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Quasilinear Schrödinger Systems Involving Critical Exponential Growth

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*Dedicated to Professor Pedro Ubilla
on the occasion of his 60th birthday*

Abstract. In this work, by using Orlicz Spaces and variational methods, we can find enough conditions for the existence of least energy solution for a class of quasilinear systems of the form

$$\begin{cases} -\Delta u + V_1(x)u - \Delta(u^2)u = h(x,u,v), & \text{in } \mathbb{R}^2 \\ -\Delta v + V_2(x)v - \Delta(v^2)v = g(x,u,v), & \text{in } \mathbb{R}^2, \end{cases} \tag{0.1}$$

where $V_1, V_2 : \mathbb{R}^2 \to \mathbb{R}$ are positive continuous potentials and $h, g : \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions with critical exponential growth in the sense of Trudinger-Moser inequality.

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1 Introduction

The main purpose of this article is to show that, using variational methods based on Orlicz spaces, we are able to obtain sufficient conditions for the existence of least energy solution for quasilinear Schröndiger systems of the form

$$\begin{cases} -\Delta u + V_1(x)u - \Delta(u^2)u = h(x,u,v), & \text{in } \mathbb{R}^2 \\ -\Delta v + V_2(x)v - \Delta(v^2)v = g(x,u,v), & \text{in } \mathbb{R}^2. \end{cases} \tag{1.1}$$

The study of the system (1.1) was in part motivated by the nonlinear Schrödinger equation

$$i\frac{\partial\psi}{\partial t} = -\Delta\psi + W(x)\psi - \eta(|\psi|^2)\psi - \left[\Delta\rho(|\psi|^2)\right]\rho'(|\psi|^2)\psi, \tag{1.2}$$

where $\psi : \mathbb{R}\times\mathbb{R}^N \to \mathbb{C}$, $W : \mathbb{R}^N \to \mathbb{R}$ is a given potential and $\rho,\ \eta : \mathbb{R}_+ \to \mathbb{R}$ are suitable functions. Quasilinear equations of the form (1.2) have received much attention in recent years and appear naturally in mathematical physics. They have been derived as models of several physical phenomena corresponding to various types of nonlinear term ρ. For instance, when $\rho(s) = s$, equation (1.2) can be used to model a superfluid film equation in *plasma physics* (see Kurihara [12] and [13]). For more mathematical models in physics described by (1.2), see [15] and references therein. In the case $\rho(s) = s$, the interest is to research by *standing wave solutions* of (1.2), that is, solutions of the type

$$\psi(t,x) = \exp(-iEt)u(x),$$

where $E \in \mathbb{R}$ and u is a real function. It is well known that ψ satisfies (1.2), with $\rho(s) = s$, if and only if the function $u(x)$ solves the following equation of elliptic type which possesses a formal variational structure:

$$-\Delta u + V(x)u - \Delta(u^2)u = p(u), \quad x \in \mathbb{R}^N, \tag{1.3}$$

where $V(x) = W(x) - E$ and $p(u) = \eta(u^2)u$, which has been intensively studied in the last years.

In the literature, there are recent mathematical studies on the existence of solutions for systems of the form (1.1) when the dimension $N \geq 3$, we can cite for instance [4, 5, 6, 7, 16]. However, as far as we know, our work is the first that leads with quasilinear systems involving critical growth exponential in the sense of Trudinger-Moser inequality. Therefore, we complement the results contained in the papers previously cited.

Here, the functions $h, g : \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous and satisfy the following conditions:

(H_1) there exists $\alpha_0 > 0$ such that

$$\lim_{|(u,v)|\to\infty} \frac{|h(x,u,v)|}{e^{\alpha(u^2+v^2)^2}} = \lim_{|(u,v)|\to\infty} \frac{|g(x,u,v)|}{e^{\alpha(u^2+v^2)^2}} = \begin{cases} 0, & \forall\, \alpha > \alpha_0 \\ +\infty, & \forall\, \alpha < \alpha_0, \end{cases}$$

uniformly in $x \in \mathbb{R}^2$;

(H_2) there exists $\theta > 4$ such that

$$\theta F(x,u,v) \leqslant U.\nabla F(x,u,v) \quad \forall\, U = (u,v) \in \mathbb{R}^2,$$

where $F : \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is of class C^1 and $\nabla F = (h, g)$;

(H_3) $\displaystyle\lim_{(u,v)\to(0,0)} \frac{h(x,u,v)}{|(u,v)|} = \lim_{(u,v)\to(0,0)} \frac{g(x,u,v)}{|(u,v)|} = 0$, uniformly in $x \in \mathbb{R}^2$;

(H_4) there exist $\xi > 0$ e $q > 2$ such that

$$F(x,t,t) \geqslant \xi t^q \quad \text{for all} \quad t \geqslant 0 \quad \text{and} \quad x \in \mathbb{R}^2,$$

with

$$\xi \geqslant \max\left\{ \xi_1, \frac{2\xi_1}{q} \left(\frac{4\xi_1 (q-2)\alpha_0 \theta}{2^{\frac{q}{2}}(\theta - 4) q} \right)^{\frac{q-2}{2}} \right\},$$

where $\xi_1 := 2^{\frac{q+2}{2}}(2 + M_1 + M_2)$, $M_1 = \max_{x\in\overline{B_2}} V_1(x)$, $M_2 = \max_{x\in\overline{B_2}} V_2(x)$ and $B_2 = B_2(0)$.

