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Some results on strongly coupled elliptic systems

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Dedicated to Professor Pedro Ubilla
on the occasion of his 60th birthday

Abstract. This work focuses on qualitative properties for nonnegative solutions to elliptic systems driven by a Gross–Pitaevskii nonlinearity on a punctured domain. We aim to present some classification results and a description of the local behavior near an isolated (nonremovable) singularity for second and fourth order systems of this class.

Keywords: coupled systems, Liouville-type results, asymptotic analysis.

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1 Introduction

In this manuscript, we discuss the classification and the local behavior near the origin for nonnegative solutions $\mathcal{U}=(u_1,\dots,u_p):B_R^*\to\mathbb{R}^p$ to vectorial equations, which we describe as follows. Here $B_R^*:=B_R^n(0)\setminus\{0\}\subset\mathbb{R}^n$ is the *punctured ball* if $R<\infty$ (resp. $B_\infty^*:=\mathbb{R}^n\setminus\{0\}$ is the *punctured space*). By *solution*, we mean a p-map $\mathcal{U}$ such that each component is smooth and solves the equation in the classical sense.

A vectorial solution $\mathcal{U}$ is said to be *singular*, if the origin is a *non-removable singularity* for $|\mathcal{U}|$. Otherwise, $\mathcal{U}$ is called *non-singular*. We also say that a p-map solution $\mathcal{U}$ is *nonnegative* (*strongly positive*) when $u_i\geqslant 0$ ($u_i>0$) and $\mathcal{U}$ is (*strictly*) *superharmonic* in case $-\Delta u_i>0$ for all $i\in I$.

First, we have the second order equation

$$-\Delta\mathcal{U}=c_1(n,a)|\mathcal{U}|^{a-1}\mathcal{U}\quad\text{in}\quad B_R^*,\tag{$\mathcal{S}^2_{p,a,R}$}$$

where $n\geqslant 3$, Δ is the Laplacian acting on p-maps, and $|\mathcal{U}|^2=\sum_{i=1}^p u_i^2$ is the Euclidean norm. This is strongly coupled by the *Gross–Pitaevskii nonlinearity* $F^a(\mathcal{U})=c_1(n,s)|\mathcal{U}|^{a-1}\mathcal{U}$, where $a\in(1,2^*-1]$ with $2^*=2n/(n-2)$ the (*upper*) *critical Sobolev exponent* of $H^1(\mathbb{R}^n)$. The constant on the right-hand side of ($\mathcal{S}^2_{p,a,R}$) is such that $c_1(n,a)=1$, if $a\in(1,2^*-1)$ is subcritical, and $c_1(n,a)=c_1(n)$, where $c_1(n)=n(n-2)/4$ is a normalizing constant, if $a=2^*-1$ is critical.

Second, we consider the fourth order equation,

$$\Delta^2\mathcal{U}=c_2(n,b)|\mathcal{U}|^{b-1}\mathcal{U}\quad\text{in}\quad B_R^*,\tag{$\mathcal{S}^4_{p,b,R}$}$$

where $n\geqslant 5$, Δ^2 is the bi-Laplacian acting on p-maps, and the nonlinearity is given by $F^b(\mathcal{U})=c_2(n,b)|\mathcal{U}|^{b-1}\mathcal{U}$, where $b\in(1,2^{**}-1]$ with $2^{**}=2n/(n-4)$ the (*upper*) *critical Sobolev exponent* of of $H^2(\mathbb{R}^n)$. The constant on the right-hand side of ($\mathcal{S}^4_{p,b,R}$) is such that $c_2(n,b)=1$, when $b\in(1,2^{**}-1)$ is subcritical, and $c_2(n,b)=c_2(n)$, where $c_2(n)=[n(n-4)(n^2-4)]/16$ is a geometric normalizing constant, when $b=2^{**}-1$ is critical.

2 Second order case

To make the exposition more comprehensible, we split our approach into two cases: blow-up limit ($R=\infty$) and local ($R<\infty$). Also, with respect to the growth, we consider four cases: subcriticals $a\in(1,2_*)$, $a=2_*$, $a\in(2_*,2^*-1)$, and critical $a=2^*-1$, where $2_*=n/(n-2)$ is the *lower Sobolev exponent* (or *Serrin exponent*). More precisely, the lower critical exponent is the greatest one for which all nonnegative singular solutions to the blow-up limit problem are trivial.

