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Existence and concentration of ground state solutions for a class of subcritical, critical or supercritical problems with steep potential well

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Dedicated to Professor Pedro Ubilla
on the occasion of his 60th birthday

Abstract. In this paper we study the quasilinear problem

$$\begin{cases} -div\left(a\left(|\nabla u|^{p}\right)|\nabla u|^{p-2}\nabla u\right)+[1+\mu V(z)]b\left(|u|^{p}\right)|u|^{p-2}u=f(u)+\varrho|u|^{\sigma-2}u, \\ u\in W^{1,p}(\mathbb{R}^N)\cap W^{1,q}(\mathbb{R}^N). \end{cases}$$

The term $1+\mu V(z)$ is the steep potential well introduced by Bartsch and Wang in [11]. With suitable hypotheses on the functions a, b and f, we show the existence of solutions and concentration behavior occurred as $\mu\to+\infty$, considering the subcritical case, the critical case and the supercritical case.

Keywords: $p\&q$ Laplacian operator, Steep potential well, subcritical or critical ou supercritical growth, Variational method, Shrödinger equation, Ground state solution.

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1 Introduction

In the last years many researchers have dedicated to study the existence of solitary waves, namely solutions of the form $\Psi(x,t) = \exp(-i\frac{E}{h}t)u(x)$, with $E \in \mathbb{R}$, for the nonlinear Schrödinger equation

$$ih\frac{\partial \Psi}{\partial t} = -h^2\Delta\Psi + \overline{V}(x)\Psi - f(\Psi), \quad \text{in } \Omega, \tag{1.1}$$

where $h > 0$ and Ω is a domain in $\mathbb{R}^N$. The equation (1.1) is related to physics problems, as nonlinear optics, plasma physics, condensed matter physics and quantum mechanics. See for example [3], for more details. A direct computation shows that Ψ is a solitary wave for (1.1) if, and only if, u is a solution of the following problem

$$-h^2\Delta u + \overline{V}(x)u = f(u), \quad \text{in } \Omega. \tag{1.2}$$

Finding a solution to problem (1.2) when Ω is an unbounded domain becomes more difficult due to the lack of compact embedding from $H^1(\Omega)$ into $L^p(\Omega)$. In general, the geometry of the potential $\overline{V}$ helps to overcome this difficulty. Interesting conditions on $\overline{V}$ to overcome the lack of compactness can be seen in [3], [4], [14], [16], [32], [33],

Bartsch and Wang [11] considered problem (1.2) with $h = 1$, $\Omega = \mathbb{R}^N$ and with steep potential well, that is, when $\overline{V}(x) = 1 + \mu V(x)$, for all $x \in \mathbb{R}^N$, $\mu > 0$ and V satisfying hypotheses (V_1), (V_2) and (V_3) that we are using in this paper and that we will put in this section. They proved that (1.2) has a positive ground state solution u_μ for μ large, and as $\mu \to +\infty$, u_μ converges strongly in $H^1(\mathbb{R}^N)$ to the ground state solution of the limiting equation

$$-h^2\Delta u + u = f(u), \quad \text{in } \Omega_0,$$

where $\Omega_0 = V^{-1}(0)$. In particular, in [2] the authors have studied the case exponential critical and in [36] the authors have studied the case polinomial critical of [11]. The existence of sign-changing solutions for (1.2) and with steep potential well was studied in [31]. In the literature, we find a lot of papers where the authors have considered elliptic problems with steep potential as [6], [7], [8], [24].

In this article, we are interested in a class of problems that includes the equation (1.2) with steep potential and subcritical, critical or supercritical growth on the nonlinearity. More precisely, we are going to study the following class of quasilinear problems $(P_{\mu,\varrho,\sigma})$ given by

$$\begin{cases} -div\left(a\left(|\nabla u|^p\right)|\nabla u|^{p-2}\nabla u\right)+[1+\mu V(z)]b\left(|u|^p\right)|u|^{p-2}u=f(u)+\varrho|u|^{\sigma-2}u, \\ \\ u\in W^{1,p}(\mathbb{R}^N)\cap W^{1,q}(\mathbb{R}^N), \end{cases}$$

where $1<p\le q<N$, $N\ge 2$, $\mu>0$. We are considering three cases. The first case is the subcritical growth on the nonlinearity, i.e. when $\varrho=0$. In this case we have

$$(P_{\mu,0})\begin{cases} -div\left(a\left(|\nabla u|^p\right)|\nabla u|^{p-2}\nabla u\right)+[1+\mu V(z)]b\left(|u|^p\right)|u|^{p-2}u=f(u), \\ \\ u\in W^{1,p}(\mathbb{R}^N)\cap W^{1,q}(\mathbb{R}^N). \end{cases}$$

The second case is the critical growth on the nonlinearity, i.e. when $\varrho=1$ and $\sigma=q^*:=\frac{Nq}{N-q}$. In this case we have $(P_{\mu,1,q^*})$, that is,

$$\begin{cases} -div\left(a\left(|\nabla u|^p\right)|\nabla u|^{p-2}\nabla u\right)+[1+\mu V(z)]b\left(|u|^p\right)|u|^{p-2}u=f(u)+|u|^{q^*-2}u, \\ \\ u\in W^{1,p}(\mathbb{R}^N)\cap W^{1,q}(\mathbb{R}^N). \end{cases}$$

The last case is the supercritical growth on the nonlinearity, i.e. when $\varrho=1$ and $\sigma>q^*:=\frac{Nq}{N-q}$. In this case we have $(P_{\mu,1,\sigma})$, that is,

$$\begin{cases} -div\left(a\left(|\nabla u|^p\right)|\nabla u|^{p-2}\nabla u\right)+[1+\mu V(z)]b\left(|u|^p\right)|u|^{p-2}u=f(u)+|u|^{\sigma-2}u, \\ \\ u\in W^{1,p}(\mathbb{R}^N)\cap W^{1,q}(\mathbb{R}^N). \end{cases}$$

In order to state the main result, we need to introduce the hypotheses on the functions a and b.

(a_1) the function a is of class C^1 and there exist constants, $k_1, k_2 \geq 0$ such that

$$k_1 t^p + t^q \leq a(t^p)t^p \leq k_2 t^p + t^q, \quad \text{for all} \quad t > 0;$$

(a_2) the mapping $t \mapsto \dfrac{a(t^p)}{t^{q-p}}$ is nonincreasing for $t > 0$.

(a_3) if $1 < p \leq q \leq 2 \leq N$ the mapping $t \mapsto a(t)$ is nondecreasing for $t > 0$. If $2 \leq p \leq q < N$ the mapping $t \mapsto a(t)t^{p-2}$ is nondecreasing for $t > 0$

As a direct consequence of (a_3) we obtain that the map a and its derivative a' satisfy

$$a'(t)t \leq \frac{(q-p)}{p} a(t) \text{ for all } t > 0. \tag{1.3}$$

Now, if we define the function $h(t) = a(t)t - \frac{q}{p}A(t)$, using (1.3) we can prove that the function h is nonincreasing, where $A(t) = \displaystyle\int_0^t a(s)ds$. Then, there exists a positive real constant $\gamma \geq \frac{q}{p}$ such that

$$\frac{1}{\gamma} a(t)t \leq A(t), \quad \text{for all } t \geq 0. \tag{1.4}$$

(b_1) The function b is of class C^1 and there exist constants $k_3, k_4 \geq 0$ such that

$$k_3 t^p + t^q \leq b(t^p)t^p \leq k_4 t^p + t^q, \quad \text{for all} \quad t > 0;$$

(b_2) the mapping $t \mapsto \dfrac{b(t^p)}{t^{q-p}}$ is nonincreasing for $t > 0$.

(b_3) if $1 < p \leq q \leq 2 \leq N$ the mapping $t \mapsto b(t)$ is nondecreasing for $t > 0$. If $2 \leq p \leq q < N$ the mapping $t \mapsto b(t)t^{p-2}$ is nondecreasing for $t > 0$

Using the hypothesis (b_3) and arguing as (1.3) and (1.4), we also can prove that there exists $\gamma \geq \frac{q}{p}$ such that

$$\frac{1}{\gamma} b(t)t \leq B(t) \quad \text{for all } t \geq 0, \tag{1.5}$$

where $B(t) = \displaystyle\int_0^t b(s)ds$. The condition in $V \in C(\mathbb{R}^N, \mathbb{R})$ are the following:

(V_1) The potential V is nonnegative, that is,

$$V(x) \geq 0, \text{ for all } x \in \mathbb{R}^N.$$

(V_2) The set $\Omega := \operatorname{int}\left\{ x \in \mathbb{R}^N \mid V(x) = 0 \right\}$ is a non-empty bounded open set with smooth boundary $\partial\Omega$.

(V_3) There exists $V^* > 0$, such that

$$meas\left(\left\{x \in \mathbb{R}^N : V(x) \leq V^*\right\}\right) < \infty.$$

Before we give the main result, we need to put some hypotheses on the nonlinearity $f \in C(\mathbb{R})$.

