varepsilon \, d\nu\right| \le C \left[B_{\varepsilon,u,q+1} + \left(\int_{|x-x_j|\le \varepsilon} u^2 \, dx\right)^{1/2}\right],$$

for all $\varepsilon$. Passing to the limit as $\varepsilon \to 0$

$$\mu(\{x_j\}) = \nu(\{x_j\}) = \nu_j.$$

Combining with part (2) of Lions Lemma,

$$\mu(\{x_j\}) \geq S\nu_j^{1/N}$$

we have

$$\nu_j \geq S\nu_j^{1/N}$$

and thus, if $\nu_j > 0$ we obtain

$$\nu_j^{\frac{N-1}{N}} \geq S,$$

which implies that

$$\mu(\{x_j\}) = \nu_j \geq S^{\frac{N}{N-1}}. \tag{3.3}$$

We know that, $c_{\lambda,\gamma} + o_n(1) = J(u_n) - \frac{1}{p}J'(u_n)u_n$, and then

$$\begin{aligned} c_{\lambda,\gamma} &= \left[\frac{1}{2} - \frac{1}{p}\right] \|\nabla u_n\|_2^2 + \left[\frac{1}{2} - \frac{1}{p}\right] \int_{\mathbb{R}^N} V_\lambda(x) u_n^2 \phi_\varepsilon \, dx \\ &\quad + \int_{\mathbb{R}^N} \left(\frac{1}{p} u_n g(x, u_n) - G(x, u_n)\right) \, dx + o_n(1) \\ &\geq \left[\frac{1}{2} - \frac{1}{p}\right] \int_{\mathbb{R}^N} |\nabla u_n|^2 \phi_\varepsilon \, dx + o_n(1) + o_\varepsilon(1). \end{aligned}$$

Passing to the limit as $n \to +\infty$, we obtain

$$c_{\lambda,\gamma} \geq \left[\frac{1}{2} - \frac{1}{p}\right] \int_{\mathbb{R}^N} \phi_\varepsilon \, d\mu + o_\varepsilon(1).$$

Taking to the limit as $\varepsilon \to 0$, we have

$$c_{\lambda,\gamma} \geq \left[\frac{1}{2} - \frac{1}{p}\right] \mu(\{x_j\}).$$

We also note that assumption (3.3) implies that, if $\nu_j > 0$ we can deduce

$$c_{\lambda,\gamma} \geq \left[\frac{1}{2} - \frac{1}{p}\right] S^{\frac{N}{N-1}},$$

which is a contradiction with the inequality $c_{\lambda,\gamma} \leq d_\gamma$ and (2.1). Then $\nu_i = 0$ for all $i$ and, $u_n$ converges to $u$ in $L^{2^*}(\mathbb{R}^N)$. □

**Lemma 3.3.** *The following limits hold for the sequence $(u_n)$:*

$$\lim_n \int_{\mathbb{R}^N} V_\lambda(x)u_n^2\,dx = \int_{\mathbb{R}^N} V_\lambda(x)u^2\,dx, \tag{3.4}$$

$$\lim_n \int_{\mathbb{R}^N} g(x,u_n)u_n\,dx = \int_{\mathbb{R}^N} g(x,u)u\,dx, \tag{3.5}$$

$$\lim_n \int_{\mathbb{R}^N} g(x,u_n)v\,dx = \int_{\mathbb{R}^N} g(x,u)v\,dx,\ \forall v\in E \tag{3.6}$$

$$\lim_n \int_{\mathbb{R}^N} G(x,u_n)\,dx = \int_{\mathbb{R}^N} G(x,u)\,dx. \tag{3.7}$$

*Proof.* We start with the following claim:

$$\lim_{r\to\infty}\int_{|x|\geq r}\left[|\nabla u_n|^2+V_\lambda(x)u_n^2\right]dx=0,\ \text{uniformly in } n. \tag{3.8}$$

In fact, let us consider a cut-off function $\eta \in C_0^\infty(B_r^c,[0,1])$ such that $\eta(x) = 1$ for all $|x| \geq 2r$ and $|\nabla\eta(x)| \leq 2/r$ for all $x \in \mathbb{R}^3$. Since $(u_n)$ is bounded in $E$, the sequence $(\eta u_n)$ is also bounded in $E$, and then $J'(u_n)(\eta u_n) = o_n(1)$, that is,

$$\int_{\mathbb{R}^N}\nabla u_n\nabla(\eta u_n)\,dx+\int_{\mathbb{R}^N}V_\lambda(x)u_n(\eta u_n)\,dx=\int_{\mathbb{R}^N}g(x,u_n)(\eta u_n)\,dx+o_n(1).$$