2.1 Scalar case

When $p=1$, ($\mathcal{S}^2_{p,a,R}$) becomes the following second order elliptic equation

$$-\Delta u=c_1(n,a)u^a \quad \text{in} \quad B^*_R, \tag{$\mathcal{S}^2_{1,a,R}$}$$

On this subject, let us mention some pioneering classification and asymptotics results due to J. Serrin [42, Theorem 11], P.-L. Lions [34, Theorem 2], P. Aviles [5, Theorem A], B. Gidas and J. Spruck [24, Theorems 1.1 and 1.2], and L. A. Caffarelli et al. [6, Theorems 1.1–1.3] with an improvement given by N. Korevaar, R. Mazzeo, F. Pacard, and R. Schoen [29, Theorem 1], which can be compiled in the statement below

Theorem 2.1. *Let u be a nonnegative solution to ($\mathcal{S}^2_{1,a,R}$). Assume that* Case (I): (*punctured space*) $R=\infty$, *and*

(i) *the origin is a removable singularity.*

  (a) *If $a\in(1,2^*-1)$, then $u\equiv0$;*

  (b) *If $a=2^*-1$, then there exist $x_0\in\mathbb{R}^n$ and $\mu>0$ such that u is radially symmetric about x_0 and, up to a constant, is given by*

$$u_{x_0,\mu}(x)=\left(\frac{2\mu}{1+\mu^2|x-x_0|^2}\right)^{\frac{n-2}{2}}; \tag{2.1}$$

these solutions are called the (second order) spherical solutions.

(ii) *the origin is a non-removable singularity.*

(a) *If* $a \in (1, 2_*]$*, then* $u \equiv 0$*;*

(b) *If* $a \in (2_*, 2^* - 1)$ *and* u *is homogeneous of degree* $-\frac{2}{a-1}$*, then*

$$u_a(x) = \left[\frac{2(n-2)(a-2_*)}{(a-1)^2}\right]^{\frac{1}{a-1}} |x|^{-\frac{2}{a-1}}; \tag{2.2}$$

(c) *If* $a = 2^* - 1$ *and* u *is radially symmetric about the origin. Moreover, there exist* $\lambda \in (0, [(n-2)/n]^{(n-2)/4}]$ *and* $T \in (0, T_\lambda]$ *such that*

$$u_{\lambda,T}(x) = |x|^{\frac{2-n}{2}} v_\lambda(-\ln|x| + T), \tag{2.3}$$

Here $v_{\lambda,T}$ *is the unique* T*-periodic bounded solution to the following second order problem*

$$\begin{cases} v^{(2)} - \frac{(n-2)^2}{4} v + \frac{n(n-2)}{4} v^{2^*-1} = 0 \\ v(0) = \lambda, \ v^{(1)}(0) = 0, \end{cases}$$

where $T_\lambda \in \mathbb{R}$ *is the fundamental period of* v_λ*. We call both* $u_{\lambda,T}$ *and* $v_{\lambda,T}$ *(second order) Emden–Fowler (or Delaunay-type) solutions*

Case (II): *(punctured ball)* $R < \infty$*, and the origin is a non-removable singularity, it follows that* $u(x) = (1 + \mathcal{O}(|x|))\overline{u}(|x|)$ *as* $x \to 0$*, where* $\overline{u}$ *is the spherical average of* u*. Moreover,*

(a) *(Serrin–Lions case) if* $a \in (1, 2_* - 1]$*, then* $u(x) \simeq |x|^{2-n}$ *as* $x \to 0$*;*

(b) *(Aviles case) if* $a = 2_* - 1$*, then*

$$u(x) = (1 + \mathrm{o}(1)) \left(\frac{n-2}{\sqrt{2}}\right)^{n-2} |x|^{2-n}(-\ln|x|)^{\frac{2-n}{2}} \quad \text{as} \quad x \to 0;$$

(c) *(Gidas–Spruck case) if* $s \in (2_*, 2^* - 1)$*, then*

$$u(x) = (1 + \mathrm{o}(1)) \left[\frac{2(n-2)(a-2_*)}{(a-1)^2}\right]^{\frac{1}{a-1}} |x|^{-\frac{2}{a-1}} \quad \text{as} \quad x \to 0;$$