(f_1)

$$\lim_{|s|\to 0} \frac{f(s)}{|s|^{q-1}} = 0 \ \text{ and } \ f(s) = 0, \quad \text{for all} \quad s \leq 0.$$

(f_2) There exists $q < r < q^* = \frac{Nq}{N-q}$ such that

$$\lim_{|s|\to\infty} \frac{f(s)}{|s|^{r-1}} = 0.$$

(f_3) There exists $\theta \in (\gamma p, q^*)$, such that

$$0 < \theta F(s) \leq f(s)s, \quad \text{for} \quad s \neq 0,$$

where $F(s) = \displaystyle\int_0^s f(t)dt$ and $\gamma > 0$ was given in (1.4).

(f_4) $s \mapsto \dfrac{f(s)}{s^{q-1}}$ is nondecreasing.

(f_5) There exist $\tau \in (q, q^*)$ and $\lambda^* > 1$ such that

$$f(s) \geq \lambda |s|^{\tau-1}, \quad \forall\, s \in \mathbb{R},$$

for a fixed $\lambda > \lambda^*$ and λ^* will be fixed latter.

The main result is:

Theorem 1.1. *Assume that $(a_1)-(a_3)$, $(b_1)-(b_3)$, $(f_1)-(f_4)$ and $(V_1)-(V_3)$ are satisfied. Then,*

(i) there exists $\mu^* > 0$ such that problem $(P_{\mu,0})$(subcritical case) has a ground state solution $u_\mu \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ for all $\mu > \mu^*$.

(ii) if the function f satisfies (f_5) there exist positive numbers λ^* and μ^{**}, such that problem $(P_{\mu,1,\sigma})$(critical or supercritical case) has a ground state solution $u_\mu \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ for all $\mu > \mu^{**}$ and for all $\lambda > \lambda^*$.

Moreover, as $\mu \to +\infty$, the sequence (u_μ) converges in $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ to a ground state solution $u_\infty \in W_0^{1,q}(\Omega)$ of the problem

$$\begin{cases} -div\left(a\left(|\nabla u|^p\right)|\nabla u|^{p-2}\nabla u\right) + b\left(|u|^p\right)|u|^{p-2}u = f(u) + \varrho|u|^{\sigma-2}u \quad in\ \Omega, \\ \\ u = 0 \quad on \quad \partial\Omega. \end{cases}$$

Our arguments were strongly influenced by [2], [9], [10], [11], [17], [18], [25], [31] and [36]. Below we list what we believe to be the main contributions of our paper.

(i) In this paper we consider a large class of quasilinear operator which includes but is not restricted to Laplacian or p-Laplacian operator. In general, this operator is not linear and nonhomogeneous. See below for several examples of operators we can consider.

(ii) Since we work with a general operator, some estimates are more refined. See for example Lemma 2.2, Lemma 4.1, Proposition 5.1 and Lemma 8.3.

(iii) Unlike the works [2], [11], [27], [31] and [36], we are also considering the supercritical case.

We are going to present some examples of functions a and b which are also interesting from the mathematical point of view and have a wide range of applications in physics and related sciences.

Problem 1: Let $a(t) = t^{\frac{q-p}{p}}$ and $b(t) = t^{\frac{q-p}{p}}$. Then conditions $(a_1) - (a_3)$ and $(b_1) - (b_3)$ are satisfied and problem (P_μ) is

$$-\Delta_q u + [1 + \mu V(x)]|u|^{q-2}u = f(u) + \varrho u^{\sigma-1} \text{ in } \quad \mathbb{R}^N.$$

Problem 2: Let $a(t) = 1 + t^{\frac{q-p}{p}}$ and $b(t) = 1 + t^{\frac{q-p}{p}}$. Then a satisfies $(a_1) - (a_3)$, b satisfies $(b_1) - (b_3)$ and problem (P_μ) is

$$-\Delta_p u - \Delta_q u + [1 + \mu V(x)][|u|^{p-2}u + |u|^{q-2}u] = f(u) + \varrho u^{\sigma-1} \quad \text{in } \mathbb{R}^N.$$

Problem 3: Let $a(t) = 1 + \frac{1}{(1+t)^{\frac{p-2}{p}}}$ and $b(t) = 1$. Note that, a satisfies $(a_1) - (a_3)$, b satisfies $(b_1) - (b_3)$ and problem (P_μ) is

$$-\operatorname{div}\left[|\nabla u|^{p-2}\nabla u + \frac{|\nabla u|^{p-2}\nabla u}{(1 + |\nabla u|^p)^{\frac{p-2}{p}}}\right] + [1+\mu V(x)]|u|^{p-2}u = f(u) + \varrho u^{\sigma-1} \text{ in } \mathbb{R}^N.$$

Problem 4: Let $a(t) = 1 + t^{\frac{q-p}{p}} + \frac{1}{(1+t)^{\frac{p-2}{p}}}$ and $b(t) = 1 + t^{\frac{q-p}{p}}$. In this case, a satisfies $(a_1) - (a_3)$, b satisfies $(b_1) - (b_3)$ and problem (P_μ) is

$$-\Delta_p u - \Delta_q u - \operatorname{div}\left[\frac{|\nabla u|^{p-2}\nabla u}{(1 + |\nabla u|^p)^{\frac{p-2}{p}}}\right] + [1+\mu V(x)][|u|^{p-2}u + |u|^{q-2}u] = f(u) + \varrho u^{\sigma-1},$$

in $\mathbb{R}^N$.

Clearly, other examples of a and b satisfying $(a_1)-(a_3)$ and $(b_1)-(b_3)$ can be provided thus generating very interesting elliptic problems from mathematical point of view and in term of applications, such as biophysics, plasma physics and chemical reaction, as it can be seen for example in [22], [23] and [35].

The interest in the study of nonlinear partial differential equations with $p\&q$ operator has increased because many applications arising in mathematical physics may be stated with an operator in this form. We cite the papers [20], [26], [28], [29], [30] and their references. Several techniques have been developed or applied in their study, such as variational methods, fixed point theory, lower and upper solutions, global branching, and the theory of multivalued mappings.

The paper is organized as follows. In section 2 we study the variational framework considering subcritical and critical problem. In section 3 we prove the existence of solution of subcritical problem. The existence of solution of critical problem is showed in section 4. In section 5 we show the concentration result considering the subcritical, critical and supercritical cases. The proof of the part of subcritical in Theorem 1.1 is proved in section 6 and the the part of critical proof of Theorem 1.1 is done is section 7. In section 8 we study the supercritical problem and define an auxiliary problem. We prove de existence and concentration of solution of supercritical problem in section 9.

2 Variational framework and some preliminary results for the subcritical case ($\varrho = 0$) and for the critical case ($\varrho = 1$ and $\sigma = q^*$)

In this chapter we are considering the cases $\varrho = 0$ or $\varrho = 1$ with $\sigma = q^*$. More specifically, we have (P_{μ,ϱ,q^*}) given by

$$\begin{cases} -div\left(a\left(|\nabla u|^p\right)|\nabla u|^{p-2}\nabla u\right) + [1+\mu V(z)]b\left(|u|^p\right)|u|^{p-2}u = f(u) + \varrho|u|^{q^*-2}u, \\ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N). \end{cases}$$

Since the approach is variational, let us consider the energy functional associated $I_{\mu,\varrho} : W \to \mathbb{R}$ given by

$$
\begin{aligned}
I_{\mu,\varrho}(u) : \quad = \quad & \frac{1}{p}\int_{\mathbb{R}^N} A\left(|\nabla u|^p\right) dx + \frac{1}{p}\int_{\mathbb{R}^N} [1+\mu V(x)] B\left(|u|^p\right) dx \\
& -\int_{\mathbb{R}^N} F(u) dx - \frac{\varrho}{q^*}\int_{\mathbb{R}^N} |u|^{q^*} dx,
\end{aligned}
$$

where

$$W := \left\{ u \in W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x) b(|u|^p)|u|^p dx < +\infty \right\}.$$

Note that W is a Banach space when endowed with the norm

$$\|u\|_{\mu} = \|u\|_{\mu,p} + \|u\|_{\mu,q},$$

where

$$\|u\|_{\mu,m} = \left(\int_{\mathbb{R}^N} |\nabla u|^m dx + \int_{\mathbb{R}^N} [1+\mu V(x)]|u|^m dx \right)^{\frac{1}{m}}, \text{ for } m \geq 1.$$

In $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ we fix the norm

$$\|u\| = \|u\|_p + \|u\|_q,$$

where

$$\|u\|_m = \left(\int_{\mathbb{R}^N} |\nabla u|^m dx + \int_{\mathbb{R}^N} |u|^m dx \right)^{\frac{1}{m}}, \text{ for } m \geq 1.$$

Note that W is continuous embedded into $L^r(\mathbb{R}^N)$, for $q < r < q^*$.