With respect to the potentials $V_1, V_2 : \mathbb{R}^2 \to \mathbb{R}$, we require that they are continuous and fulfill the conditions

(ν_1) $0 < V_0 := \min\left\{\inf_{\mathbb{R}^2} V_1, \inf_{\mathbb{R}^2} V_2\right\}$;

(ν_2) there exists $M_0 > 0$ such that for all $M \geqslant M_0$,

$$\mu(\{x \in \mathbb{R}^2; V_i(x) \leqslant M\}) < \infty, \quad i = 1, 2,$$

where $\mu(A)$ denotes the Lebesgue measure of a measurable subset $A \subset \mathbb{R}^2$.

Hereafter, $H^1(\mathbb{R}^2)$ denotes the usual Sobolev space and we consider the space $E := H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ endowed with the norm

$$\|(u,v)\|_E = (\|u\|_{1,2}^2 + \|v\|_{1,2}^2)^{\frac{1}{2}},$$

where $\|u\|_{1,2} := (\|\nabla u\|_2^2 + \|u\|_2^2)^{1/2}$ is the usual norm of $H^1(\mathbb{R}^2)$. We say that a pair (u,v) is a weak solution for the System (1.1) if $(u,v) \in E \cap [L^\infty_{loc}(\mathbb{R}^2)]^2$ and for all $\varphi, \psi \in C_0^\infty(\mathbb{R}^2)$ it holds

$$\int_{\mathbb{R}^2} [(1+2u^2)\nabla u \nabla \varphi + (1+2v^2)\nabla v \nabla \psi + 2u|\nabla u|^2\varphi + 2v|\nabla v|^2\psi$$
$$+ V_1(x)u\varphi + V_2(x)v\psi] = \int_{\mathbb{R}^2} [h(x,u,v)\varphi + g(x,u,v)\psi].$$

By condition (H_3), we can see that $h(x,0,0) = g(x,0,0) = 0$ and hence $(0,0)$ is the trivial solution of (1.1). Thus, we look for by nontrivial solutions for (1.1). Our first main result has the following statement:

Theorem 1.1. *Suppose that $(H_1) - (H_4)$ and $(\nu_1) - (\nu_2)$ are satisfied. Then, System (1.1) has a weak nontrivial solution.*

We are also interested in the existence of least energy solution for (1.1), that is, a solution whose energy is minimal among all others nontrivial solutions. For that, we need to consider another hypothesis on h and g, namely,

(H_5) For each $x \in \mathbb{R}^2$ and $t > 0$, $\frac{h(x,s,t)}{s^3}$ is nondecreasing in $s > 0$ and, for each $x \in \mathbb{R}^2$ and $s > 0$, $\frac{g(x,s,t)}{t^3}$ is nondecreasing in $t > 0$.

Now, our second result can be stated as follows:

Theorem 1.2. *Under the same hypotheses of Theorem 1.1 and supposing condition* (H_5)*, the solution obtained in Theorem 1.1 is a least energy solution.*

A main difficulty in treating this class of quasilinear Schrödinger systems in $\mathbb{R}^2$ is the lack of compactness, typical for elliptic systems in unbounded domains, the critical exponential growth of the nonlinearities g, h and the appearance of the terms $\Delta(u^2)u$ and $\Delta(v^2)v$, which cause problems in using the minimax techniques. To overcome these difficulties that have arisen from these features, we present an approach based on a convenient Orlicz space and Trudinger-Moser inequality in the whole $\mathbb{R}^2$.

The underling idea for proving Theorem 1.1: motivated by arguments used in [8, 10, 14], we use a change of variables to reformulate the problem, obtaining a semilinear one which has an associated functional well defined and Gateaux differentiable in a suitable Orlicz space. This functional satisfies the geometric hypotheses of the Mountain-Pass Theorem and by using the Trudinger-Moser inequality we are able to show a compactness condition on an interval. We achieve the existence results by using a version of the Mountain-Pass Theorem, which is a consequence of the Ekeland Variational Principle.