Since $\eta(x) = 0$ for all $|x| \leq r$, using $(g_3)$ we obtain

$$\begin{aligned}\int_{|x|\geq r}\eta\left[|\nabla u_n|^2+V_\lambda(x)u_n^2\right]\,dx \leq{}& \frac{1}{p}\int_{|x|\geq r}\eta V_\lambda(x)u_n^2\,dx\\ &-\int_{|x|\geq r}u_n\nabla u_n\nabla\eta\,dx+o_n(1),\end{aligned}$$

which implies

$$\begin{aligned}\left(1-\frac{1}{p}\right)\int_{|x|\geq r}\eta\left[|\nabla u_n|^2+V_\lambda(x)u_n^2\right]\,dx&\\ \leq \int_{r\leq|x|\leq 2r}|u_n||\nabla u_n|\,dx+o_n(1).&\end{aligned} \tag{3.9}$$

Using Hölder inequality, we can estimate

$$\int_{r\leq|x|\leq 2r}|u_n||\nabla u_n|\,dx\leq\|\nabla u_n\|_{L^2(\mathbb{R}^3)}\left(\int_{r\leq|x|\leq 2r}|u_n|^2\,dx\right)^{1/2}$$

Since $u_n \to u$ strongly in $L^2(B_{2r} \setminus B_r)$ and $\|\nabla u_n\|_{L^2(\mathbb{R}^3)} \leq 1$ (see Lemma 3.1), it follows that

$$\limsup_n \int_{r \leq |x| \leq 2r} |u_n||\nabla u_n| \, dx \leq \left( \int_{r \leq |x| \leq 2r} |u|^2 \, dx \right)^{1/2} \tag{3.10}$$

On the other hand, Hölder inequality implies

$$\left( \int_{r \leq |x| \leq 2r} |u|^2 \, dx \right)^{1/2} \leq \left( \int_{r \leq |x| \leq 2r} |u|^{2^*} \, dx \right)^{1/2^*} |B_{2r} \setminus B_r|^{1/N}$$

which together with (3.10) yields

$$\limsup_n \int_{r \leq |x| \leq 2r} |u_n||\nabla u_n| \, dx \leq |B_{2r} \setminus B_r|^{1/N} \left( \int_{r \leq |x| \leq 2r} |u|^{2^*} \, dx \right)^{1/2^*} \tag{3.11}$$

(3.9) and (3.11) show the claim.

Since $u_n \to u$ strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$, (3.4) follows from (3.8). To prove (3.5)–(3.6), we can use (3.4) together with condition ($g_3$). $\square$

Using Lemmas 3.2 and 3.3 we can show that $u$ is a weak solution for the problem

$$-\Delta u + V_\lambda(x) u = |u|^{2^*-2} u + g(x,u), \mathbb{R}^N$$

and

$$\|u\|_\lambda^2 = \int_{\mathbb{R}^N} |u|^{2^*} \, dx + \int_{\mathbb{R}^N} g(x,u) u \, dx.$$

Now passing to the limit in

$$\|\nabla u_n\|_2^2 + \int_{\mathbb{R}^N} V_\lambda(x) u_n^2 \, dx = \int_{\mathbb{R}^N} |u_n|^{2^*} \, dx + \int_{\mathbb{R}^N} g(x,u_n) u_n \, dx + o_n(1),$$

we conclude that

$$\lim_n \|u_n\|_\lambda^2 = \|u\|_\lambda^2.$$

Then $u_n$ converges to $u$ in $E$ and $J(u) = c_{\lambda,\gamma}$. Therefore $u$ is a ground state solution to auxiliary problem ($\mathcal{AP}$) which depends on $R$ and satisfies

$$\|\nabla u\|_2^2 \leq d_\gamma \frac{p-2}{2p}, \text{ for all } R > 1.$$

Furthermore, it follows

$$\|u\|_{2^*}^2 \le S^{-1}\|\nabla u\|_2^2 \le d_\gamma \frac{p-2}{2pS} \tag{3.12}$$

independent on the choice of $R > 1$. Combining Proposition 2.1 and (3.12) we have that

$$\|u\|_{2^*} \le C\gamma^{-\frac{1}{p-2}}. \tag{3.13}$$

## 4 A priori estimates in the $L^\infty(\mathbb{R}^N)$ norm

We derive some a priori $L^\infty -$ *estimates* for the solutions of Auxiliary Problem ($\mathcal{AP}$). For that we follow some extraordinary ideas due to E. De Giorgi, J. Nash and J. Moser, to obtain regularity results that were discovered in the mid 1950's and early 1960's. For more details see for example [9, 12, 13].