By standard arguments, it is possible to prove that $I_{\mu,\varrho} \in C^1(W, \mathbb{R})$. Note that (f_1) and (f_2) imply that for any given $\xi > 0$, there is a constant $C_\xi > 0$, such that

$$|f(s)| \leq \xi |s|^{q-1} + C_\xi |s|^{r-1}, \quad \forall \;\; s \in \mathbb{R}. \tag{2.1}$$

Moreover, by (f_3) there exist positive constants $D_1,\ D_2$ such that

$$F(s) \geq D_1|s|^\theta - D_2, \quad \forall \quad s \in \mathbb{R}. \tag{2.2}$$

To use the Mountain Pass Theorem [5], we define the Palais-Smale compactness condition. We say that a sequence $(u_n) \subset W$ is a Palais-Smale sequence at level $c_{\mu,\varrho}$ for the functional $I_{\mu,\varrho}$ if

$$I_{\mu,\varrho}(u_n) \to c_{\mu,\varrho}$$

and

$$\|I'_{\mu,\varrho}(u_n)\| \to 0 \text{ in } (W)',$$

where

$$c_{\mu,\varrho} = \inf_{\eta\in\Gamma} \max_{t\in[0,1]} I_{\mu,\varrho}(\eta(t)) > 0 \tag{2.3}$$

and

$$\Gamma := \{\eta \in C([0,1], W) : \eta(0) = 0,\ I_{\mu,\varrho}(\eta(1)) < 0\}.$$

If every Palais-Smale sequence of $I_{\mu,\varrho}$ has a strong convergent subsequence, then one says that $I_{\mu,\varrho}$ satisfies the Palais-Smale condition ((PS) for short). Now let us show that the functional $I_{\mu,\varrho}$ has the mountain pass geometry.

We say that a solution $u_{\mu,\varrho} \in W \setminus \{0\}$ of (P_{μ,ϱ,q^*}) is a ground solution if $I_{\mu,\varrho}(u_{\mu,\varrho}) = \inf\limits_{\mathcal{N}_\mu} I_{\mu,\varrho}(u_{\mu,\varrho})$, where $\mathcal{N}_{\mu,\varrho}$ is the Nehari manifold associated to $I_{\mu,\varrho}$ given by

$$\mathcal{N}_{\mu,\varrho} := \{u \in W : u \neq 0 : {I_{\mu,\varrho}}'(u)u = 0\}.$$

Lemma 2.1. *The functional $I_{\mu,\varrho} : W \to \mathbb{R}$ and the constant $c_{\mu,\varrho}$ satisfy the following conditions:*

(i) There are positive numbers α and ρ, such that

$$I_{\mu,\varrho}(u) \geq \alpha \quad \textit{if} \quad \|u\|_\mu = \rho.$$

(ii) For any positive function $w \in C_0^\infty(\Omega)$, *we have*

$$\lim_{t\to\infty} I_{\mu,\varrho}(tw) = -\infty.$$

(iii) There exists a positive constant Υ_1 *which does not depend of* μ, *such that* $c_{\mu,\varrho} \le \Upsilon_1$.

Proof. Using (a_1), (b_1) and (2.1), we have

$$\begin{aligned} I_{\mu,\varrho}(u) \quad &\ge \quad \frac{\min\{k_1,k_3\}}{p}\|u\|_{\mu,p}^p + \frac{1}{q}\|u\|_{\mu,q}^q - \frac{\xi}{q}\int_{\mathbb{R}^N}|u|^q dx \\ &\quad -\frac{C_\xi}{r}\int_{\mathbb{R}^N}|u|^r dx - \frac{\varrho}{q^*}\int_{\mathbb{R}^N}|u|^{q^*}dx. \end{aligned}$$

Therefore, using the Sobolev embeddings and taking ξ and $\|u\|_\mu$ sufficiently small, there are constants $C_1,\ C_2 > 0$ such that

$$I_{\mu,\varrho}(u) \ge C_1\|u\|_\mu^q - C_2\|u\|_\mu^r - C_3\varrho\|u\|_\mu^{q^*}$$

and the item (i) is proved.

Now we are going to show that the item (ii) holds. Since for all $x \in \Omega$, we have $\mu V(x) = 0$, for a positive function $w \in C_0^\infty(\Omega)$ and $t > 0$, we can use (a_1), (b_1), (2.2) to obtain

$$I_{\mu,\varrho}(tw) \le \frac{t^p}{p}\max\{k_2,k_4\}\|w\|_p^p + \frac{t^q}{q}\|w\|_q^q - D_1 t^\theta\int_{\mathbb{R}^N}|w|^\theta dx + D_2|\Omega|.$$

Since $q < \theta$, this completes the proof of the item (ii). The proof of the item (iii) follows by the last inequality and the item (i) because

$$\begin{aligned} 0 \quad &< \quad c_{\mu,\varrho} \\ &\le \quad \max_{t\ge 0}\left[\frac{t^p}{p}\max\{k_2,k_4\}\|w\|_p^p + \frac{t^q}{q}\|w\|_q^q - D_1 t^\theta\int_{\mathbb{R}^N}|w|^\theta dx + D_2|\Omega|\right] \\ &:= \quad \Upsilon_1, \end{aligned}$$

where D_1, D_2 were defined in (2.2). $\square$

From [34, Lemma 1.15], Lemma 2.1 ensures that there exists a sequence $(PS)_{c_{\mu,\varrho}}$ for the functional $I_{\mu,\varrho}$, where $c_{\mu,\varrho}$ is set in (2.3).

Lemma 2.2. *Let (u_n) be a $(PS)_{c_{\mu,\varrho}}$ sequence of the functional $I_{\mu,\varrho}$. Then the following statements hold.*

(i) The sequence (u_n) is bounded in W.

(ii) There exists a positive constant Υ_2, which does not depend on μ, such that

$$\limsup_{\mu\to\infty} \|u_n\|_\mu \le \Upsilon_2.$$

Consequently, $\liminf\limits_{\mu\to+\infty} c_{\mu,\varrho} > 0$.

Proof. Since (u_n) is a $(PS)_{c_{\mu,\varrho}}$ sequence of the functional $I_{\mu,\varrho}$, then, by (1.3) and (1.5),

$$\begin{aligned} & o_n(1) + c_{\mu,\varrho} + o_n(1)\|u_n\|_\mu = I_{\mu,\varrho}(u_n) - \frac{1}{\theta} I'_{\mu,\varrho}(u_n)u_n \\ \ge\ & \left(\frac{1}{p\gamma} - \frac{1}{\theta}\right) \int\limits_{\mathbb{R}^N} \left[a(|\nabla u_n|^p)|\nabla u_n|^p + [1+\mu V(x)]\, b(|u_n|^p)|u_n|^p\right] dx \\ +\ & \frac{1}{\theta}\int\limits_{\mathbb{R}^N} \left[f(u_n)(u_n) - \theta F(u_n)\right] dx + \varrho\left(\frac{1}{\theta} - \frac{1}{q^*}\right)\int\limits_{\mathbb{R}^N} |u_n|^{q^*} dx \\ \ge\ & \left(\frac{1}{p\gamma} - \frac{1}{\theta}\right)\left[\min\{k_1,k_3\}\|u_n\|^p_{\mu,p} + \|u_n\|^q_{\mu,q}\right]. \end{aligned} \tag{2.4}$$

Then, arguing as [1, Lemma 2.3] we can concluded that (u_n) is bounded in W.

Let us show that the item (ii) holds. Using the item (i) we can consider $R_{\mu,\varrho} := \limsup\limits_{n\to\infty} \|u_n\|_\mu$. We suppose, by contradiction, that $R_{\mu,\varrho} \to +\infty$ when $\mu \to +\infty$. Hence for μ large enough we can guarantee that there exists $m_{\mu,\varrho} \in \mathbb{N}$ such that

$$\|u_{m_{\mu,\varrho}}\|_\mu \ge \frac{R_{\mu,\varrho}}{2} \to +\infty \quad \text{when} \quad \mu \to +\infty.$$

Therefore, using (2.4) and the item (iii) of Proposition 2.1, we conclude that

$$\frac{\Upsilon_1}{\|u_{m_{\mu,\varrho}}\|_\mu} + o_\mu(1) \geq \left(\frac{1}{p\gamma} - \frac{1}{\theta}\right) \frac{\min\{k_1, k_3, 1\}}{2^p} \|u_{m_{\mu,\varrho}}\|_\mu^{p-1}.$$