(d) (*Caffarelli–Gidas–Spruck case*) *if* $a = 2^* - 1$, *then there exists a second order Emden–Fowler solution* $u_{\lambda,T}$ *as in* (2.3) *such that*

$$u(x) = (1 + \mathrm{o}(1))u_{\lambda,T}(|x|) \quad \text{as} \quad x \to 0;$$

(e) (*Korevaar–Mazzeo–Pacard–Schoen case*) *Furthermore, one can find and* $\beta_0 > 0$ *such that*

$$u(x) = (1 + \mathcal{O}(|x|^{\beta_0}))u_{\lambda,T}(|x|) \quad \text{as} \quad x \to 0. \tag{2.4}$$

2.2 Vectorial case

Now we will be based on the information obtained previously to discuss the case of systems, that is, $p > 1$. Here it is convenient to define $\mathbb{S}^{p-1}_+ = \{x \in \mathbb{S}^{p-1} : x_i \geqslant 0\}$. In this case, O. Druet, E. Hebey, and J. Vétois [16, Proposition 1.1], M. Ghergu, S. Kim and H. Shahgholian [23, Theorems 1.1–1.5] and R. Caju, J. M. do Ó and A. Santos [7, Theorem 1.2] provided the following results

Theorem 2.2. *Let* $\mathcal{U}$ *be a nonnegative solution to* ($\mathcal{S}^2_{p,a,R}$). *Assume that* Case (I): (*punctured space*) $R = \infty$, *and*

(i) *the origin is a removable singularity.*

 (a) *If* $a \in (1, 2^* - 1)$, *then* $\mathcal{U} \equiv 0$;

 (b) *If* $a = 2^* - 1$, *then there exist* $\Lambda \in \mathbb{S}^{p-1}_+$, $x_0 \in \mathbb{R}^n$ *and* u_{μ,x_0} *given by* (2.1) *such that* $\mathcal{U} = \Lambda u_{\mu,x_0}$.

(ii) *the origin is a non-removable singularity.*

 (a) *If* $a \in (1, 2_*]$, *then* $\mathcal{U} \equiv 0$;

 (b) *If* $a \in (2_*, 2^* - 1)$ *and* $\mathcal{U}$ *is homogeneous of degree* $-\frac{2}{a-1}$, *then there exists* $\Lambda \in \mathbb{S}^{p-1}_+$ *and* u_a *given by* (2.2) *such that* $\mathcal{U} = \Lambda u_a$;

 (c) *If* $a = 2^* - 1$, *then there exists* $u_{\lambda,T}$ *given by* (2.3) *such that* $\mathcal{U} = \Lambda u_{\lambda,T}$.

Case (II): (*punctured ball*) $R < \infty$, *and the origin is a non-removable singularity, then* $|\mathcal{U}(x)| = (1 + \mathcal{O}(|x|))|\overline{\mathcal{U}}(x)|$ *as* $x \to 0$, *where* $|\overline{\mathcal{U}}|$ *is the spherical average of* $|\mathcal{U}|$. *Moreover,*

(a) *if* $a \in (1, 2_*]$, *then* $|\mathcal{U}(x)| \simeq |x|^{2-n}$ *as* $x \to 0$;

(b) *if* $a = 2_*$, *then*

$$|\mathcal{U}(x)| = (1 + \mathrm{o}(1)) \left(\frac{n-2}{\sqrt{2}}\right)^{n-2} |x|^{2-n}(-\ln|x|)^{\frac{2-n}{2}} \quad \text{as} \quad x \to 0;$$

(c) *if* $a \in (2_*, 2^* - 1)$, *then*

$$|\mathcal{U}(x)| = (1 + \mathrm{o}(1)) \left[\frac{2(n-2)(a-2_*)}{(a-1)^2}\right]^{\frac{1}{a-1}} |x|^{-\frac{2}{a-1}} \quad \text{as} \quad x \to 0;$$

(d) *if* $a = 2^* - 1$, *then there exists a second order Emden–Fowler* $u_{\lambda,T}$ *as in* (2.3) *such that*

$$|\mathcal{U}(x)| = (1 + \mathrm{o}(1))u_{\lambda,T}(|x|) \quad \text{as} \quad x \to 0.$$

Furthermore, one can find $\beta_0 > 0$ *such that*

$$|\mathcal{U}(x)| = (1 + \mathcal{O}(|x|^{\beta_0}))u_{\lambda,T}(|x|) \quad \text{as} \quad x \to 0.$$

3 Fourth order case

As before, we split our approach into four cases: subcriticals $b \in (1, 2_{**})$, $b = 2_{**}$, $b \in (2_{**}, 2^{**} - 1)$, and critical $b = 2^{**} - 1$, where $2_{**} = n/(n-4)$ is the *lower Sobolev exponent* (or *Serrin exponent*). This is a fourth order analog of the one found by J. Serrin [42] in the context of second order problems.