**Theorem 4.1.** *Let $u$ be a solution to ($\mathcal{AP}$) then*

$$\|u\|_\infty \le C\gamma^{\frac{2Np-(8+4N)}{(2^*-p)(p-2)(N-2)}},$$

*where $C$ is a positive constant.*

Before we prove the above estimate we will need to provide some crucial results. First let us state a version of

**Lemma 4.2.** *Let $b : \mathbb{R}^N \mapsto \mathbb{R}$ be a nonnegative measurable function and let $h \in L^q_{loc}(\mathbb{R}^N)$ such that*

$$[h]_q = \sup_{z\in\mathbb{R}^N}\left(\int_{B_2(z)} |h|^q \, dx\right)^{1/q} < \infty,$$

*where $3 \le N < 2q$. Suppose that $v \in E$ is a weak solution to the problem*

$$-\Delta v + b(x)v = h(x) \text{ in } \mathbb{R}^N. \tag{4.1}$$

*Then we have*

$$\sup_{x\in B_1(z)} |v(x)| \le C\,[h]_q \left(\int_{B_2(z)} |v|^{2^*} \, dx\right)^{1/2^*} \text{ for all } z \in \mathbb{R}^N,$$

*where $C$ depends only on $q$ (it does not depend on $b$ or $v$).*

**Proposition 4.3.** *Let the potential $V_o : \mathbb{R}^N \mapsto \mathbb{R}$ be a nonnegative measurable function and the nonlinear term $g(x,s)$ be a Caratheodory function such that some $\alpha_o, \beta_o > 0$,*

$$|g(x,s)| \leq \alpha_o |s|^{2^*-1} + \beta_o |s| \quad \textit{for all} \quad (x,s) \in \mathbb{R}^N \times \mathbb{R}.$$

*Suppose that $u \in E$ is a weak solution to the problem*

$$-\Delta u + V_o(x) u = g(x,u) \ \textit{in}\ \mathbb{R}^N \tag{4.2}$$

*satisfying*

$$(\mathcal{C}) \qquad \frac{2N^2}{N^2-4} \alpha_o \|u\|_{2^*}^{2^*-2} \leq S.$$

*Then there is $\Lambda$ such that*

$$\|u\|_\infty \leq \Lambda \|u\|_{2^*}^2,$$

*where $\Lambda$ does not depend on $V$ or $u$, indeed $\Lambda$ depends only on $\beta_o$. In addition we have $\Lambda = O(\beta_o)$ as $\beta_o \to \infty$.*

(See [10] - Proposition 4.3).

*Proof.* For each $n \in \mathbb{N}$, let us consider the sets

$$A_n = \{x \in \mathbb{R}^N : |u|^{2^*-2} \leq n^2\} \quad \text{and} \quad B_n = \mathbb{R}^N \setminus A_n.$$

and define the function $v_n \in E$ by

$$v_n = |u|^{2^*-2} u \quad \text{in } A_n \quad \text{and} \quad v_n = n^2 u \text{ in } B_n.$$

Observe that $v_n \in E$, $v_n u \leq |u|^{2^*}$ in $\mathbb{R}^N$,

$$\nabla v_n = (2^*-1)|u|^{2^*-2} \nabla u \text{ in } A_n \text{ and } \nabla v_n = n^2 \nabla u \text{ in } B_n. \tag{4.3}$$

Then, using $v_n$ as a test function in (4.2),

$$\int_{\mathbb{R}^N} [\nabla u \nabla v_n + V_o(x) u v_n] \ dx = \int_{\mathbb{R}^3} g(x,u) v_n \, dx.$$

From (4.3) we have

$$\int_{\mathbb{R}^N} \nabla u \nabla v_n \, dx = (2^* - 1) \int_{A_n} |u|^{2^*-2} |\nabla u|^2 \, dx + n^2 \int_{B_n} |\nabla u|^2 \, dx. \quad (4.4)$$

Now consider

$$\omega_n = |u|^{\frac{2}{N-2}} u \;\text{ in } A_n \text{ and } \omega_n = nu \text{ in } B_n.$$

