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Self-similar solutions of k-Hessian evolution equations

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To Pedro Ubilla, for his 60th birthday anniversary, with appreciation and gratitude

Abstract

In this note we construct self-similar solutions of the k-Hessian evolution equation

$$u_t = (-1)^{k-1} S_k(D^2 u)$$

in $(0, \infty) \times \mathbb{R}^n$, providing a new class of explicit and radially symmetric self-similar solutions that we call *k-Barenblatt solutions*. These solutions present some common properties as those of well-known Barenblatt solutions for the porous media equation and the p-Laplacian evolution equation as well.

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1 The k-Hessian operator

We briefly introduce the class of operators under study. For a twice-differentiable function u defined on a domain $\Omega \subset \mathbb{R}^n$, the *$k$-Hessian operator* ($k = 1, ..., n$) is defined by the formula

$$S_k(D^2u) = \sigma_k(\Lambda) = \sum_{1\le i_1<...<i_k\le n} \lambda_{i_1}...\lambda_{i_k},$$

where $\Lambda = \Lambda(D^2u) := (\lambda_1, ..., \lambda_n)$, the λ's are the eigenvalues of D^2u and σ_k is the k-th elementary symmetric function. Equivalently, $S_k(D^2u)$ is the sum of the k-th principal minors of the Hessian matrix. See, *e.g.*, X.-J. Wang [20, 21]. These operators form an important class of second order operators that contains, as the most relevant examples, the Laplace operator $S_1(D^2u) = \Delta u$ and the Monge-Ampère operator $S_n(D^2u) = \det D^2u$. They are fully nonlinear when $k > 1$. In particular, $S_2\left(D^2u\right) = \frac{1}{2}\left((\Delta u)^2 - \left|D^2u\right|^2\right)$. The study of k-Hessian equations has many applications in geometry, optimization theory and in other related fields. See [21]. There exists a large literature about existence, regularity and qualitative properties of solutions for the k-Hessian equations, starting with the seminal work of L. Caffarelli, L. Nirenberg and J. Spruck [3].

We point out that the k-Hessian operators are k-homogeneous and also invariant under rotations of coordinates. For more details about these operators we refer to X.-J. Wang [21].

We construct self-similar solutions of a k-Hessian evolution equation posed on the whole Euclidean space. This study is a first step towards understanding important properties of the underlying equations which can be captured by these special solutions. We point out that there is a vast literature concerning evolution equations that generalize the standard heat equation. This literature addresses among others, the p-Laplacian equation, the porous medium equation and the space-fractional porous medium equation. See *e.g.* [2, 4, 5, 6, 8, 9, 10, 12, 13, 19].

Concerning exact solutions of some nonlinear diffusion equations, in [11] new closed-form similarity solutions of N-dimensional radially symmetric equations were given, which are generalizations of the classical Barenblatt solutions. In [9], the authors study an explicit equivalence between radially symmetric solutions for two basic nonlinear degenerate diffusion equations, namely, the porous medium equation and the p-Laplacian equation. In particular, they derive the existence of new self-similar solutions for the evolution p-Laplacian equation. In [8] several one-parameter families of explicit self-similar solutions were constructed for the porous medium equations with fractional operators.

2 The family of self-similar solutions

We construct special positive solutions $u = u(t, x)$, called self-similar solutions, of equation

$$u_t = (-1)^{k-1} S_k(D^2 u) \tag{2.1}$$

As we can see, when $k = 1$ equation (2.1) is reduced to the classical heat equation. When $k > 1$ in (2.1), we have found an explicit one-parameter family of positive self-similar solutions on $\mathbb{R}^n$ with compact support in space for every fixed time. We describe them now:

$$U_C(t, x) = t^{-\alpha} \left(C - \gamma \left(\frac{|x|}{t^\beta} \right)^2 \right)_+^{\frac{k}{k-1}}, \tag{2.2}$$

where $(\cdot)_+$ denotes the positive part, $C > 0$ is an arbitrary constant (the parameter), and α, β and γ have precise values, namely

$$\alpha = \frac{n}{n(k-1)+2k},\ \beta = \frac{1}{n(k-1)+2k},\ \gamma = \frac{k-1}{2k}\left(\frac{\beta}{c_{n,k}}\right)^{\frac{1}{k}},\ c_{n,k} = \frac{\binom{n}{k}}{n}.$$

Note that this family, whose elements we call *k-Barenblatt solutions*, is well defined for the full range of k-Hessian operators with $k > 1$. See [14]. Moreover, these solutions are similar to those known for the porous medium equation and the p-Laplacian equation as well. See, *e.g.*, [9] and the references therein. We also note that the relation between the similarity exponents α and β, $\alpha = n\beta$, is an *a priori* condition that reflects the mass conservation of these special solutions.