3.1 Scalar case

Let us start our analysis with $p = 1$. In this situation, we have the following fourth order equation

$$\Delta^2 u = c_2(n, b)u^b \quad \text{in} \quad B_R^*. \qquad (\mathcal{S}_{1,b,R}^4)$$

The next result presents a holistic picture of the classification and local behavior for solutions to this equation. Namely, we summarize some recent contributions due to C. S. Lin [33, Theorem 1.3], Z. Guo, J. Wei and F. Zhou [26, Theorem 1.2], R. Frank and T. König [22, Theorem 2], R. Soranzo [43, Theorems 3 and 5], H. Yang [45, Theorem 1.1], T. Jin and J. Xiong [28, Theorem 1.1], J. Ratzkin [38, Theorem 1] and the present authors in [2, Theorem 1] and [4, Theorem 2].

Theorem 3.1. *Let u be a nonnegative solution to* ($\mathcal{S}^4_{1,b,R}$). *Assume that* Case (I): (*punctured space*) $R=\infty$, *and*

(i) *the origin is a removable singularity.*

  (a) *If* $b\in(1,2^{**}-1)$, *then* $u\equiv 0$;

  (b) *If* $b=2^{**}-1$, *then there exist* $x_0\in\mathbb{R}^n$ *and* $\mu>0$ *such that u is radially symmetric about x_0 and, up to a constant, is given by*

$$u_{x_0,\mu}(x)=\left(\frac{2\mu}{1+\mu^2|x-x_0|^2}\right)^{\frac{n-4}{2}}. \tag{3.1}$$

  These are called the (*fourth order*) *spherical solutions* (*or bubbles*).

(ii) *the origin is a non-removable singularity.*

  (a) *If* $b\in(1,2_{**}]$, *then* $u\equiv 0$;

  (b) *If* $b\in(2_{**},2^{**}-1)$ *and u is homogeneous of degree* $-\frac{4}{b-1}$, *then*

$$u_b(x)=K_0(n,b)^{\frac{1}{b-1}}|x|^{-\frac{4}{b-1}}, \tag{3.2}$$

  where

$$K_0(n,b)=8(b-1)^{-4}\left[(n-2)(n-4)(b-1)^3+\right.$$
$$\left.2\left(n^2-10n+20\right)(b-1)^2-16(n-4)(b-1)+32\right],$$

  (c) *If* $b=2^{**}-1$, *then u is radially symmetric about the origin. Moreover, there exist* $\lambda\in(0,[n(n-4)/(n^2-4)]^{n-4/8}]$ *and* $T\in(0,T_\lambda]$ *such that*

$$u_{\lambda,T}(x)=|x|^{\frac{4-n}{2}}v_\lambda(\ln|x|+T). \tag{3.3}$$

Here $T_\lambda \in \mathbb{R}$ is the fundamental period of the unique T-periodic bounded solution v_λ to the following fourth order Cauchy problem,

$$\begin{cases} v^{(4)} + K_2^*(n)v^{(2)} + K_0^*(n)v = c_4(n)v^{2^{**}-1} \\ v(0) = \lambda, \ v^{(1)}(0) = 0, \ v^{(2)}(0) = \gamma, \ v^{(3)}(0) = 0, \end{cases}$$

where

$$K_0^*(n) = \frac{n^2(n-4)^2}{16} \quad \text{and} \quad K_2^*(n) = -\frac{n^2-4n+8}{2}.$$

We call both $u_{\lambda,T}$ and $v_{\lambda,T}$ (fourth order) Emden–Fowler (or Delaunay-type) solutions.