Note that $\omega_n^2 = uv_n \leq |u|^{2^*}$, $0 \leq V_o(x)\omega_n^2 = V_o(x)uv_n$ in $\mathbb{R}^N$. Moreover,

$$\nabla \omega_n = \frac{N}{N-2} |u|^{\frac{2}{N-2}} \nabla u \text{ in } A_n \quad \text{and} \quad \nabla \omega_n = n \nabla u \text{ in } B_n.$$

Thus,

$$\int_{\mathbb{R}^N} |\nabla \omega_n|^2 \, dx = \frac{N^2}{(N-2)^2} \int_{A_n} u^{2^*-2} |\nabla u|^2 \, dx + n^2 \int_{B_n} |\nabla u|^2 \, dx. \quad (4.5)$$

Combining (4.4) and (4.5), we obtain

$$\int_{\mathbb{R}^N} \left[ (|\nabla \omega_n|^2 + V_o(x)\omega_n^2 \right] \, dx - \int_{\mathbb{R}^N} \left[ \nabla u \nabla v_n + V_o(x) u v_n \right] \, dx$$
$$= \frac{4}{(N-2)^2} \int_{A_n} u^{2^*-2} |\nabla u|^2 \, dx.$$

From (4.4), we extract the inequality

$$(2^* - 1) \int_{A_n} u^{2^*-2} |\nabla u|^2 \, dx \leq \int_{\mathbb{R}^N} \left[ \nabla u \nabla v_n + V_o(x) u v_n \right] \, dx,$$

and then

$$\int_{\mathbb{R}^N} \left[ |\nabla \omega_n|^2 + V_o(x)\omega_n^2 \right] \, dx \leq \frac{N^2}{N^2 - 4} \int_{\mathbb{R}^N} \left[ \nabla u \nabla v_n + V_o(x) u v_n \right] \, dx.$$

Since $u$ a weak solution to (4.2), we have

$$\int_{\mathbb{R}^N} \left[ |\nabla \omega_n|^2 + V_o(x)\omega_n^2 \right] \, dx \leq \frac{N^2}{N^2 - 4} \int_{\mathbb{R}^N} g(x,u) v_n \, dx. \quad (4.6)$$

Observe that $g(x,u)v_n \leq \alpha_o|u|^{2^*-1}|v_n| + \beta_o|u||v_n| = \alpha_o|u|^{2^*-2}w_n^2 + \beta_o w_n^2$ in $\mathbb{R}^N$. From the Hölder inequality, we get

$$\begin{aligned}\int_{\mathbb{R}^N} \left[|\nabla\omega_n|^2 + V_o(x)\omega_n^2\right] dx &\leq \frac{N^2}{N^2-4}\alpha_o \int_{\mathbb{R}^N} |u|^{2^*-2}w_n^2 \, dx \\ &\quad + \frac{N^2}{N^2-4}\beta_o \int_{\mathbb{R}^N} w_n^2 \, dx \\ &\leq \frac{N^2}{N^2-4}\alpha_o\|u\|_{2^*}^{2^*-2}\|w_n\|_{2^*}^2 + \frac{N^2}{N^2-4}\beta_o \int_{\mathbb{R}^N} \omega_n^2 \, dx.\end{aligned}$$

Combining this last inequality with the Sobolev inequality bellow

$$S\|w_n\|_{2^*}^2 \leq \int_{\mathbb{R}^N} |\nabla\omega_n|^2 \, dx \leq \int_{\mathbb{R}^N} \left[|\nabla\omega_n|^2 + V_o(x)\omega_n^2\right] \, dx,$$

under hypothesis $(\mathcal{C})$, we have

$$\left[\int_{A_n} |\omega_n|^{2^*} \, dx\right]^{\frac{2}{2^*}} \leq \left[\int_{\mathbb{R}^N} |\omega_n|^{2^*} \, dx\right]^{\frac{2}{2^*}} \leq \frac{2N^2}{N^2-4}\beta_o S^{-1} \int_{\mathbb{R}^N} \omega_n^2 \, dx,$$

which together with the fact that $|\omega_n| \leq |u|^{\frac{N}{N-2}}$ in $\mathbb{R}^N$ and $|\omega_n| = |u|^{\frac{N}{N-2}}$ in $A_n$ implies

$$\left[\int_{A_n} |u|^{\frac{2N^2}{(N-2)^2}} \, dx\right]^{\frac{2}{2^{*2}}} \leq \left(\frac{2N^2}{(N-2)^2}\beta_o S^{-1}\right)^{\frac{1}{2^*}} \left[\int_{\mathbb{R}^N} |u|^{2^*} \, dx\right]^{\frac{1}{2^*}}. \tag{4.7}$$