This absurd shows the first part of item (ii). To conclude the item (ii) let us suppose by contradiction that $\liminf\limits_{\mu\to+\infty} c_{\mu,\varrho} = 0$. Then using the inequality (2.4) and that $\limsup\limits_{\mu\to\infty} \|u_n\|_\mu \leq \Upsilon_2$, we can conclude that

$$\|u_n\|_\mu = o_n(1) + o_\mu(1). \tag{2.5}$$

Since $I'_{\mu,\varrho}(u_n)u_n = o_n(1)$, we get

$$\begin{aligned}
&\int_{\mathbb{R}^N} a\left(|\nabla u_n|^p\right)|\nabla u_n|^p dx + \int_{\mathbb{R}^N} [1+\mu V(x)] b\left(|u_n|^p\right)|u_n|^p dx \\
=& \int_{\mathbb{R}^N} f(u_n)u_n dx + \frac{\varrho}{q^*}\int_{\mathbb{R}^N} |u|^{q^*} dx + o_n(1).
\end{aligned}$$

Using Sobolev embeddings, (a_1), (a_2) and (2.1) there exists a constant $C > 0$ which is independent of μ such that

$$\begin{aligned}
o_n(1) + [\min\{k_1,k_3,1\} - \xi]\,\|u_n\|^q &\leq \min\{k_1,k_3,1\}\left[\|u_n\|_p^p + \|u_n\|_q^q\right] \\
&\leq \min\{k_1,k_3,1\}\left[\|u_n\|_{\mu,p}^p + \|u_n\|_{\mu,q}^q\right] \\
&\leq C_\xi \int_{\mathbb{R}^N} |u_n|^r + \varrho \int_{\mathbb{R}^N} |u_n|^{q^*} \leq C[\|u_n\|^r + \varrho\|u_n\|^{q^*}].
\end{aligned}$$

Hence

$$o_n(1) + [\min\{k_1,k_3,1\} - \xi] \leq C\left[\|u_n\|^{r-q} + \varrho\|u_n\|^{q^*-q}\right],$$

which is a contradiction with (2.5). Then, we conclude that

$$\liminf_{\mu\to+\infty} c_{\mu,\varrho} > 0.$$

$\square$

3 The proof of the item (i) of Theorem 1.1 for the subcritical case ($\varrho = 0$)

From Lemma 2.1 and Lemma 2.2 there exists a bounded $(PS)_{c_{\mu,0}}$ sequence (u_n) for $I_{\mu,0}$. Then, by Sobolev embeddings, there exists $u_\mu \in W$ such that, up to a subsequence, we have

$$\begin{cases} u_n \rightharpoonup u_\mu \text{ in } W; \\ u_n \to u_\mu \text{ in } L^s_{loc}(\Omega),\ 1 \le s \le q; \\ u_n \to u_\mu\ a.e \text{ in } \mathbb{R}^N. \end{cases} \tag{3.1}$$

Moreover, using the ideias contained in [1, Lemma 2.3], we can conclude that u_μ is a critical point of $I_{\mu,0}$.

Now we prove that u_μ is a critical point of $I_{\mu,0}$ at Mountain Pass level $c_{\mu,0}$, for μ large enough. First of all, some technical lemmas.

Lemma 3.1. *Consider $(u_\mu) \subset W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$, then there exists a positive constant Υ_3 which does not depend on μ such that*

$$\liminf_{\mu \to +\infty} \int_{\mathbb{R}^N} |u_\mu|^r dx \ge \Upsilon_3.$$

Proof. Let us suppose, by contradiction, that $\liminf_{\mu \to +\infty} \int_{\mathbb{R}^N} |u_\mu|^r dx = 0$. Using Sobolev embeddings, (a_1), (b_1) and (2.1) we obtain

$$\min\left\{k_1, k_2, \frac{1}{2}\right\}[\|u_\mu\|^p_{\mu,p} + \|u_\mu\|^q_{\mu,q}] \le o_\mu(1). \tag{3.2}$$

Hence, $\lim_{\mu\to\infty} c_{\mu,0} = 0$ which contradicts the item (ii) of Lemma 2.2. □

Proposition 3.2. *There exists $\mu^* > 0$ such that $I_{\mu,0}$ has a critical point $u_\mu \in W$ at mountain pass level $c_{\mu,0}$, for $\mu \ge \mu^*$.*

Proof. By Lemma 3.1 there exists $\mu^* > 0$ such that $I_{\mu,0}$ has a nontrivial critical point, for $\mu \ge \mu^*$. On the other hand, the assumptions (a_3), (b_3)

and (f_4) imply the following monotonicity conditions:

$$t \longmapsto \frac{1}{p}A(t) - \frac{1}{q}a(t)t \text{ is increasing for } t \in (0, +\infty),$$

$$t \longmapsto \frac{1}{p}B(t) - \frac{1}{q}b(t)t \text{ is increasing for } t \in (0, +\infty),$$

$$t \longmapsto \frac{1}{q}f(t)t - F(t) \text{ is increasing for } t \in (0, +\infty).$$

Therefore, by (3.1) and Fatou's Lemma, we obtain

$$\begin{aligned}
I_{\mu,0}(u_\mu) &= I_{\mu,0}(u_\mu) - \frac{1}{q}I'_{\mu,0}(u_\mu) \\
&\leq \int\limits_{\mathbb{R}^N} \left(\frac{1}{p}A\left(|\nabla u_\mu|^p\right) - \frac{1}{q}a\left(|\nabla u_\mu|^p\right)|\nabla u_\mu|^p\right) dx \\
&\quad + \int\limits_{\mathbb{R}^N} (1+\mu V(x))\left(\frac{1}{p}B\left(|u_\mu|^p\right) - \frac{1}{q}b\left(|u_\mu|^p\right)|u_\mu|^p\right) dx \\
&\quad + \int\limits_{\mathbb{R}^N} \left(\frac{1}{q}f(u_\mu)u_\mu - F(u_\mu)\right) dx \\
&\leq \liminf_{n\to+\infty} \left[\int\limits_{\mathbb{R}^N} \left(\frac{1}{p}A\left(|\nabla u_n|^p\right) - \frac{1}{q}a\left(|\nabla u_n|^p\right)|\nabla u_n|^p\right) dx \right. \\
&\quad + \int\limits_{\mathbb{R}^N} (1+\mu V(x))\left(\frac{1}{p}B\left(|u_n|^p\right) - \frac{1}{q}b\left(|u_n|^p\right)|u_n|^p\right) dx \\
&\quad \left. + \int\limits_{\mathbb{R}^N} \left(\frac{1}{q}f(u_n)u_n - F(u_n)\right) dx\right] \\
&= \lim_{n\to+\infty} I_{\mu,0}(u_n) = c_{\mu,0}.
\end{aligned}$$

Hence, using the characterization (2.3) of the mountain pass level $c_{\mu,0}$ and (f_4), we conclude

$$c_{\mu,0} \leqslant I_{\mu,0}(u_\mu) \leq \lim_{n\to+\infty} I_{\mu,0}(u_n) = c_{\mu,0}, \quad \mu \geq \mu^*.$$

□

4 The proof of the item (ii) of Theorem 1.1 for the critical case ($\varrho = 1$ and $\sigma = q^*$)

To find a nontrivial solution for the case critical of the problem $(P_{\mu,1,q^*})$ it is necessary to control the level critical $c_{\mu,1}$. For this, we need to consider an auxiliary problem given by

$$\begin{cases} -k_2\Delta_p u - \Delta_q u + k_4|u|^{p-2}u + |u|^{q-2}u = |u|^\tau \ \text{ in } \Omega, \\ u \in W_0^{1,q}(\Omega), \end{cases} \tag{P_Ω}$$

where τ is the constant that appeared in the hypothesis (f_5) and Ω is the bounded domain that appeared in the hypothesis (V_2). The Euler-Lagrange functional associated to (P_Ω) is given by

$$\Phi_0(u) = \frac{1}{p}\int_\Omega [k_2|\nabla u|^p + k_4|u|^p]\,dx + \frac{1}{q}\int_\Omega [|\nabla u|^q + |u|^q]\,dx - \frac{1}{\tau}\int_\Omega |u|^\tau dx$$

and the Nehari manifold

$$\mathcal{N}_{\Phi_0} = \{u \in W_0^{1,q}(\Omega) : \ u \neq 0 \text{ and } \Phi_0'(u)u = 0\}.$$

Then, from [15, Apendix] there exists $w_\tau \in W_0^{1,q}(\Omega)$ such that

$$\Phi_0(w_\tau) = c_0, \ \Phi_0'(w_\tau) = 0$$

and

$$c_0 \geq \left(\frac{\tau - q}{\tau q}\right)\int_\Omega |w_\tau|^\tau dx. \tag{4.1}$$

Lemma 4.1. *There exists a positive number λ^* such that the level $c_{\mu,1}$ satisfies*

$$c_{\mu,1} < \left(\frac{1}{p\gamma} - \frac{1}{q^*}\right)S^{N/q}, \quad \forall\ \mu \geq 0 \ \text{ and } \ \forall\ \lambda > \lambda^*.$$