Case (II): (*punctured ball*) *$R < \infty$, and the origin is a non-removable singularity. Suppose that u is superharmonic. Then, $u(x) = (1+\mathcal{O}(|x|))\overline{u}(|x|)$ as $x \to 0$. Moreover,*

(*a*) *if $b \in (1, 2_{**})$, then $u(x) \simeq |x|^{4-n}$ as $x \to 0$;*

(*b*) *if $b = 2_{**}$, then*

$$|\mathcal{U}(x)| = (1+\mathrm{o}(1))\widehat{K}_0(n)^{\frac{n-4}{4}}|x|^{4-n}(\ln|x|)^{\frac{4-n}{4}} \quad \text{as} \quad x \to 0,$$

where $\widehat{K}_0(n) = \frac{(n-2)(n-4)^2}{2}$;

(*c*) *if $b \in (2_{**}, 2^{**}-1)$, then*

$$u(x) = (1+\mathrm{o}(1))K_0(n,b)^{\frac{1}{b-1}}|x|^{-\frac{4}{b-1}} \quad \text{as} \quad x \to 0;$$

(*d*) *if $b = 2^{**}-1$, then there exists $u_{\lambda,T}$ as in (3.3) such that*

$$u(x) = (1+\mathrm{o}(1))u_{\lambda,T}(|x|) \quad \text{as} \quad x \to 0.$$

Furthermore, one can find $\beta_0 > 0$ such that

$$u(x) = (1+\mathcal{O}(|x|^{\beta_0}))u_{\lambda,T}(|x|) \quad \text{as} \quad x \to 0.$$

3.2 Vectorial case

Finally, let us study the vectorial equation ($\mathcal{S}^4_{p,b,R}$). In this fashion, the theorem below is the compilation of the results of the present authors, which can be found in [3, Theorems 1 and 2], [2, Theorem 1] and [4, Thereoms 1 and 2].

Theorem 3.2. *Let $\mathcal{U}$ be a nonnegative solution to ($\mathcal{S}^4_{p,b,R}$). Assume that* Case (I): (*punctured space*) $R = \infty$, *and*

(i) *the origin is a removable singularity.*

  (a) *If $b \in (1, 2^{**} - 1)$, then $\mathcal{U} \equiv 0$;*

  (b) *If $b = 2^{**} - 1$, then there exist $\Lambda \in \mathbb{S}^{p-1}_+$ and $u_{x_0,\mu}$ given by (3.1) such that $\mathcal{U} = \Lambda u_{x_0,\mu}$.*

(ii) *the origin is a non-removable singularity.*

  (a) *If $b \in (1, 2_{**}]$, then $\mathcal{U} \equiv 0$;*

  (b) *If $b \in (2_{**}, 2^{**} - 1)$ and $\mathcal{U}$ is homogeneous of degree $-\frac{4}{b-1}$, then there exists $\Lambda \in \mathbb{S}^{p-1}_+$ and u_b given by (3.2) such that $\mathcal{U} = \Lambda u_b$;*

  (c) *If $b = 2^{**} - 1$, then there exist $\Lambda \in \mathbb{S}^{p-1}_+$ and $u_{\lambda,T}$ given by (3.3) such that $\mathcal{U} = \Lambda u_{\lambda,T}$.*

Case (II): (*punctured ball*) *$R < \infty$, and the origin is a non-removable singularity. Suppose that $\mathcal{U}$ is superharmonic. Then, it follows that $|\mathcal{U}(x)| = (1 + \mathcal{O}(|x|))|\overline{\mathcal{U}}(x)|$ as $x \to 0$. Moreover,*

(a) *if $b \in (1, 2_{**})$, then*

$$|\mathcal{U}(x)| \simeq |x|^{4-n} \quad \text{as} \quad x \to 0;$$

(b) *if $b = 2_{**}$, then*

$$|\mathcal{U}(x)| = (1 + \mathrm{o}(1))\widehat{K}_0(n)^{\frac{n-4}{4}}|x|^{4-n}(\ln|x|)^{\frac{4-n}{4}} \quad \text{as} \quad x \to 0;$$

(c) *if* $b \in (2_{**}, 2^{**}-1)$*, then*

$$|\mathcal{U}(x)| = (1+\mathrm{o}(1))K_0(n,b)^{\frac{1}{b-1}}|x|^{-\frac{4}{b-1}} \quad \text{as} \quad x \to 0;$$