The outline of this paper is as follows: in the forthcoming section is the reformulation of the problem and some preliminary results, including the appropriate variational setting to study the quasilinear system, the regularity of the dual energy functional and properties of its critical points. In Section 3, we obtain the main properties of the Orlicz space that we consider in our approach. Section 4 is dedicated to the proof of some technical results involving the geometric conditions of a version of the Mountain-Pass Theorem, as well as the campactness condition of the dual functional. In Section 5, we derive an important estimate for the minimax level of the functional. Finally, the last two sections are dedicated to the proof of the main results of this work.

In this work, C, C_0, C_1, C_2, ... denote positive (possibly different) constants and, for $1 \leq p < \infty$, $L^p(\mathbb{R}^N)$ is the usual Lebesgue space with norm $\|u\|_p := \left(\int_{\mathbb{R}^N} |u|^p \, dx\right)^{1/p}$. We denote $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$ by $[L^p(\mathbb{R}^N)]^2$ and for $(u,v) \in [L^p(\mathbb{R}^N)]^2$ we consider the norm

$$\|(u,v)\|_p = (\|u\|_p^2 + \|v\|_p^2)^{1/2}.$$

We also write $\int_{\mathbb{R}^N} u$ instead of $\int_{\mathbb{R}^N} u(x)dx$ and $|U|$ will denote the Euclidian norm of $U \in \mathbb{R}^2$.

2 Preliminary Results

In this section, we introduce some facts that will be useful in the sequel. By the conditions (H_1) and (H_3), for each $\varepsilon > 0$, $q > 2$ and $\alpha > \alpha_0$, there exists $C = C(\varepsilon, q, \alpha) > 0$ such that

$$|h(x,u,v)| + |g(x,u,v)| \leqslant \varepsilon |(u,v)|^2 + C|(u,v)|^{q-1}(e^{\alpha(u^2+v^2)^2} - 1) \quad (2.1)$$

and, consequently, for all $(x,u,v) \in \mathbb{R}^2 \times \mathbb{R}^2$

$$|F(x,u,v)| \leqslant \frac{\varepsilon}{2}|(u,v)|^2 + C_1|(u,v)|^q(e^{\alpha(u^2+v^2)^2} - 1). \quad (2.2)$$

Associated to the System (1.1), we have the formal functional I given by

$$I(u,v) = \frac{1}{2}\int_{\mathbb{R}^2} [(1+2u^2)|\nabla u|^2 + (1+2v^2)|\nabla v|^2 + V_1(x)u^2 + V_2(x)v^2] - \int_{\mathbb{R}^2} F(x,u,v),$$

where $\nabla F(x,u,v) = (h(x,u,v), g(x,u,v))$. Observe that this functional is not well defined in E, because the integral $\int_{\mathbb{R}^2}(|u|^2|\nabla u|^2)$ may not be finite if $u \in H^1(\mathbb{R}^2)$. To overcome this difficulty, as in [8, 16], we use the change of variable $z = f^{-1}(u)$ and $w = f^{-1}(v)$, where f is defined by

$$f'(t) = \frac{1}{[1+2f^2(t)]^{\frac{1}{2}}}, \quad t \in [0, +\infty)$$
$$f(t) = -f(-t), \qquad t \in (-\infty, 0].$$

For an easy reference, next we collect some properties of the function f (see proofs for example in [8, 11]).

Lemma 2.1. *The function $f(t)$ and its derivative enjoy the following properties:*

(1) *f is uniquely defined C^∞ function and invertible;*

(2) $|f'(t)| \leqslant 1$ *for all* $t \in \mathbb{R}$;

(3) $|f(t)| \leqslant t$ *for all* $t \in \mathbb{R}$;

(4) $\frac{f(t)}{t} \to 1$ *as* $t \to 0$;

(5) $\frac{f(t)}{\sqrt{t}} \to 2^{1/4}$ *as* $t \to +\infty$;

(6) $\frac{f(t)}{2} \leqslant tf'(t) \leqslant f(t)$ *for all* $t \geqslant 0$ *and* $\frac{f^2(t)}{2} \leqslant tf(t)f'(t) \leqslant f^2(t)$ *for all* $t \in \mathbb{R}$;

(7) $|f(t)| \leqslant 2^{1/4}|t|^{1/2}$ *for all* $t \in \mathbb{R}$;

(8) *The function $f^2(t)$ is strictly convex;*

(9) *There is a positive constant C such that*

$$|f(t)| \geqslant \begin{cases} C|t|, |t| \leqslant 1; \\ C|t|^{1/2}, |t| \geqslant 1; \end{cases} \tag{2.3}$$

(10) *There are positive constants C_1 and C_2 such that*

$$|t| \leqslant C_1|f(t)| + C_2|f(t)|^2 \quad for \quad all \quad t \in \mathbb{R};$$

(11) $|f(t)f'(t)| \leqslant \frac{1}{\sqrt{2}}$ *for all* $t \in \mathbb{R}$;

(12) $f^2(\rho t) \leqslant \rho^2 f^2(t)$ *for all* $0 \leqslant \rho \leqslant 1$ *and* $t \in \mathbb{R}$;

(13) $f^2(\rho t) \leqslant \rho f^2(t)$ *for all* $\rho \geqslant 1$ *and* $t \in \mathbb{R}$.