Proof. Since $V(x) = 0$ for $x \in \Omega$, and the hypotheses (a_1), (b_1) and (f_5)

hold, we deduce that

$$\begin{aligned}
&\int_{\mathbb{R}^N} a(|\nabla w_\tau|^p)|\nabla w_\tau|^p dx + \int_{\mathbb{R}^N} (1+\mu V(x))b(|w_\tau|^p)|w_\tau|^p dx \\
&= \int_{\mathbb{R}^N} a(|\nabla w_\tau|^p)|\nabla w_\tau|^p dx + \int_{\mathbb{R}^N} b(|w_\tau|^p)|w_\tau|^p dx \\
&\le \int_\Omega \left(k_2|\nabla w_\tau|^p + k_4|w_\tau|^p\right)\, dx + \int_\Omega \left(|\nabla w_\tau|^q + |w_\tau|^q\right)\, dx \\
&= \int_\Omega |w_\tau|^\tau dx \le \frac{1}{\lambda}\int_\Omega f(w_\tau)w_\tau dx \le \int_{\mathbb{R}^N} f(w_\tau)w_\tau dx + \int_{\mathbb{R}^N} |w_\tau|^{q^*} dx.
\end{aligned}$$

This inequality implies that $I'_{\mu,1}(w_\tau)w_\tau \le 0$. After that, by (a_3), (b_3) and (f_4) there exists $t \in (0,1]$, such that

$$I_{\mu,1}(t_\mu w_\tau) = \sup_{t>0} I_{\mu,1}(t w_\tau).$$

Therefore, using (a_1), (b_1), (g_3), (f_5) and that $\Phi'_0(w_\tau)w_\tau = 0$, we obtain

$$\begin{aligned}
c_{\mu,1} &\le I_{\mu,1}(t_\mu w_\tau) \\
&\le \frac{t^p}{p}\int_\Omega \left[k_2|\nabla w_\tau|^p + k_4|w_\tau|^p\right] dx + \frac{t^q}{q}\int_\Omega \left[|\nabla w_\tau|^q + |w_\tau|^q\right] dx \\
&\quad -\frac{\lambda}{\tau} t^\tau \int_\Omega |w_\tau|^\tau dx \\
&\le \frac{t^p}{p}\int_\Omega \left[k_2|\nabla w_\tau|^p + k_4|w_\tau|^p\right] dx + \frac{t^p}{p}\int_\Omega \left[|\nabla w_\tau|^q + |w_\tau|^q\right] dx \\
&\quad -\frac{\lambda}{\tau} t^\tau \int_\Omega |w_\tau|^\tau dx \\
&\le \left[\frac{t^p}{p} - \lambda\frac{t^\tau}{\tau}\right]\int_\Omega |w_\tau|^\tau dx \le \max_{s\ge 0}\left[\frac{s^p}{p} - \lambda\frac{s^\tau}{\tau}\right]\int_\Omega |w_\tau|^\tau dx.
\end{aligned}$$

Then, using (4.1) and some straight forward algebric manipulations, we get

$$c_{\mu,1} \leq \max_{s\geq 0}\left[\frac{s^p}{p} - \lambda\frac{s^\tau}{\tau}\right]\frac{c_0 q\tau}{(\tau-q)} = \left[\frac{\tau-p}{p\lambda^{p/(\tau-p)}}\right]\frac{c_0 q}{(\tau-q)}.$$

Hence, choosing $\lambda > \lambda^* := \left[\dfrac{(\tau-p)c_0 q q^*\theta}{(\tau-q)(q^*-\theta)pS^{\frac{N}{q}}}\right]^{\frac{\tau-p}{p}}$ in (f_5), the result follows. □

Let us introduce the notation which we are going to use in the next results. From Lemma 2.1 and Lemma 2.2 there exists a bounded $(PS)_{c_{\mu},1}$ sequence (u_n) for $I_{\mu,1}$. Then, by Sobolev embeddings, there exists $u_\mu \in W$ such that, up to a subsequence, we have

$$\begin{cases} u_n \rightharpoonup u_\mu \text{ in } W; \\ u_n \to u_\mu \text{ in } L^s_{loc}(\Omega),\ 1 \leq s \leq q; \\ u_n \to u_\mu\ a.e \text{ in } \mathbb{R}^N. \end{cases} \tag{4.2}$$

Moreover, using the ideias contained in [1, Lemma 2.3], we can conclude that u_μ is a critical point of $I_{\mu,1}$.

First of all, using the notation above, we are going to prove some technical result.

Lemma 4.2. *Let $u_\mu \in W$ be the weak limit of the sequence defined in (4.2). For $\lambda > \lambda^*$, there exists a positive constant Υ_4, which does not depend on μ, such that*

$$\liminf_{\mu\to+\infty} \int_{\mathbb{R}^N} |u_\mu|^r dx \geq \Upsilon_4.$$

Proof. Let us suppose, by contradiction, that $\liminf_{\mu\to+\infty} \int_{\mathbb{R}^N} |u_\mu|^r dx = 0$. By (f_3), we obtain

$$\int_{\mathbb{R}^N} f(u_\mu)\, u_\mu dx = o_\mu(1). \tag{4.3}$$

Since $I'_{\mu,1}(u_\mu)u_\mu = 0$, then

$$\int_{\mathbb{R}^N} \left[a\left(|\nabla u_\mu|^p\right)|\nabla u_\mu|^p + (1+\mu V(x))b\left(|u_\mu|^p\right)|u_\mu|^p\right]dx = \int_{\mathbb{R}^N} |u_\mu|^{q^*}\,dx + o_\mu(1).$$

Setting

$$l := \int_{\mathbb{R}^N} |u_\mu|^{q^*}\,dx + o_\mu(1),$$

we have that $l > 0$, from Lemma 2.2 we have $c_{\mu,1} > 0$, for all $\mu > 0$. By definition of the best constant S in the embedding from $D^{1,q}(\mathbb{R}^N)$ into $L^q(\mathbb{R}^N)$, we get

$$S \leq \frac{\displaystyle\int_{\mathbb{R}^N} |\nabla u_\mu|^q dx}{\left(\displaystyle\int_{\mathbb{R}^N} |u_\mu|^{q^*}dx\right)^{q/q^*}} \leq l^{q/N}. \tag{4.4}$$

Using (2.4) and (4.4), we obtain $c_{\mu,1} \geq \left(\frac{1}{p\gamma} - \frac{1}{q^*}\right)S^{N/q}$, which contradicts the Lemma 4.1. □

Proposition 4.3. *There exist positive numbers μ^{**} and λ^*, which are independent each other, such that $I_{\mu,1}$ has a nontrivial critical point $u_\mu \in W$ at mountain pass level $c_{\mu,1}$, for $\mu \geq \mu^{**}$ and for $\lambda \geq \lambda^*$.*

Proof. The proof follows using the same reasoning that can be found in Proposition 3.2. □

5 Concentration Results

We are going to investigate the behavior of a sequence of ground solution (u_{μ_n}) of (P_{μ,ϱ,q^*}) when $\mu_n \to \infty$. For simplicity of notation such sequence will be denoted just by (u_n). For this goal, let us consider the problem limit (P_{0,ϱ,q^*}) given by

$$\begin{cases} -div\left(a\left(|\nabla u|^p\right)|\nabla u|^{p-2}\nabla u\right)+b\left(|u|^p\right)|u|^{p-2}u=f(u)+\varrho|u|^{q^*-2}u \ \text{ in } \Omega,\\ \\ u=0 \ \text{ on } \ \partial\Omega. \end{cases}$$

The functional associated to (P_{0,ϱ,q^*}) is

$$J_\varrho(u)=\frac{1}{p}\int_\Omega A\left(|\nabla v|^p\right)dx+\frac{1}{p}\int_\Omega B\left(|v|^p\right)dx-\int_\Omega F(v)dx-\frac{\varrho}{q^*}\int_\Omega |v|^{q^*}dx,$$

which is differentiable on $W_0^{1,q}(\Omega)$, and let $\mathcal{N}_\varrho$ be the Nehari manifold associated to J_ϱ given by

$$\mathcal{N}_\varrho=\left\{u\in W_0^{1,q}(\Omega)/\{0\}:\ J_\varrho'(u)u=0\right\}.$$

Proposition 5.1. *Let* $(u_n)\subset W\setminus\{0\}$ *be a sequence of ground states solutions for* $(P_{\mu_n,\varrho,q^*})_{\mu_n\geq 1}$*. Then, up to a subsequence, there exists* $u_\infty\in W^{1,p}(\mathbb{R}^N)\cap W^{1,q}(\mathbb{R}^N)$ *such that* $u_n\rightharpoonup u_\infty$ *in* $W^{1,p}(\mathbb{R}^N)\cap W^{1,q}(\mathbb{R}^N)$*. Furthermore,*

(i) $u_\infty=0$ *in* $\mathbb{R}^N\setminus\Omega$, $u_\infty(x)\geq 0$, $u_\infty(x)\neq 0$.