Passing to the liminf in (4.7) and using Fatou's lemma we obtain

$$\|u\|_{\frac{2N^2}{(N-2)^2}} \leq \left(\frac{2N^2}{(N-2)^2}\beta_o S^{-1}\right)^{\frac{1}{2^*}} \|u\|_{2^*}. \tag{4.8}$$

Thus $u \in L^{\frac{2N^2}{(N-2)^2}}(\mathbb{R}^N) \cap L^{2^*}(\mathbb{R}^N)$, which implies that $h = \alpha_o|u|^{2^*-1} + \beta_o|u| \in L_{loc}^{\frac{2N^2}{N^2-4}}(\mathbb{R}^N)$. Moreover, from (4.8) and condition $(\mathcal{C})$, we obtain

$$[h]_{\frac{2N^2}{N^2-4}} \leq \alpha_o\|u\|_{\frac{2N^2}{(N-2)^2}}^{2^*-1} + C\beta_o[u]_{\frac{2N^2}{N^2-4}}.$$

Since $\frac{2N^2}{N^2-4} < 2^*$, we have

$$\left(\int_{B_2(z)} |u|^{\frac{2N^2}{N^2-4}} \, dx\right)^{\frac{N^2-4}{2N^2}} \leq \left[\left(\int_{B_2(z)} |u|^{2^*} \, dx\right)^{\frac{N}{N+2}} \left(\int_{B_2(z)} dx\right)^{\frac{2}{N+2}}\right]^{\frac{N^2-4}{2N^2}}$$

and then $[u]_{\frac{2N^2}{N^2-4}} \leq |B_2(0)|^{\frac{N-2}{N^2}} \|u\|_{2^*}$. So, using $(\mathcal{C})$ once more we have

$$\begin{aligned}
[h]_{\frac{2N^2}{N^2-4}} &\leq \alpha_o \left( \frac{2N^2}{(N-2)^2} \beta_o S^{-1} \right)^{\frac{N+2}{2N}} \|u\|_{2^*}^{2^*-1} + C\beta_o |B_2(0)|^{\frac{N-2}{N^2}} \|u\|_{2^*} \\
&= \alpha_o \left( \frac{2N^2}{(N-2)^2} \beta_o S^{-1} \right)^{\frac{N+2}{2N}} \|u\|_{2^*}^{2^*-2} \|u\|_{2^*} + C\beta_o |B_2(0)|^{\frac{N-2}{N^2}} \|u\|_{2^*} \\
&\qquad C \left( \beta_o^{\frac{N+2}{2N}} + \beta_o \right) \|u\|_{2^*}.
\end{aligned}$$

From Lemma 4.2 there exists a positive constant $\Lambda$ which depends only on $\alpha_o$ and $\beta_o$ such that

$$\|u\|_\infty \leq \Lambda \|u\|_{2^*}^2$$

and the proof is completed. □

## 4.1 Proof of Theorem 4.1 completed

From the Young's inequality we see that

$$\gamma |s|^{p-2} \leq \frac{(p-2)}{2^*-2} |s|^{2^*-2} + \frac{(2^*-p)}{2^*-2} \gamma^{\frac{2^*-2}{2^*-p}},$$

for all real $s$. This implies that

$$|g(x,s)| \leq |s|^{2^*-1} + \gamma |s|^{p-1} \leq \frac{(2^*+p-4)}{2^*-2} |s|^{2^*-1} + \frac{(2^*-p)}{2^*-2} \gamma^{\frac{2^*-2}{2^*-p}} |s|,$$

The choice of $d_\gamma$ in (2.1) together (3.12) show that a solution $u = u_R$ above satisfies

$$-\Delta u + V_\lambda(x) u = |u|^{2^*-2} u + \gamma |u|^{p-2} u$$

and

$$\begin{aligned}
\frac{2N^2}{N^2-4} \cdot \frac{2^*+p-4}{2^*-2} \|u\|_{2^*}^{2^*-2} S^{-1} \leq \frac{2N^2}{N^2-4} \cdot \frac{2^*+p-4}{(2^*-2)S} \cdot \left( C\gamma^{-\frac{1}{p-2}} \right)^{\frac{4}{N-2}} \\
\leq 1,
\end{aligned}$$

for $\gamma$ large enough. We will use Proposition 4.3 with $\alpha_o = \frac{2^*+p-4}{2^*-2}$. From Proposition 4.3 with $\beta_o = \frac{(2^*-p)}{2^*-2}\gamma^{\frac{2^*-2}{2^*-p}}$ we have:

$$\|u\|_\infty \leq C\Lambda\|u\|_{2^*}^2 \leq C\gamma^{\frac{2^*-2}{2^*-p}}\gamma^{\frac{-2}{p-2}}.$$

Now we have a family of solutions $u = u_R$ of the auxiliary problems $(\mathcal{AP})$ in $L^\infty$ and

$$\|u\|_\infty \leq C\gamma^{\frac{2Np-(8+4N)}{(2^*-p)(p-2)(N-2)}}. \tag{4.9}$$

where $C$ is a positive constant.