Through the change of variable $z = f^{-1}(u)$ and $w = f^{-1}(v)$, we get a new functional, namely,

$$\begin{aligned} J(z,w) &:= I(f(z), f(w)) \\ &= \frac{1}{2}\int_{\mathbb{R}^2}(|\nabla z|^2 + |\nabla w|^2) + \frac{1}{2}\int_{\mathbb{R}^2}[V_1(x)f^2(z) + V_2(x)f^2(w)] \\ &\quad - \int_{\mathbb{R}^2} F(x, f(z), f(w)), \end{aligned}$$

In view of the inequality

$$ab - 1 \leqslant \frac{1}{2}(a^2-1) + \frac{1}{2}(b^2-1), \quad \forall\, a, b \geq 0, \tag{2.4}$$

property (7) of Lemma 2.7 and Trudinger-Moser Inequality in all $\mathbb{R}^2$ (see [3, 9]), for each $\alpha > 0$ and $z, w \in H^1(\mathbb{R}^2)$, we have

$$\int_{\mathbb{R}^2}(e^{\alpha(f^2(z)+f^2(w))^2} - 1) \leqslant \frac{1}{2}\int_{\mathbb{R}^2}(e^{8\alpha z^2} - 1) + \frac{1}{2}\int_{\mathbb{R}^2}(e^{8\alpha w^2} - 1) < \infty. \tag{2.5}$$

Next, we consider the Orlicz space

$$W = \left\{(z,w) \in E; \int_{\mathbb{R}^2}[V_1(x)f^2(z) + V_2(x)f^2(w)] < \infty\right\},$$

endowed with the norm

$$\|(z,w)\|_W = \|z\|_{V_1} + \|w\|_{V_2}, \quad (z,w) \in W,$$

where

$$\|z\|_{V_1} := \|\nabla z\|_2 + \inf_{\xi>0} \frac{1}{\xi}\left[1 + \int_{\mathbb{R}^2} V_1(x) f^2(\xi z)\right] \tag{2.6}$$

and

$$\|w\|_{V_2} := \|\nabla w\|_2 + \inf_{\xi>0} \frac{1}{\xi}\left[1 + \int_{\mathbb{R}^2} V_2(x) f^2(\xi w)\right]. \tag{2.7}$$

According to Proposition 3.1 (see Section 3), $(W, \|(\cdot,\cdot)\|_W)$ is a Banach space. Moreover, under the assumptions (H_1) and (H_3), as in [16] we can see that the J functional possesses the following properties:

(1) J is well defined in W;

(2) J is continuous in W;

(3) J is Gateaux-differentiable in W and for $(z,w),(\varphi,\psi) \in W$ its derivative is given by

$$\begin{aligned}&\langle J'(z,w),(\phi,\psi)\rangle \\ &= \int_{\mathbb{R}^2} [\nabla z \nabla \phi + \nabla w \nabla \psi] + \int_{\mathbb{R}^2} \Big[V_1(x) f(z) f^{'}(z)\phi + V_2(x) f(w) f^{'}(w)\psi\Big] \\ &\quad - \int_{\mathbb{R}^2} [(h(x,f(z),f(w))\phi + g(x,f(z),f(w))\psi)]\,;\end{aligned}$$

(4) For $(z,w) \in W$, $J'(z,w) \in W'$ (dual space of W) and if $(z_n,w_n) \to (z,w)$ in W then $J'(z_n,w_n) \rightharpoonup J'(z,w)$ in the weak topology-$*$ of W', that is, for each $(\varphi,\psi) \in W$ we have

$$\langle J'(z_n,w_n),(\phi,\psi)\rangle \to \langle J'(z,w),(\phi,\psi)\rangle.$$

Notice that $J(z,w)$ is the Euler-Lagrange functional associated with the following elliptical system:

$$\begin{cases} -\Delta z + V_1(x) f(z) f'(z) = h(x,f(z),f(w)) f'(z), & \text{in } \mathbb{R}^2 \\ -\Delta w + V_2(x) f(w) f'(w) = g(x,f(z),f(w)) f'(w), & \text{in } \mathbb{R}^2. \end{cases} \tag{2.8}$$

Furthermore, it can be shown that if $(z,w) \in W$ is a critical point of J, then (u,v), where $u = f(z)$ and $v = f(w)$, is a weak solution of System (1.1) (see Proposition 2.5 in [16]). Therefore, in order to obtain our existence result, it is sufficient to show that J has a nontrivial critical point.