(ii) Setting $d_{\mu_n,\varrho}:=\inf_{\mathcal{N}_{\mu_n}} I_{\mu_n,\varrho}(u_n)$*, then*

$$\lim_{n\to+\infty} d_{\mu_n,\varrho}=\lim_{n\to+\infty} I_{\mu_n,\varrho}(u_n)=J_\varrho(u_\infty).$$

Moreover, $u_n\to u_\infty$ *in* $W^{1,p}(\mathbb{R}^N)\cap W^{1,q}(\mathbb{R}^N)$ *and*

$$J_\varrho(u_\infty)=d_\varrho:=\inf_{\mathcal{N}_\varrho} J_\varrho.$$

Proof. Using Lemma 2.2, (ii), we conclude that $(\|u_n\|_{\mu_n})$ is bounded in $\mathbb{R}$ and (u_n) is bounded in $W^{1,p}(\mathbb{R}^N)\cap W^{1,q}(\mathbb{R}^N)$. So, up to a subsequence, there exists $u_\infty\in W^{1,p}(\mathbb{R}^N)\cap W^{1,q}(\mathbb{R}^N)$ such that

$$u_n\rightharpoonup u_\infty \text{ in } W^{1,p}(\mathbb{R}^N)\cap W^{1,q}(\mathbb{R}^N) \ \text{ and } \ u_n(x)\to u_\infty(x) \text{ for a.e. } x\in\mathbb{R}^N. \tag{5.1}$$

Now, for each $m \in \mathbb{N}$, we define $C_m = \left\{ x \in \mathbb{R}^N \,;\, V(x) \geq \dfrac{1}{m} \right\}$. Thus

$$\int_{C_m} b(|u_n|^p)|u_n|^p \, dx \leq \frac{m}{\mu_n} \int_{C_m} \big(\mu_n V(x) + 1\big) b(|u_n|^p)|u_n|^p \, dx \leq \frac{C}{\mu_n}. \quad (5.2)$$

Taking $n \to \infty$, we have by Fatou's lemma,

$$\int_{C_m} b(|u_\infty|^p)|u_\infty|^p \, dx = 0,$$

implying that $u_\infty = 0$ in C_m and consequence, $u_\infty = 0$ in $\mathbb{R}^N \setminus \overline{\Omega}$, which implies $u_\infty \in W_0^{1,p}(\Omega) \cap W_0^{1,q}(\Omega) = W_0^{1,q}(\Omega)$ (see Proposition 9.18 in [12]).

Next we claim that the limit u_∞ is a nontrivial solution for $(P_{0,\varrho})$. To prove this let us consider the following sets

$$\widetilde{A}_R = \{ x \in \mathbb{R}^N \backslash B_R(0) : \; V(x) \geq V^* \}$$

and

$$A_R = \{ x \in \mathbb{R}^N \backslash B_R(0) : \; V(x) < V^* \}.$$

Using Lemma 2.2 and (V_3) we can ensure, by Hölder's inequality and Sovolev embeddings, that there exists $\Upsilon_5 > 0$ such that

$$\begin{aligned} \int\limits_{\widetilde{A}_R} |u_n|^q \, dx \;\; &\leq \;\; \frac{1}{1+\mu V^*} \int\limits_{\mathbb{R}^N} [1 + \mu V(x)] \, |u_n|^q \, dx \\ &\leq \;\; \frac{1}{1+\mu V^*} \|u_n\|_\mu^q \leq \frac{\Upsilon_5}{1+\mu_n V^*} \end{aligned}$$

and

$$\int\limits_{A_R} |u_n|^q \, dx \leq \left(\int\limits_{A_R} |u_n|^{q_*} \, dx \right)^{\frac{q}{q_*}} meas(A_R)^{\frac{q_*-q}{q}} \leq \Upsilon_5 o_R(1).$$

Hence, by the interpolation argument there exists $\Upsilon_6 > 0$ such that

$$\limsup_{n \to +\infty} \int\limits_{\widetilde{A}_R} |u_n|^r \, dx = 0 \quad \text{and} \quad \limsup_{n \to +\infty} \int\limits_{A_R} |u_n|^r \, dx \leq \Upsilon_6 o_R(1). \quad (5.3)$$

Observe that, from Lemma 2.2, the constants Υ_5 and Υ_6 are independent on the parameter μ. Since, up to a subsequence, $u_n \to u_\infty$ in $L^r_{loc}(\mathbb{R}^N)$ and (5.3) holds, we obtain that

$$\begin{aligned}\liminf_{n\to+\infty}\int\limits_{\mathbb{R}^N}|u_n|^r\,dx &\leq \limsup_{n\to+\infty}\int\limits_{\mathbb{R}^N}|u_n|^r\,dx\\ &\leq \limsup_{n\to\infty}\left[\int\limits_{B_R(0)}|u_n|^r\,dx+\int\limits_{\widetilde{\Lambda}_R}|u_n|^r\,dx+\int\limits_{\Lambda_R}|u_n|^r\,dx\right]\\ &\leq \int\limits_{B_R(0)}|u_\infty|^r\,dx+\Upsilon_6 o_R(1).\end{aligned} \tag{5.4}$$

Hence, by Lemma 3.1 (for $\varrho = 0$) or Lemma 4.2 (for $\varrho = 1$) the claim follows, for R large enough. Moreover, using (f_1) and u_∞^- a test function, we get $u_\infty \geq 0$ and $u_\infty \neq 0$.

We now prove the second item (ii). Observe that since $V = 0$ in Ω, we obtain, for all $u \in W_0^{1,q}(\Omega)$,

$$\int\limits_{\mathbb{R}^N} V(x)B(|u|^p)dx = \int\limits_{\mathbb{R}^N\setminus\Omega} V(x)B(|u|^p)dx + \int\limits_{\Omega} V(x)B(|u|^p)dx = 0,$$

which implies

$$I_{\mu_n,\varrho}(u) = J_\varrho(u) \quad \text{and} \quad I'_{\mu_n,\varrho}(u)u = J'_\varrho(u)u, \qquad \forall\, u \in W_0^{1,q}(\Omega). \tag{5.5}$$

Then, from (5.5), we have that $u \in \mathcal{N}_{\mu_n,\varrho}$, for all $u \in \mathcal{N}_\varrho$. Hence,

$$d_{\mu_n,\varrho} \leq d_\varrho. \tag{5.6}$$

On the other hand, since $u_n \rightharpoonup u_\infty$ in $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ we have, by

(1.4), (1.5) and the Fatou's Lemma,

$$
\begin{aligned}
0 \leq \frac{1}{p}\int\limits_{\mathbb{R}^N} [A(|\nabla u|^p) + B(|u|^p)]\,dx - \frac{1}{\theta}\int\limits_{\mathbb{R}^N} [a(|\nabla u|^p)|\nabla u|^p + b(|u|^p)|u|^p]\,dx \\
\leq \liminf_{n\to+\infty}\Bigg\{\frac{1}{p}\int\limits_{\mathbb{R}^N} [A(|\nabla u_n|^p) + B(|u_n|^p)]\,dx \\
- \frac{1}{\theta}\int\limits_{\mathbb{R}^N} [a(|\nabla u_n|^p)|\nabla u_n|^p + b(|u_n|^p)|u_n|^p]\,dx\Bigg\}
\end{aligned}
\tag{5.7}
$$

Therefore, using the fact that $u_\infty \in \mathcal{N}_\varrho$, we obtain, by (5.5), (5.6) and (5.7),

$$
\begin{aligned}
d_{\mu_n,\varrho} \leq d_\varrho \leq J_\varrho(u_\infty) &= I_{\mu_n,\varrho}(u_\infty) - I'_{\mu_n,\varrho}(u_\infty)u_\infty \\
&\leq \liminf_{n\to\infty}\left[I_{\mu_n,\varrho}(u_n) - \frac{1}{\theta}I'_{\mu_n,\varrho}(u_n)u_n\right] \\
&= I_{\mu_n,\varrho}(u_n) + o_n(1) = d_{\mu_n,\varrho} + o_n(1),
\end{aligned}
\tag{5.8}
$$

which implies

$$
\lim_{n\to+\infty} d_{\mu_n,\varrho} = \lim_{n\to+\infty} I_{\mu_n,\varrho}(u_n) = J_\varrho(u_\infty). \tag{5.9}
$$

Assume, by contradiction, that

$$
u_n \to u_\infty \quad \text{in} \;\; W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), \tag{5.10}
$$

does not hold. Then, the inequality (5.7) is strict and hence, arguing as (5.8), there exists $n_0 \in \mathbb{N}$

$$
d_\varrho < d_{\mu_n,\varrho} + \frac{d_\varrho}{2}, \quad n \geq n_0.
$$

This contradicts (5.9). □

6 Theorem 1.1 (subcritical case)

Proof of Theorem 1.1(subcritical case). From Proposition 3.2 and (f_4) we can guarantee that there exists $\mu^* > 0$ such that $(P_{\mu,0})$ has a positive ground state solution $u_\mu \in W$, for $\mu \geq \mu^*$. Then, using Proposition 5.1, we obtain, up to a subsequence, $u_\mu \to u_\infty$ in $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ when $\mu \to +\infty$, where u_∞ is a ground state solution to problem

$$(P_{0,0}) \begin{cases} -div\left(a\left(|\nabla u|^p\right)|\nabla u|^{p-2}\nabla u\right) + b\left(|u|^p\right)|u|^{p-2}u = f(u) \ \text{ in } \Omega, \\ \\ u = 0 \ \text{ on } \ \partial\Omega. \end{cases}$$

