Self-similar solutions of \( k \)-Hessian evolution equations

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To Pedro Ubilla, for his 60th birthday anniversary, with appreciation and gratitude

Abstract

In this note we construct self-similar solutions of the \( k \)-Hessian evolution equation

\[
    u_t = (-1)^{k-1} S_k(D^2 u)
\]

in \( (0, \infty) \times \mathbb{R}^n \), providing a new class of explicit and radially symmetric self-similar solutions that we call \( k \)-Barenblatt solutions. These solutions present some common properties as those of well-known Barenblatt solutions for the porous media equation and the \( p \)-Laplacian evolution equation as well.

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1 The $k$-Hessian operator

We briefly introduce the class of operators under study. For a twice-differentiable function $u$ defined on a domain $\Omega \subset \mathbb{R}^n$, the $k$-Hessian operator ($k = 1, \ldots, n$) is defined by the formula

$$S_k(D^2u) = \sigma_k(\Lambda) = \sum_{1 \leq i_1 < \ldots < i_k \leq n} \lambda_{i_1} \ldots \lambda_{i_k},$$

where $\Lambda = \Lambda(D^2u) := (\lambda_1, \ldots, \lambda_n)$, the $\lambda$’s are the eigenvalues of $D^2u$ and $\sigma_k$ is the $k$-th elementary symmetric function. Equivalently, $S_k(D^2u)$ is the sum of the $k$-th principal minors of the Hessian matrix. See, e.g., X.-J. Wang [20, 21]. These operators form an important class of second order operators that contains, as the most relevant examples, the Laplace operator $S_1(D^2u) = \Delta u$ and the Monge-Ampère operator $S_n(D^2u) = \det D^2u$. They are fully nonlinear when $k > 1$. In particular, $S_2(D^2u) = \frac{1}{2} ((\Delta u)^2 - |D^2u|^2)$. The study of $k$-Hessian equations has many applications in geometry, optimization theory and in other related fields. See [21]. There exists a large literature about existence, regularity and qualitative properties of solutions for the $k$-Hessian equations, starting with the seminal work of L. Caffarelli, L. Nirenberg and J. Spruck [3].

We point out that the $k$-Hessian operators are $k$-homogeneous and also invariant under rotations of coordinates. For more details about these operators we refer to X.-J. Wang [21].

We construct self-similar solutions of a $k$-Hessian evolution equation posed on the whole Euclidean space. This study is a first step towards understanding important properties of the underlying equations which can be captured by these special solutions. We point out that there is a vast literature concerning evolution equations that generalize the standard heat equation. This literature addresses among others, the $p$-Laplacian equation, the porous medium equation and the space-fractional porous medium equation. See e.g. [2, 4, 5, 6, 8, 9, 10, 12, 13, 19].

Concerning exact solutions of some nonlinear diffusion equations, in [11] new closed-form similarity solutions of $N$-dimensional radially sym-
metric equations were given, which are generalizations of the classical Barenblatt solutions. In [9], the authors study an explicit equivalence between radially symmetric solutions for two basic nonlinear degenerate diffusion equations, namely, the porous medium equation and the \( p \)-Laplacian equation. In particular, they derive the existence of new self-similar solutions for the evolution \( p \)-Laplacian equation. In [8] several one-parameter families of explicit self-similar solutions were constructed for the porous medium equations with fractional operators.

2 The family of self-similar solutions

We construct special positive solutions \( u = u(t, x) \), called self-similar solutions, of equation

\[
 u_t = (-1)^{k-1} S_k(D^2 u)
\]

(2.1)

As we can see, when \( k = 1 \) equation (2.1) is reduced to the classical heat equation. When \( k > 1 \) in (2.1), we have found an explicit one-parameter family of positive self-similar solutions on \( \mathbb{R}^n \) with compact support in space for every fixed time. We describe them now:

\[
 U_C(t, x) = t^{-\alpha} \left( C - \gamma \left( \frac{|x|}{t^{\beta}} \right)^{\frac{2}{k-1}} \right)_+, \quad (2.2)
\]

where \( (\cdot)_+ \) denotes the positive part, \( C > 0 \) is an arbitrary constant (the parameter), and \( \alpha, \beta \) and \( \gamma \) have precise values, namely

\[
 \alpha = \frac{n}{n(k-1)+2k}, \quad \beta = \frac{1}{n(k-1)+2k}, \quad \gamma = \frac{k-1}{2k} \left( \frac{\beta}{c_{n,k}} \right)^{\frac{1}{k}}, \quad c_{n,k} = \left( \frac{n}{k} \right)^{\frac{1}{n}}.
\]

Note that this family, whose elements we call \( k \)-Barenblatt solutions, is well defined for the full range of \( k \)-Hessian operators with \( k > 1 \). See [14]. Moreover, these solutions are similar to those known for the porous medium equation and the \( p \)-Laplacian equation as well. See, e.g., [9] and
the references therein. We also note that the relation between the similarity exponents $\alpha$ and $\beta$, $\alpha = n\beta$, is an \textit{a priori} condition that reflects the mass conservation of these special solutions.