Before we finish this section, note that as an immediate consequence of items (12) and (13) of Lemma 2.7, we have

$$\min\{\rho,\rho^2\} f^2(t) \leqslant f^2(\rho t) \leqslant \max\{\rho,\rho^2\} f^2(t), \quad \forall\, t \geqslant 0. \tag{2.9}$$

Also, taking $t = 1$ in item (6) of Lemma 2.7, we get $f^2(1) \geqslant [f'(1)]^2 = 1/(1+2f^2(1))$, that is, $2f^4(1) + f^2(1) \geqslant 1$. Since $f(1) > 0$, a simple calculation shows that $f(1) \geqslant 1/\sqrt{2}$ and it is immediate to check again by (6) of Lemma 2.7 that $f(t)/t$ is decreasing. Therefore,

$$f(t) \geqslant \frac{1}{\sqrt{2}} t, \quad \forall \;\; t \in [0,1]. \tag{2.10}$$

3 Some properties of the space W

In this section, concisely we present some important properties of the space W that are relevant in our arguments for proving the existence of nontrivial weak solutions for (1.1). First, we consider

$$X := \left\{(u,v) \in E; \int_{\mathbb{R}^2} [V_1(x)u^2 + V_2(x)v^2] < \infty\right\},$$

which is a Hilbert space endowed with the inner product given by

$$\langle (u_1,v_1),(u_2,v_2)\rangle = \int_{\mathbb{R}^2} (\nabla u_1 \nabla u_2 + \nabla v_1 \nabla v_2 + V_1(x)u_1u_2 + V_2(x)v_1v_2),$$

whose corresponding norm is

$$\|(u,v)\|_X^2 = \int_{\mathbb{R}^2} [|\nabla u|^2 + |\nabla v|^2 + V_1(x)u^2 + V_2(x)v^2], \quad (u,v) \in X.$$

As in [2], we can see that the embedding

$$X \hookrightarrow L^q(\mathbb{R}^2) \times L^q(\mathbb{R}^2) \ \text{ is compact for all } \ q \in [2,\infty). \tag{3.1}$$

The main properties of W are presented below.

Proposition 3.1. *(i) W is a normed space with respect to the norm given in (2.6);*

(ii) There exists a positive constant C such that, for all $(u,v) \in W$,

$$\frac{\int_{\mathbb{R}^2}(V_1(x)f^2(u) + V_2(x)f^2(v))}{1 + [\int_{\mathbb{R}^2}(V_1(x)f^2(u) + V_2(x)f^2(v))]^{\frac{1}{2}}} \leqslant C\|(u,v)\|; \tag{3.2}$$

(iii) If $(u_n, v_v) \to (u,v)$ in W, then

$$\int_{\mathbb{R}^2} V_1(x)|f^2(u_n) - f^2(u)| + \int_{\mathbb{R}^2} V_2(x)|f^2(v_n) - f^2(v)| \to 0$$

and

$$\int_{\mathbb{R}^2} V_1(x)|f(u_n) - f(u)|^2 + \int_{\mathbb{R}^2} V_2(x)|f(v_n) - f(v)|^2 \to 0;$$

(iv) If $u_n \to u$ and $v_n \to v$ almost everywhere in $\mathbb{R}^2$ and

$$\int_{\mathbb{R}^2} (V_1(x)f^2(u_n) + V_2(x)f^2(v_n)) \to \int_{\mathbb{R}^2} (V_1(x)f^2(u) + V_2(x)f^2(v)),$$

then

$$\inf_{\xi>0} \frac{1}{\xi} \left[1 + \int_{\mathbb{R}^2} V_1(x)f^2(\xi(u_n - u)) + V_2(x)f^2(\xi(v_n - v))\right] \to 0.$$

Proof. It is just an adaptation of the proof of Proposition 2.4 in [10] and we omit. □

Proposition 3.2. *The application $(u, v) \to (f(u), f(v))$ from W into $L^q(\mathbb{R}^2) \times L^q(\mathbb{R}^2)$ is continuous for each $2 \leqslant q < \infty$.*

Proof. It is similar to the proof of Proposition 2.2 in [11]. □

Proposition 3.3. *If (ν_1) and (ν_2) are satisfied, then the map $(u, v) \to (f(u), f(v))$ from W into $L^q(\mathbb{R}^2) \times L^q(\mathbb{R}^2)$ is compact for all $2 \leqslant q < \infty$.*