□

7 Theorem 1.1 (critical case)

Proof of Theorem 1.1(critical case). From Proposition 4.3 and (f_4) we can guarantee that there exist $\mu^{**} > 0$ and $\lambda^* > 0$ such that $(P_{\mu,1,q^*})$ has a positive ground state solution $u_\mu \in W$, for all $\mu \geq \mu^{**}$ and $\lambda \geq \lambda^*$. Then, using Proposition 5.1, we obtain, up to a subsequence, $u_\mu \to u_\infty$ in $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ when $\mu \to +\infty$, where u_∞ is a ground state solution to problem $(P_{0,1,q^*})$ given by

$$\begin{cases} -div\left(a\left(|\nabla u|^p\right)|\nabla u|^{p-2}\nabla u\right) + b\left(|u|^p\right)|u|^{p-2}u = f(u) + |u|^{q^*-2}u \ \text{ in } \Omega, \\ u = 0 \ \text{ on } \ \partial\Omega. \end{cases}$$

□

8 Case supercritical

In this section we are going to study the supercritical case of the problem $(P_{\mu,\varrho,\sigma})$, that is, when $\varrho = 1$ and $\sigma > q^*$, observe that in this case $\displaystyle\int_{\mathbb{R}^N} |u|^\sigma \, dx$ is not well defined in W. Then, inspired by [13] and [19], we are going to consider in this section the function $\psi : \mathbb{R} \to \mathbb{R}$ given by

$$\psi(s) := \begin{cases} 0 & \text{if} \quad s < 0, \\ s^{\sigma-1} & \text{if} \quad 0 \le s \le 1, \\ s^{q^*-1} & \text{if} \quad s > 1. \end{cases}$$

It follows immediately that

$$\psi(s) \le |s|^{q^*-1}, \ \forall \ s \in \mathbb{R}, \tag{8.1}$$

and

$$\begin{aligned} \frac{1}{\theta} \int\limits_{\mathbb{R}^N} [\psi(u)u - \theta \Psi(u)] \ dx \ &\ge \left(\frac{1}{\theta} - \frac{1}{\sigma}\right) \left[\int\limits_{\{|u| \le 1\}} |u|^{\sigma} \ dx + \int\limits_{\{|u|>1\}} |u|^{q^*} \ dx \right] \\ &> 0, \end{aligned} \tag{8.2}$$

where $\Psi(s) := \displaystyle\int_0^s \psi(t)dt$. We also consider the auxiliary problem $(P_{\mu,\sigma})$, in $\mathbb{R}^N$, given by

$$\begin{cases} -div \left(a \left(|\nabla u|^p \right) |\nabla u|^{p-2} \nabla u \right) + [1 + \mu V(x)] b \left(|u|^p \right) |u|^{p-2} u = f(u) + \psi(u), \\ \\ u \in W. \end{cases}$$

Remark 8.1. If u_μ is a nonnegative solution of $(P_{\mu,\sigma})$ with $\|u_\mu\|_\infty \le 1$, then u_μ is also a nonnegative solution of $(P_{\mu,1,\sigma})$.

8.1 Existence of positive solution for problem $(P_{\mu,\sigma})$

The nonnegative weak solutions for the problem $(P_{\mu,\sigma})$ are the critical points of the functional $I_{\mu,\sigma} : W \to \mathbb{R}$ given by

$$\begin{aligned} I_{\mu,\sigma}(v) \ &= \ \frac{1}{p} \int\limits_{\mathbb{R}^N} A \left(|\nabla v|^p \right) dx + \frac{1}{p} \int\limits_{\mathbb{R}^N} [1 + \mu V(x)] B \left(|v|^p \right) dx \\ &\quad - \int\limits_{\mathbb{R}^N} F(v) dx - \int\limits_{\mathbb{R}^N} \Psi(v) dx, \end{aligned}$$

where $\Psi(s) := \int_0^s \psi(t)dt$. Now we are going to find a nontrivial and nonnegative solution for $(P_{\mu,\sigma})$.

Using the same arguments of Lemma 4.2 and Proposition 4.3 with short modifications we can prove the following results

Proposition 8.2. *There exist $\mu^{**} > 0$ and $\lambda^* > 0$ such that the functional $I_{\mu,\sigma}$ has a nontrivial critical point $u_\mu \in W$ at the mountain pass level $c_{\mu,\sigma}$, for all $\mu \geq \mu^{**}$ and $\lambda \geq \lambda^*$.*

The next result relates the critical points of the functional $I_{\mu,\sigma}$ with solutions to the problem $(P_{\mu,1,\sigma})$, the arguments used here are inspired by [1, Lemma 5.5] and [21, Theorem 3].

Lemma 8.3. *Let $u_\mu \in W$ be a nonnegative solution for problem $(P_{\mu,\sigma})$. Then,*

$$\|u_\mu\|_{L^\infty(\mathbb{R}^N)} \leq 1, \quad \forall\, \lambda > \lambda^*.$$

Moreover, the function u_μ is a solution of $(P_{\mu,1,\sigma})$.

Proof. For each $L > 0$, let

$$u_L(x) = \begin{cases} u_\mu(x), & u_\mu(x) \leq L, \\ \\ L, & u_\mu(x) > L. \end{cases} \tag{8.3}$$

and

$$z_L := u_L^{q(\gamma-1)} u_\mu$$

with $\gamma > 1$ will be determined later.

Taking z_L as a test function, we obtain that $I'_{\mu,\sigma}(u_\mu)z_L = 0$. That is,

$$\int_{\mathbb{R}^N} u_L^{q(\gamma-1)} a(|\nabla u_\mu|^p)|\nabla u_\mu|^p dx$$

$$+q(\gamma-1)\int_{\mathbb{R}^N} u_L^{q(\gamma-1)-1} u_\mu a(|\nabla u_\mu|^p)|\nabla u_\mu|^{p-2}\nabla u_\mu \nabla u_L dx$$

$$+\int_{\mathbb{R}^N}[1+\mu V(x)]b(|u_\mu|^p)|u_\mu|^p u_L^{q(\gamma-1)}dx$$

$$=\int_{\mathbb{R}^N} f(u_\mu)u_\mu u_L^{q(\gamma-1)}dx + \int_{\mathbb{R}^N}\psi(u_\mu)u_\mu u_L^{q(\gamma-1)}dx.$$

Using (a_1), (b_1), (f_1), (f_2) and (8.1) we obtain that given $\xi > 0$ there exists $C_\xi > 0$, such that

$$\int_{\mathbb{R}^N} u_L^{q(\gamma-1)}[k_1|\nabla u_\mu|^p + |\nabla u_\mu|^q]dx + q(\gamma-1)\int_{\mathbb{R}^N} u_L^{q(\gamma-1)}[k_1|\nabla u_L|^p + |\nabla u_L|^q]dx$$

$$+\int_{\mathbb{R}^N} u_L^{q(\gamma-1)}[k_3|u_\mu|^p + |u_\mu|^q]dx$$

$$\le \xi\int_{\mathbb{R}^N} u_L^{q(\gamma-1)}|u_\mu|^q dx + (C_\xi+1)\int_{\mathbb{R}^N} u_L^{q(\gamma-1)}|u_\mu|^{q^*}dx.$$

Let us now consider the function $w_L := u_\mu u_L^{\gamma-1}$. Hence, by inequality above,

$$\begin{aligned}\int_{\mathbb{R}^N}|\nabla w_L|^q dx &\le 2^q\int_{\mathbb{R}^N} u_L^{q(\gamma-1)}|\nabla u_\mu|^q dx + 2^q(\gamma-1)^q\int_{\mathbb{R}^N} u_L^{q(\gamma-1)}|\nabla u_L|^q dx \\ &\le 4^q\gamma^q\xi\int_{\mathbb{R}^N} u_L^{q(\gamma-1)}|u_\mu|^q dx \qquad (8.4)\\ &\quad +4^q\gamma^q(C_\xi+1)\int_{\mathbb{R}^N} u_L^{q(\gamma-1)}|u_\mu|^{q^*}dx\end{aligned}$$

Therefore, since $u_L \leq u_\mu$,

$$\begin{aligned} \|w_L\|^q_{L^{q^*}(\mathbb{R}^N)} &\leq S \int\limits_{\mathbb{R}^N} |\nabla w_L|^q dx \\ &\leq 4^q \gamma^q S \xi \int\limits_{\mathbb{R}^N} |u_\mu|^{q\gamma} dx \\ &\quad + 4^q \gamma^q S (C_\xi + 1) \int\limits_{\mathbb{R}^N} |u_\mu|^{\gamma q} |u_\mu|^{q^*-q} dx \end{aligned} \tag{8.5}$$

where S is the best Sobolev constant of the embedding $D^{1,q}(\mathbb{R}^N) \hookrightarrow L^{q^*}(\mathbb{R}^N)$.