3 The scaling group and self-similarity for the k-Hessian equation

Before we start the construction of the Barenblatt solutions we review some basic facts following some arguments given in [18]. Let first observe that for every solution $u(t,x)$ and positive constants a, b, c the function

$$\tilde{u}(t,x) = cu(at, bx)$$

is again a solution of (2.1) if

$$ac^{1-k} = b^{2k}.$$

So we obtain a two-parametric transformation group $T = T(a,b)$ (scaling group) acting on the set of solutions of the k-Hessian equation (2.1):

$$(Tu)(t,x) = \left(\frac{b^{2k}}{a}\right)^{\frac{1}{k-1}} u(at, bx).$$

Those special solutions that are themselves invariant under the scaling group are called *self-similar solutions*: this means that $(Tu)(t,x) = u(t,x)$ for all (t,x) in the domain of definition. If moreover, we impose the condition that the solutions have constant finite mass (i.e., integral in space), then $c = b^n$, which implies that the group is reduced to the one-parameter family $(T_\lambda u)(t,x) = b^n u(b^{n(k-1)+2k}t, bx)$, which may be written in terms of $\lambda = b^{n(k-1)+2k}$ as

$$(T_\lambda u)(t,x) = \lambda^\alpha u(\lambda t, \lambda^\beta x),$$

with scaling exponents given by

$$\alpha = \frac{n}{n(k-1)+2k}, \; \beta = \frac{1}{n(k-1)+2k}.$$

Thus, if $u(t,x)$ is a self-similar solution, then for all $x \in \mathbb{R}^n$ and $t > 0$ we have

$$u(t,x) = \lambda^{\alpha} u(\lambda t, \lambda^{\beta} x).$$

Fix now $t_1 > 0$ and let $\lambda = \frac{1}{t_1}$. We get

$$u(t,x) = t_1^{-\alpha} u\left(\frac{t}{t_1}, t_1^{-\beta} x\right).$$

Since t_1 is arbitrary we can replace it with t. Calling now $u(1,x) \equiv \theta(x)$ we get

$$u(t,x) = t^{-\alpha}\theta(t^{-\beta}x),$$

where θ is called the *profile* of the solution.

4 The construction of the k-Barenblatt solutions

In this section we will derive the compactly supported family of mass conserving k-Barenblatt solutions given in (2.2) for the equation

$$u_t = (-1)^{k-1} S_k(D^2 u). \tag{4.1}$$

Thus, we are looking for a positive solution of the above evolution equation with constant mass in $\mathbb{R}^n$, that is

$$\int_{\mathbb{R}^n} u(t,x)dx = M > 0, \text{ for all } t > 0. \tag{4.2}$$

According to the previous section, we will actually look for a self-similar solution u to (4.1) of the form:

$$u(t,x) = t^{-\alpha}\theta(\xi),\ \xi = \frac{x}{t^{\beta}},\ t > 0,\ x \in \mathbb{R}^n, \tag{4.3}$$

for some profile θ and the exponents α and β to be determined. Inserting (4.3) into the left-hand side of (4.1), we have

$$\begin{aligned} u_t &= -\alpha t^{-\alpha-1}\theta(\xi) + t^{-\alpha}\frac{d\theta}{d\xi}\cdot\frac{d\xi}{dt} \\ &= -\alpha t^{-\alpha-1}\theta(\xi) + t^{-\alpha}\nabla_{\xi}\theta(\xi)\cdot(-\beta)t^{-\beta-1}x \\ &= t^{-\alpha-1}(-\alpha\theta(\xi) - \beta\nabla_{\xi}\theta(\xi)\cdot\xi). \end{aligned}$$

Inserting (4.3) into the right-hand side of (4.1) (omitting the scalar factor) we have

$$\begin{aligned}S_k(D^2u) &= t^{-k\alpha-2k\beta}S_k(D^2\theta(\xi))\\ &= t^{-k(\alpha+2\beta)}S_k(D^2\theta(\xi)).\end{aligned}$$

Then, from the condition $\alpha(k-1)+2k\beta = 1$ (self-similarity condition), we get the following profile equation

$$\alpha\theta(\xi) + \beta\nabla_\xi\theta(\xi)\cdot\xi = (-1)^k S_k(D^2\theta(\xi)). \tag{4.4}$$

We also have from (4.2)

$$M = \int_{\mathbb{R}^n} u(t,x)\,dx = \int_{\mathbb{R}^n} t^{-\alpha}\theta\left(\frac{x}{t^\beta}\right)dx = t^{n\beta-\alpha}\int_{\mathbb{R}^n}\theta(\xi)\,d\xi$$

(it is assumed that $\theta \in L^1(\mathbb{R}^n)$), which yields $n\beta - \alpha = 0$ (mass-preserving condition). Solving the relations between the similarity exponents α and β we obtain $\alpha = \frac{n}{n(k-1)+2k}$ and $\beta = \frac{1}{n(k-1)+2k}$, the same values found through scaling arguments.