3 The scaling group and self-similarity for the $k$-Hessian equation

Before we start the construction of the Barenblatt solutions we review some basic facts following some arguments given in [18]. Let first observe that for every solution $u(t, x)$ and positive constants $a, b, c$ the function

$$\tilde{u}(t, x) = cu(at, bx)$$

is again a solution of (2.1) if

$$ac^{1-k} = b^{2k}.$$ 

So we obtain a two-parametric transformation group $T = T(a, b)$ (scaling group) acting on the set of solutions of the $k$-Hessian equation (2.1):

$$(Tu)(t, x) = \left(\frac{b^{2k}}{a}\right)^{\frac{1}{k-1}} u(at, bx).$$

Those special solutions that are themselves invariant under the scaling group are called \textit{self-similar solutions}: this means that $(Tu)(t, x) = u(t, x)$ for all $(t, x)$ in the domain of definition. If moreover, we impose the condition that the solutions have constant finite mass (i.e., integral in space), then $c = b^n$, which implies that the group is reduced to the one-parameter family $(T\lambda u)(t, x) = b^nu(b^{n(k-1)+2k}t, bx)$, which may be written in terms of $\lambda = b^{n(k-1)+2k}$ as

$$(T\lambda u)(t, x) = \lambda^\alpha u(\lambda t, \lambda^\beta x),$$

with scaling exponents given by

$$\alpha = \frac{n}{n(k-1) + 2k}, \quad \beta = \frac{1}{n(k-1) + 2k}.$$
Thus, if \( u(t, x) \) is a self-similar solution, then for all \( x \in \mathbb{R}^n \) and \( t > 0 \) we have
\[
u(t, x) = \lambda^\alpha u(\lambda t, \lambda^\beta x).
\]
Fix now \( t_1 > 0 \) and let \( \lambda = \frac{1}{t_1} \). We get
\[
u(t, x) = t_1^{-\alpha} u \left( \frac{t}{t_1}, \frac{x}{t_1^\beta} \right).
\]
Since \( t_1 \) is arbitrary we can replace it with \( t \). Calling now \( u(1, x) \equiv \theta(x) \) we get
\[
u(t, x) = t^{-\alpha} \theta(t^{-\beta} x),
\]
where \( \theta \) is called the profile of the solution.

4 The construction of the \( k \)-Barenblatt solutions

In this section we will derive the compactly supported family of mass conserving \( k \)-Barenblatt solutions given in (2.2) for the equation
\[
u_t = (-1)^{k-1} S_k(D^2 u).
\] (4.1)
Thus, we are looking for a positive solution of the above evolution equation with constant mass in \( \mathbb{R}^n \), that is
\[
\int_{\mathbb{R}^n} u(t, x) dx = M > 0, \text{ for all } t > 0.
\] (4.2)
According to the previous section, we will actually look for a self-similar solution \( u \) to (4.1) of the form:
\[
u(t, x) = t^{-\alpha} \theta(\xi), \quad \xi = \frac{x}{t^\beta}, \quad t > 0, \quad x \in \mathbb{R}^n,
\] (4.3)
for some profile \( \theta \) and the exponents \( \alpha \) and \( \beta \) to be determined. Inserting (4.3) into the left-hand side of (4.1), we have
\[
u_t = -\alpha t^{-\alpha-1} \theta(\xi) + t^{-\alpha} \frac{d\theta}{d\xi} \cdot \frac{d\xi}{dt}
\]
\[
= -\alpha t^{-\alpha-1} \theta(\xi) + t^{-\alpha} \nabla_\xi \theta(\xi) \cdot (-\beta) t^{-\beta-1} x
\]
\[
= t^{-\alpha-1} (-\alpha \theta(\xi) - \beta \nabla_\xi \theta(\xi) \cdot \xi).
\]
Inserting (4.3) into the right-hand side of (4.1) (omitting the scalar factor) we have

\[ S_k(D^2u) = t^{-k\alpha - 2k\beta}S_k(D^2\theta(\xi)) = t^{-k(\alpha + 2\beta)}S_k(D^2\theta(\xi)). \]

Then, from the condition \( \alpha(k - 1) + 2k\beta = 1 \) (self-similarity condition), we get the following profile equation

\[ \alpha \theta(\xi) + \beta \nabla_{\xi} \theta(\xi) \cdot \xi = (-1)^k S_k(D^2\theta(\xi)). \] (4.4)

We also have from (4.2)

\[
M = \int_{\mathbb{R}^n} u(t, x) \, dx = \int_{\mathbb{R}^n} t^{-\alpha} \theta \left( \frac{x}{t^\beta} \right) \, dx = t^{n\beta - \alpha} \int_{\mathbb{R}^n} \theta(\xi) \, d\xi
\]

(it is assumed that \( \theta \in L^1(\mathbb{R}^n) \)), which yields \( n\beta - \alpha = 0 \) (mass-preserving condition). Solving the relations between the similarity exponents \( \alpha \) and \( \beta \) we obtain \( \alpha = \frac{n}{n(k-1) + 2k} \) and \( \beta = \frac{1}{n(k-1) + 2k} \), the same values found through scaling arguments.