## 5 Proof of Theorem 1.1

We need to show that a solution $u \in E$ of the auxiliary problem satisfies

$$f(u) \leq \frac{V_\lambda(x)}{p}u \quad \text{in } |x| \geq R. \tag{5.1}$$

**Lemma 5.1.** *For any ground state solution to $(\mathcal{AP})$, it holds*

$$u(x) \leq \frac{R\|u\|_\infty}{|x|}, \quad \textit{for all} \quad |x| \geq R. \tag{5.2}$$

*Proof.* It is an usual approach and you can find in Lemma 5.1 - [10]. □

**Lemma 5.2.** *There exists $C_o > 0$ such that for any ground state solution to Problem $(\mathcal{AP})$ it holds*

$$\frac{f(u)}{u} \leq C_o\left(\frac{R}{|x|}\right)^{p-2}\gamma^{\frac{(2^*-2)[2Np-(8+4N)]}{(2^*-p)(p-2)(N-2)}}, \quad \textit{for all} \quad |x| \geq R. \tag{5.3}$$

*Proof.* From Lemma 5.1, we have

$$\frac{f(u)}{u} = u^{2^*-2} + \gamma|u|^{p-2} \leq \frac{R^{2^*-2}\|u\|_\infty^{2^*-2}}{|x|^{2^*-2}} + \gamma\frac{R^{p-2}\|u\|_\infty^{p-2}}{|x|^{p-2}},$$

which together with (4.9) gives

$$
\begin{aligned}
\frac{f(u)}{u} &\leq \frac{R^{2^*-2}C^{2^*-2}\gamma^{\frac{(2^*-2)[2Np-(8+4N)]}{(2^*-p)(p-2)(N-2)}}}{|x|^{2^*-2}} + \gamma\frac{R^{p-2}C^{p-2}\gamma^{\frac{(p-2)[2Np-(8+4N)]}{(2^*-p)(p-2)(N-2)}}}{|x|^{p-2}} \\
&\leq \left[\frac{R^{2^*-2}C^{2^*-2}}{|x|^{2^*-2}} + \frac{R^{p-2}C^{p-2}}{|x|^{p-2}}\right]\gamma^{\frac{(2^*-2)[2Np-(8+4N)]}{(2^*-p)(p-2)(N-2)}} \\
&= \frac{R^{p-2}}{|x|^{p-2}}\left[C^{p-2} + C^{2^*-2}\frac{R^{2^*-p}}{|x|^{2^*-p}}\right]\gamma^{\frac{(2^*-2)[2Np-(8+4N)]}{(2^*-p)(p-2)(N-2)}} \\
&\leq C_o\left(\frac{R}{|x|}\right)^{p-2}\gamma^{\frac{(2^*-2)[2Np-(8+4N)]}{(2^*-p)(p-2)(N-2)}},
\end{aligned}
$$

where $C_o = (C^{p-2} + C^{2^*-2})$ and we have used $|x| \geq R$ and $\gamma \geq 1$. □

## 5.1 Proof of Theorem 1.1 completed

From condition $(V_3)$ there exists $R_1 > 0$ and $c_1 > 0$ such that

$$
|x|^{p-2}V(x) \geq c_1 \quad \text{for all} \quad |x| \geq R_1. \tag{5.4}
$$

On the other hand, since $V_\lambda(x) \geq \lambda V(x)$, using (5.3) and taking $R > R_1$ we can see that

$$
\frac{f(u)}{u} \leq \frac{V_\lambda(x)}{p} \quad \text{for all} \quad |x| \geq R,
$$

provided that

$$
\lambda \geq \frac{c_o p}{c_1}\gamma^{\frac{(2^*-2)[2Np-(8+4N)]}{(2^*-p)(p-2)(N-2)}}
$$

and consequently $u$ solution to auxiliary Problem $(\mathcal{AP})$ is indeed solution to original Problem $(\mathcal{P}_{\lambda,\gamma})$.

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