Proof. Let $(u_n, v_n) \subset W$ be a bounded sequence in W. Thus, $(\|\nabla u_n\|_2 + \|\nabla v_n\|_2)$ is bounded and by (3.2) it follows that $\int_{\mathbb{R}^2}(V_1(x)f^2(u_n) + V_2(x)f^2(v_n))$ is also bounded. Thus, $(f(u_n), f(v_n))$ is bounded in X and by using the compact embedding (3.1), there exists $(w_1, w_2) \in L^q(\mathbb{R}^2) \times L^q(\mathbb{R}^2)$ such that $(f(u_n), f(v_n)) \to (w_1, w_2)$ in $L^q(\mathbb{R}^2) \times L^q(\mathbb{R}^2)$ and the proof is done. □

Proposition 3.4. *W is a Banach space with respect to the norm*

$$\|(u, v)\|_W = \|u\|_{V_1} + \|v\|_{V_2}.$$

Proof. It is similar to the proof of Proposition 2.7 in [9]. □

4 Palais-Smale Condition

In this section, our main goal is to show that the functional J satisfies the Palais-Smale condition in a convenient interval. For this, we need some lemmas.

Lemma 4.1. *Suppose that* (H_1) *and* (H_3) *are satisfied. Given* $\alpha > \alpha_0$, $q \geqslant 2$ *and* $0 < \rho < \sqrt{\pi/(2\alpha)}$, *if* $\|(u,v)\|_W \leqslant \rho$ *then*

$$\int_{\mathbb{R}^2} (e^{\alpha(|f(u)|^2+|f(v)|^2)^2} - 1)|(f(u),f(v))|^q \leqslant C\|(f(u),f(v))\|_X^q. \qquad (4.1)$$

Proof. By using (7) of Lemma 2.7, (2.5), Hölder's Inequality and (3.1), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^2} (e^{\alpha(|f(u)|^2+|f(v)|^2)^2} - 1)|(f(u),f(v))|^q \\
&\leqslant \int_{\mathbb{R}^2} (e^{2\alpha(2|u|^2+2|v|^2)} - 1)|(f(u),f(v))|^q \\
&\leqslant \left[\int_{\mathbb{R}^2} (e^{4\alpha(u^2+v^2)} - 1)\right]^{1/2} \left[\int_{\mathbb{R}^2} |(f(u),f(v))|^{2q}\right]^{1/2} \\
&= \left[\int_{\mathbb{R}^2} \frac{1}{2}(e^{8\alpha u^2} - 1) + \int_{\mathbb{R}^2} \frac{1}{2}(e^{8\alpha v^2} - 1)\right]^{1/2} \|(f(u),f(v))\|_{2q}^q \\
&\leqslant C_1 \left[\int_{\mathbb{R}^2} \frac{1}{2}(e^{8\alpha\|u\|^2\left(\frac{u}{\|u\|}\right)^2} - 1) + \int_{\mathbb{R}^2} \frac{1}{2}(e^{8\alpha\|v\|^2\left(\frac{v}{\|v\|}\right)^2} - 1)\right] \|(f(u),f(v))\|_X^q.
\end{aligned}$$

Since $8\alpha\|u\|^2 \leqslant 8\alpha\|(u,v)\|^2 \leqslant 8\alpha\rho^2 < 4\pi$ and similarly $8\alpha\|v\|^2 < 4\pi$, by invoking the Trundinger-Moser inequality, the proof is done. □

Now, for $\rho > 0$ consider the set

$$S_\rho := \{(u,v) \in W;\ Q(u,v) = \rho^2\},$$

where $Q : W \to \mathbb{R}$ is given by $Q(u,v) := Q_1(u) + Q_2(v)$,

$$Q_1(u) = \int_{\mathbb{R}^2} (|\nabla u|^2 + V_1(x) f^2(u)) \quad \text{and} \quad Q_2(v) = \int_{\mathbb{R}^2} (|\nabla v|^2 + V_2(x) f^2(v)).$$

Since $Q(u,v)$ is continuous, then S_ρ is a closed subset that disconnects the space W.

Lemma 4.2. *The functional* J *satisfies the following geometric conditions:*

i) there exist $\rho, \sigma_0 > 0$ *such that* $J(u,v) \geq \sigma_0$ *for all* $(u,v) \in S_\rho$;

ii) there exists $(u_0,v_0) \in W$ *satisfying* $Q(u_0,v_0) > \rho^2$ *and* $J(u_0,v_0) < 0$.

Proof. The proof is standard (see [16]). □

Next, we will obtain some properties of the sequence $(PS)_c$ associated with the functional energy J.