Now let \( \theta \) be a radially symmetric function, say \( \theta = \theta(r), r = |\xi| \geq 0 \) (abuse of notation). We point out that for a radially symmetric \( C^2 \)-function \( \theta \) the \( k \)-Hessian operator take the one-dimensional form

\[ S_k(D^2\theta) = c_{n,k} r^{1-n}(r^{n-k}(\theta'(r))^k)' \], \( r > 0 \)

where \( c_{n,k} = \frac{1}{n\binom{n}{k}} \). Then the governing equation (4.4) takes the form

\[ \alpha \theta(r) + \beta r\theta'(r) = (-1)^k c_{n,k} r^{1-n}(r^{n-k}(\theta'(r))^k)' \], \( r > 0 \) \hspace{1cm} (4.5)

with the symmetry condition \( \theta'(0) = 0 \). From this and the equality \( \alpha = n\beta \), the equation (4.5) can be integrated once (fortunately) and then simplified as

\[ \beta \theta(r) = (-1)^k c_{n,k} r^{-k}(\theta'(r))^k \], \( r > 0; \ \theta'(0) = 0 \) \hspace{1cm} (4.6)

We observe that, when \( k = 1 \) in (4.6), an explicit integration shows that \( \theta(r) = Ce^{-\frac{r^2}{4}} \), where \( C \) is a positive constant. Thus from (4.3) we recover
the Gaussian function of the classical heat equation. Now let \( k > 1 \).

A necessary condition for the existence of a solution with the required properties is that the profile \( \theta \) be decreasing with limit zero at infinity. Thus integrating (4.6) we have

\[
\theta(r) = \left( C - \frac{k - 1}{k} \left( \frac{\beta}{c_{n,k}} \right) \frac{1}{k} \frac{r^2}{2} \right)^{\frac{k}{k-1}}, \quad r \geq 0. \tag{4.7}
\]

Finally, putting \( \gamma = \frac{k - 1}{2k} \left( \frac{\beta}{c_{n,k}} \right)^{\frac{1}{k}} \) and inserting \( \theta(r) \) in (4.3), we obtain (2.2).

Note that the positive constant \( C \) in (4.7) may easily be put in correspondence with the mass of the solution, \( C = C(M) \), by (4.2). In fact, introducing the constant \( r_0 = \sqrt{\frac{2C}{\gamma}} \), the self-similar solution with constant mass has the explicit form

\[
u(t, x) = t^{-\frac{n}{n(k-1)+2k}} \left[ \frac{k - 1}{4k[c_{n,k}(n(k - 1) + 2k)]^{\frac{1}{k}}} \left( r_0^2 - \frac{|x|^2}{t^{\frac{2}{n(k-1)+2k}}} \right) \right]^{\frac{k}{k-1}}, \tag{4.8}
\]

where

\[
r_0 = r_0(M) = \left\{ \pi^{\frac{n}{2}} D(n, k) \left( \frac{4k}{k-1} \right)^{\frac{k}{k-1}} \Gamma \left( \frac{n}{2} + \frac{2k-1}{k-1} \right) \Gamma \left( \frac{2k-1}{k-1} \right) M \right\}^{\frac{k-1}{2(k-1)+2k}},
\]

\( D(n, k) = [c_{n,k}(n(k - 1) + 2k)]^{\frac{1}{k-1}} \) and where \( \Gamma(\cdot) \) is the Gamma-function.

We have the following properties of the self-similar solutions given in (2.2):

\begin{itemize}
  \item \text{supp} \ U_C(t, \cdot) \subseteq B \left( 0, t^\beta \left[ \frac{2k}{k-1} \left( \frac{c_{n,k}}{\beta} \right)^{\frac{k}{2}} C \right]^{\frac{1}{2}} \right).
  \item Finite propagation speed.
  \item Mass conservation.
\end{itemize}
• $U_C(t, x) \to M\delta_0(x)$ as $t \to 0^+$, where $\delta_0(x)$ is the Dirac delta function concentrated at 0. In other words,
\[
\lim_{t \to 0^+} \int_{\mathbb{R}^n} U_C(t, x)f(x)dx = Mf(0)
\]
for all test functions $f \in C^\infty_c(\mathbb{R}^n)$.

• Everywhere, except on the degeneracy surface
\[
[0, \infty) \times \left\{ |x| = t^\beta \left[ \frac{2k}{k-1} \left( \frac{c_{n,k}}{\beta} \right)^{\frac{1}{k}} C \right]^{\frac{1}{2}} \right\},
\]
it is classical (and infinitely differentiable).

References