(d) *if* $b = 2^{**}-1$*, then there exists* $u_{\lambda,T}$ *as in* (3.3) *such that*

$$|\mathcal{U}(x)| = (1+\mathrm{o}(1))u_{\lambda,T}(|x|) \quad \text{as} \quad x \to 0.$$

Furthermore, one can find $\beta_0 > 0$ *such that*

$$|\mathcal{U}(x)| = (1+\mathcal{O}(|x|^{\beta_0}))u_{\lambda,T}(|x|) \quad \text{as} \quad x \to 0.$$

4 Some applications

Applications for higher order strongly coupled elliptic systems are ubiquitous in several mathematical physics branches. For instance, the Gross–Pitaevskii coupling is one of the first approximations to consider in the Hartree–Fock theory to model Bose–Einstein double condensates, where the component solutions represent the state (or wave) functions of a many-body quantum-mechanical system [1, 25, 19]. They can also describe the behavior of deep-water and Rogue waves in the ocean [17, 35].

For the scalar case $p = 1$, the study of equations like ($\mathcal{S}^2_{1,a,R}$) dates back to the classical papers [18, 21, 31] regarding the Lane–Emden–Fowler equation, which models the distribution of mass density for spherical polytropic stars in hydrostatic equilibrium (see [8] for more details). In addition, fourth order equations like ($\mathcal{S}^4_{1,b,R}$), the Swift–Hohenberg equation, and the Extended–Fisher–Kolmogorov equation are frequently used on the theory of pattern formation [12, 36, 37]. Surprisingly, these models were recently applied to construct the new counterexamples for Kelvin's three-dimensional honeycomb conjecture [27, 39, 44].

Apart from the aforementioned applications, in the critical growth cases $a = 2^*-1$ and $b = 2^{**}-1$, Systems ($\mathcal{S}^2_{p,a,R}$) and ($\mathcal{S}^4_{p,b,R}$) are the most natural generalization of the singular Yamabe and Q-curvature equations on its conformally flat case; these are famous problems in conformal geometry [40, 41, 30].

5 Further problems

In this section, we discuss some still undeveloped extensions for the results presented in this paper. Here we focus on changing the order of the equation and the nonlinear coupling term. In this context, most results are not proved on their full generality even in the second order case.

It would be reasonable to extend the classification and asymptotics results to the study of nonnegative p-map solutions $\mathcal{U} : B_R^* \to \mathbb{R}^p$ to the following class of higher order p-systems,

$$(-\Delta)^k \mathcal{U} = c_k(q,n)|\mathcal{U}|^{q-1}\mathcal{U} \quad \text{in} \quad B_R^*, \qquad (\mathcal{S}_{p,q,R}^k)$$

Here $q \in (1, 2_k^* - 1]$, where $2_k^* := 2n/(n-2k)$ is the (upper critical Sobolev exponent) of $H^k(\mathbb{R}^n)$ with $n > 2k$ and $k \geqslant 1$, $(-\Delta)^k$ is the poly-Laplacian, and $c_k(n,q)$ is a normalizing constant. Notice that, by using the Green representation theorem for the poly-Laplacian, it follows that ($\mathcal{S}_{p,q,R}^k$) can also be realized as an integral system. For some references on this, see [9, 10, 14, 15, 20, 28] and the references therein.

It also be interesting to consider other type of couplings. Namely, to understand nonnegative solutions $\mathcal{U}$ to the strongly coupled higher order 2-system,

$$\begin{cases} (-\Delta)^k u_1 = \mu_1 u_1^{q-1} + \beta u_1^{q/2-1} u_2^{q/2} \\ (-\Delta)^k u_2 = \mu_2 u_2^{q-1} + \beta u_2^{q/2-1} u_1^{q/2} \end{cases} \quad \text{in} \quad B_R^*,$$

where $k \geqslant 1$, $n > 2k$, $\mu_1, \mu_2, \beta > 0$, and $q \in (1, 2_k^* - 1]$. This type of coupling system on the right-hand side of this system seems to be harder to tackle than the one in ($\mathcal{S}_{p,a,R}^2$) and ($\mathcal{S}_{p,b,R}^4$) since the behavior of the coupling term depends strongly on the dimension. In fact, even for the second order case not much is known on the asymptotic behavior near the isolated singularity. For more details, see [11, 13, 32, 46] and the references therein.

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