The next step is to show that $u_\mu \in L^{\frac{(q^*)^2}{q}}(\mathbb{R}^N)$. For this, we choose $\gamma = \dfrac{q^*}{q}$ in (8.5) then, by Hölder's inequality,

$$\begin{aligned} \|w_L\|^q_{L^{q^*}(\mathbb{R}^N)} &\leq \left(\frac{4q^*}{q}\right)^q S \xi \|u_\mu\|^{q^*}_{L^{q^*}(\mathbb{R}^N)} \\ &\quad + \left(\frac{4q^*}{q}\right)^q S (C_\xi + 1) \|u_\mu\|^{q^*-q}_{q^*} \|w_L\|^q_{L^{q^*}(\mathbb{R}^N)}. \end{aligned}$$

Using (8.2) and Lemma 4.1 and that the function u_μ is a critical point of $I_{\mu,\sigma}$, we have that

$$\begin{aligned} \left[\frac{\tau - p}{p\lambda^{p/(\tau-p)}}\right] \frac{c_\Lambda q}{(\tau - q)} \geq c_\mu &= I_{\mu,\sigma}(u_\mu) - \frac{1}{\theta} I'_{\mu,\sigma}(u_\mu) u_\mu \\ &= \left(\frac{\theta - p\gamma}{p\gamma\theta}\right) \int\limits_{\mathbb{R}^N} |\nabla u_\mu|^q \, dx \\ &\geq \left(\frac{\theta - p\gamma}{p\gamma\theta S}\right) \|u_\mu\|^q_{L^{q^*}(\mathbb{R}^N)}. \end{aligned} \tag{8.6}$$

Choosing $\xi = \dfrac{1}{2}$ in (8.1) there exists $D_3 > 0$ such that using the inequality

(8.6) and Fatou's Lemma in (8.1), we obtain that

$$\begin{aligned}\frac{1}{2}\left[\int\limits_{\mathbb{R}^N}|u_\mu|^{\frac{(q^*)^2}{q}}\,dx\right]^{\frac{q}{q^*}} &\leq \left(\frac{4q^*}{q}\right)^q\frac{S}{2}\left[\frac{(\tau-p)c_\infty q\theta p\gamma S}{p(\tau-q)(\theta-p\gamma)}\right]^{\frac{q^*}{q}}\frac{1}{\lambda^{\frac{pq^*}{q(\tau-p)}}}\\ &< \infty, \end{aligned} \tag{8.7}$$

whenever $\lambda > D_3$. Note that from (8.4) and previous arguments there exists a positive constant K, such that

$$\|w_L\|^q_{L^{q^*}(\mathbb{R}^N)} \leq 4^q\gamma^q S(K+1)\int\limits_{\mathbb{R}^N}|u_\mu|^{\gamma q}|u_\mu|^{q^*-q}dx. \tag{8.8}$$

We are now going to consider $\gamma = \gamma_0 := \dfrac{q^*}{q}\dfrac{(t-1)}{t}$ in (8.8), where $t := \dfrac{(q^*)^2}{q(q^*-q)} > 1$. Then, by Hölder inequality and Fatou's Lemma,

$$\begin{aligned}\|u_\mu\|^{q\gamma_0}_{L^{q^*\gamma_0}(\mathbb{R}^N)} &\leq \liminf_{L\to+\infty}\|w_L\|^q_{L^{q^*}(\mathbb{R}^N)}\\ &\leq \|w_L\|^q_{L^{q^*}(\mathbb{R}^N)} \leq 4^q\gamma_0^q S(K+1)\|u_\mu\|^{q^*-q}_{L^{\frac{(q^*)^2}{q}}(\mathbb{R}^N)}\|u_\mu\|^{\gamma_0 q}_{L^{q^*}(\mathbb{R}^N)}.\end{aligned}$$

Hence,

$$\|u_\mu\|^{q\gamma_0}_{L^{q^*\gamma_0}(\mathbb{R}^N)} \leq \left[4S^{\frac{1}{q}}(K+1)^{\frac{1}{q}}\|u_\mu\|^{\frac{q^*-q}{q}}_{L^{\frac{(q^*)^2}{q}}(\mathbb{R}^N)}\right]^{\frac{1}{\gamma_0}}\gamma_0^{\frac{1}{\gamma_0}}\|u_\mu\|_{L^{q^*}(\mathbb{R}^N)}. \tag{8.9}$$

Already when $\gamma = \gamma_0^2$ in (8.5) we obtain, by (8.9), that

$$\|u_\mu\|_{L^{q^*\gamma_0^2}(\mathbb{R}^N)} \leq \left[4S^{\frac{1}{q}}(K+1)^{\frac{1}{q}}\|u_\mu\|^{\frac{q^*-q}{q}}_{L^{\frac{(q^*)^2}{q}}(\mathbb{R}^N)}\right]^{\sum\limits_{i=1}^{2}\frac{1}{\gamma_0^i}}\gamma_0^{\sum\limits_{i=1}^{2}\frac{i}{\gamma_0^i}}\|u_\mu\|_{L^{q^*}(\mathbb{R}^N)}.$$

Repeating the arguments above for $\gamma_0^3,\ \gamma_0^4, \ldots$ we can concluded that

$$\|u_\mu\|_{L^{q^*\gamma_0^m}(\mathbb{R}^N)} \leq \left[4S^{\frac{1}{q}}(K+1)^{\frac{1}{q}}\|u_\mu\|^{\frac{q^*-q}{q}}_{L^{\frac{(q^*)^2}{q}}(\mathbb{R}^N)}\right]^{\sum\limits_{i=1}^{m}\frac{1}{\gamma_0^i}}\gamma_0^{\sum\limits_{i=1}^{m}\frac{i}{\gamma_0^i}}\|u_\mu\|_{L^{q^*}(\mathbb{R}^N)}. \tag{8.10}$$

Once that

$$\sum_{i=1}^{\infty}\frac{1}{\gamma_0^i} \quad \text{and} \quad \sum_{i=1}^{\infty}\frac{i}{\gamma_0^i},$$

are convergent series it follows from (8.10) that

$$\begin{aligned}
\|u_\mu\|_{L^\infty(\mathbb{R}^N)} &\leq \left[4S^{\frac{1}{q}}(K+1)^{\frac{1}{q}}\|u_\mu\|_{L^{\frac{(q^*)^2}{q}}(\mathbb{R}^N)}^{\frac{q^*-q}{q}}\right]^{\sum_{i=1}^{\infty}\frac{1}{\gamma_0^i}} \gamma_0^{\sum_{i=1}^{\infty}\frac{i}{\gamma_0^i}}\|u_\mu\|_{L^{q^*}(\mathbb{R}^N)} \\
&= \|u_\mu\|_{L^{\frac{(q^*)^2}{q}}(\mathbb{R}^N)}^{\frac{(q^*-q)}{q}\sum_{i=1}^{\infty}\frac{1}{\gamma_0^i}}\left[4S^{\frac{1}{q}}(K+1)^{\frac{1}{q}}\right]^{\sum_{i=1}^{\infty}\frac{1}{\gamma_0^i}}\gamma_0^{\sum_{i=1}^{\infty}\frac{i}{\gamma_0^i}}\|u_\mu\|_{L^{q^*}(\mathbb{R}^N)}.
\end{aligned}$$

Finally there exists $\lambda^* > 1$ such that, by (8.7) and the last inequality, we have that

$$\|u_\mu\|_\infty \leq 1, \quad \forall\, \lambda > \lambda^*.$$

Hence, $\psi(u_\mu) = |u_\mu|^{\sigma-2}u_\mu$ which implies that the function u_μ is a solution of the problem $(P_{\mu,1,\sigma})$. □

9 Theorem 1.1 (supercritical case)

Proof of Theorem 1.1(supercritical case). From Proposition 8.2 and (f_4) we can guarantee that there exists $\mu^{**} > 0$ such that $(P_{\mu,1,\sigma})$ has a positive ground state solution $u_\mu \in W$, for all $\mu \geq \mu^{**}$ and $\lambda \geq \lambda^*$. Then, using Proposition 5.1 with short modifications, we obtain, up to a subsequence, $u_\mu \to u_\infty$ in $W^{1,p}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ when $\mu \to +\infty$, where u_∞ is a ground state solution to problem $(P_{0,1,\sigma})$ given by

$$\begin{cases} -div\left(a\left(|\nabla u|^p\right)|\nabla u|^{p-2}\nabla u\right) + b\left(|u|^p\right)|u|^{p-2}u = f(u) + |u|^{\sigma-2}u \ \text{ in } \Omega, \\ \\ u = 0 \ \text{ on } \ \partial\Omega. \end{cases}$$

□

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