Proposition 4.3. *If $(u_n, v_n) \subset W$ is a Palais-Smale sequence for the functional J at level $c \in \mathbb{R}$, then (u_n, v_n) is bounded in W.*

Proof. Let $(u_n, v_n) \subset W$ be a Palais-Smale sequence for J at the level c, that is,

$$J(u_n, v_n) \to c \ \ \text{and} \ \ J'(u_n, v_n) \to 0. \tag{4.2}$$

Thus,

$$|J'(u_n, v_n)(u_n, v_n)| \leqslant o_n(1)\|(u_n, v_n)\|_W. \tag{4.3}$$

By (6) of Lemma 2.7, (H_2) and (4.3), we obtain

$$\begin{aligned}
&c + o_n(1) + o_n(1)\|(u_n, v_n)\|_W \\
&\geqslant J(u_n, v_n) - \frac{2}{\theta} J'(u_n, v_n)(u_n, v_n) \\
&\geqslant \left(\frac{1}{2} - \frac{2}{\theta}\right) \left[\int_{\mathbb{R}^2} (|\nabla u_n|^2 + |\nabla v_n|^2) + \int_{\mathbb{R}^2} (V_1(x) f^2(u_n) + V_2(x) f^2(v_n))\right]
\end{aligned}$$

and therefore

$$\begin{aligned}
&\left(\frac{1}{2} - \frac{2}{\theta}\right) \left[\int_{\mathbb{R}^2} (|\nabla u_n|^2 + |\nabla v_n|^2) + \int_{\mathbb{R}^2} (V_1(x) f^2(u_n) + V_2(x) f^2(v_n))\right] \\
&\leqslant c + o_n(1) + o_n(1) \left(\int_{\mathbb{R}^2} (|\nabla u_n|^2 + |\nabla v_n|^2)\right)^{1/2} + o_n(1) \int_{\mathbb{R}^2} V_1(x) f^2(u_n) \\
&\quad + o_n(1) \int_{\mathbb{R}^2} V_2(x) f^2(v_n).
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\left(\frac{1}{2} - \frac{2}{\theta}\right) \int_{\mathbb{R}^2} (|\nabla u_n|^2 + |\nabla v_n|^2) \\
&+ \left(\frac{1}{2} - \frac{2}{\theta} - o_n(1)\right) \int_{\mathbb{R}^2} (V_1(x) f^2(u_n) + V_2(x) f^2(v_n)) \\
&\leqslant c + o_n(1) + o_n(1) \left(\int_{\mathbb{R}^2} (|\nabla u_n|^2 + |\nabla v_n|^2)\right)^{1/2}
\end{aligned} \tag{4.4}$$

which shows that $\int_{\mathbb{R}^2}(|\nabla u_n|^2 + |\nabla v_n|^2)$ is bounded. In view of (4.4) we have $\left(\int_{\mathbb{R}^2}\left[V_1(x)f^2(u_n) + V_2(x)f^2(v_n)\right]\right)$ is also bounded. Now, since

$$\|(u_n, v_n)\|_W \leqslant \left(\int_{\mathbb{R}^2}(|\nabla u_n|^2 + |\nabla v_n|^2)\right)^{1/2} + 1 + \int_{\mathbb{R}^2}(V_1(x)f^2(u_n) + V_2(x)f^2(v_n))$$

we conclude that (u_n, v_n) is bounded in W. □

Before proving the Palais-Smale condition, let us show the following corollary:

Corollary 4.4. *If $(u_n, v_n) \subset W$ is a Palais-Smale sequence for J at level c, then*

$$Q(u_n, v_n) \leqslant \frac{4\theta}{\theta - 4}c + o_n(1). \tag{4.5}$$

Proof. Since (u_n, v_n) is bounded in W, the conclusion follows directly from (4.4). □

Lemma 4.5. *Let (u_n, v_n) be a Palais-Smale sequence for J at level $c \in \mathbb{R}$ with*

$$c < \frac{(\theta - 4)\pi}{2\alpha_0\theta}.$$

If $(u_n, v_n) \rightharpoonup (u, v)$ in W, then

$$\int_{\mathbb{R}^2} h(x, f(u_n), f(v_n))f'(u_n)(u_n - u) \to 0 \quad \text{and}$$
$$\int_{\mathbb{R}^2} g(x, f(u_n), f(v_n))f'(v_n)(v_n - v) \to 0.$$

Proof. We will prove the first convergence, the proof the second one is analogous. Using (H_1), (H_3) and (2), (7), (11) of Lemma 2.7, given $\varepsilon > 0$