Now let θ be a radially symmetric function, say $\theta = \theta(r)$, $r = |\xi| \geq 0$ (abuse of notation). We point out that for a radially symmetric C^2-function θ the k-Hessian operator take the one-dimensional form

$$S_k(D^2\theta) = c_{n,k}\,r^{1-n}(r^{n-k}(\theta'(r))^k)',\ r>0$$

where $c_{n,k} = \frac{1}{n}\binom{n}{k}$. Then the governing equation (4.4) takes the form

$$\alpha\theta(r) + \beta r\theta'(r) = (-1)^k c_{n,k}\,r^{1-n}(r^{n-k}(\theta'(r))^k)',\ r>0, \tag{4.5}$$

with the symmetry condition $\theta'(0) = 0$. From this and the equality $\alpha = n\beta$, the equation (4.5) can be integrated once (fortunately) and then simplified as

$$\beta\theta(r) = (-1)^k c_{n,k}\,r^{-k}(\theta'(r))^k,\ r>0;\ \theta'(0) = 0. \tag{4.6}$$

We observe that, when $k = 1$ in (4.6), an explicit integration shows that $\theta(r) = Ce^{-\frac{r^2}{4}}$, where C is a positive constant. Thus from (4.3) we recover the Gaussian function of the classical heat equation. Now let $k > 1$. A necessary condition for the existence of a solution with the required properties is that the profile θ be decreasing with limit zero at infinity. Thus integrating (4.6) we have

$$\theta(r) = \left(C - \frac{k-1}{k} \left(\frac{\beta}{c_{n,k}} \right)^{\frac{1}{k}} \frac{r^2}{2} \right)_+^{\frac{k}{k-1}}, \quad r \geq 0. \tag{4.7}$$

Finally, putting $\gamma = \frac{k-1}{2k} \left(\frac{\beta}{c_{n,k}} \right)^{\frac{1}{k}}$ and inserting $\theta(r)$ in (4.3), we obtain (2.2).

Note that the positive constant C in (4.7) may easily be put in correspondence with the mass of the solution, $C = C(M)$, by (4.2). In fact, introducing the constant $r_0 = \sqrt{\frac{2C}{\gamma}}$, the self-similar solution with constant mass has the explicit form

$$u(t,x) = t^{-\frac{n}{n(k-1)+2k}} \left[\frac{k-1}{4k[c_{n,k}(n(k-1)+2k)]^{\frac{1}{k}}} \left(r_0^2 - \frac{|x|^2}{t^{\frac{2}{n(k-1)+2k}}} \right)_+ \right]^{\frac{k}{k-1}}, \tag{4.8}$$

where

$$\begin{aligned} r_0 &= r_0(M) \\ &= \left\{ \pi^{-\frac{n}{2}} D(n,k) \left(\frac{4k}{k-1} \right)^{\frac{k}{k-1}} \frac{\Gamma\left(\frac{n}{2} + \frac{2k-1}{k-1}\right)}{\Gamma\left(\frac{2k-1}{k-1}\right)} M \right\}^{\frac{k-1}{n(k-1)+2k}}, \end{aligned}$$

$D(n,k) = [c_{n,k}(n(k-1)+2k)]^{\frac{1}{k-1}}$ and where $\Gamma(\cdot)$ is the Gamma-function.

We have the following properties of the self-similar solutions given in (2.2):

- $\operatorname{supp} U_C(t,\cdot) \subseteq B\left(0,\, t^{\beta} \left[\frac{2k}{k-1} \left(\frac{c_{n,k}}{\beta} \right)^{\frac{1}{k}} C \right]^{\frac{1}{2}} \right)$.
- Finite propagation speed.
- Mass conservation.

- $U_C(t,x) \to M\delta_0(x)$ as $t \to 0^+$, where $\delta_0(x)$ is the Dirac delta function concentrated at 0. In other words,
$$\lim_{t\to 0^+} \int_{\mathbb{R}^n} U_C(t,x) f(x) dx = M f(0)$$
for all test functions $f \in C_c^\infty(\mathbb{R}^n)$.

- Everywhere, except on the degeneracy surface
$$[0,\infty) \times \left\{ |x| = t^\beta \left[\frac{2k}{k-1} \left(\frac{c_{n,k}}{\beta} \right)^{\frac{1}{k}} C \right]^{\frac{1}{2}} \right\},$$
it is classical (and infinitely differentiable).

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