and $\alpha > \alpha_0$, we get

$$\begin{aligned}
&\left|\int_{\mathbb{R}^2} h(x, f(u_n), f(v_n)) f'(u_n)(u_n - u)\right| \leqslant \int_{\mathbb{R}^2} |h(x, u_n, v_n)||f'(u_n)||u_n - u| \\
&\leqslant \varepsilon \int_{\mathbb{R}^2} |(f(u_n), f(v_n)||u_n - u| + C \int_{\mathbb{R}^2} (e^{2\alpha(f(u_n)^4 + f(v_n)^4)} - 1)|u_n - u| \\
&\leqslant \varepsilon \left(\int_{\mathbb{R}^2} |f(u_n), f(v_n)|^2\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} |u_n - u|^2\right)^{\frac{1}{2}} \\
&\quad + C \left(\int_{\mathbb{R}^2} (e^{4\alpha r_1 (u_n^2 + v_n^2)} - 1)\right)^{\frac{1}{r_1}} \left(\int_{\mathbb{R}^2} |u_n - u|^{r_2}\right)^{\frac{1}{r_2}} \\
&\leqslant C\|u_n - u\|_2 + C \left(\int_{\mathbb{R}^2} (e^{8\alpha r_1 u_n^2} - 1) + \int_{\mathbb{R}^2} (e^{8\alpha r_1 v_n^2} - 1)\right)^{\frac{1}{r_1}} \|u_n - u\|_{r_2},
\end{aligned} \tag{4.6}$$

where $r_1, r_2 > 1$ and $\frac{1}{r_1} + \frac{1}{r_2} = 1$. If the sequences

$$\left(\int_{\mathbb{R}^2} (e^{8r_1 \alpha u_n^2} - 1)\right) \quad \text{and} \quad \left(\int_{\mathbb{R}^2} (e^{8r_1 \alpha v_n^2} - 1)\right) \tag{4.7}$$

are bounded, then by (4.6) it follows that

$$\left|\int_{\mathbb{R}^2} h(x, f(u_n), f(v_n)) f'(u_n)(u_n - u)\right| \to 0, \tag{4.8}$$

since $u_n \to u$ in $L^q(\mathbb{R}^2)$ for all $q \geqslant 2$. Similarly,

$$\left|\int_{\mathbb{R}^2} g(x, f(u_n), f(v_n)) f'(v_n)(v_n - v)\right| \to 0.$$

We will prove the boundedness of the first integral in (4.7), the other is analogous. We can write

$$\int_{\mathbb{R}^2} (e^{8\alpha r_1 u_n^2} - 1) = \int_{\mathbb{R}^2} (e^{8\alpha r_1 Q_1(u_n)\widetilde{u}_n^2} - 1),$$

with $\widetilde{u}_n = u_n / \sqrt{Q_1(u_n)}$. Note that

$$\|\nabla \widetilde{u}_n\|_2^2 = \int_{\mathbb{R}^2} |\nabla \widetilde{u}_n|^2 = \frac{1}{\int_{\mathbb{R}^2} |\nabla u_n|^2 + \int_{\mathbb{R}^2} V_1(x) f^2(u_n)} \int_{\mathbb{R}^2} |\nabla u_n|^2 \leqslant 1.$$

Moreover, we claim $\|\widetilde{u}_n\|_2 \leq M < \infty$. Indeed,

$$Q_1(\widetilde{u}_n) = \frac{1}{Q_1(u_n)} \int_{\mathbb{R}^2} |\nabla u_n|^2 + \int_{\mathbb{R}^2} V_1(x) f^2 \left(\frac{1}{\sqrt{Q_1(u_n)}} u_n\right).$$

By using (2.9), we get

$$\begin{aligned} Q_1(\widetilde{u}_n) &\leqslant \left(\frac{1}{Q_1(u_n)} + \frac{1}{\sqrt{Q_1(u_n)}}\right)\int_{\mathbb{R}^2} |\nabla u_n|^2 \\ &\quad + \left(\frac{1}{Q_1(u_n)} + \frac{1}{\sqrt{Q_1(u_n)}}\right)\int_{\mathbb{R}^2} V_1(x) f^2(u_n) \\ &= \left(\frac{1}{Q_1(u_n)} + \frac{1}{\sqrt{Q_1(u_n)}}\right) Q_1(u_n) = 1 + \sqrt{Q_1(u_n)} \leqslant 1 + C. \end{aligned}$$

Once $\|f(u)\|_q \leqslant C\sqrt{Q_1(u)}$ for $q \geqslant 2$ and for all $u \in H^1(\mathbb{R}^2)$ with $\int_{\mathbb{R}^2} V_1(x) f^2(u) < \infty$, we conclude that $\|f(\widetilde{u}_n)\|_q \leqslant C$ for all $n \in \mathbb{N}$. Using item (10) of Lemma 2.7, it follows that $\|\widetilde{u}_n\|_2 \leqslant C$. Now, in view of Corollary 4.4

$$Q_1(u_n) \leqslant Q(u_n, v_n) \leqslant \frac{2\theta}{\theta - 4} c + o_n(1).$$

Since $c < \frac{(\theta-4)\pi}{4\alpha_0\theta}$, we can choose $\alpha > \alpha_0$ close to α_0 and $r_1 > 1$ close to 1 so that $c\alpha r_1 < \frac{(\theta-4)\pi}{4\theta\alpha_0}$. Thus, for n sufficiently large

$$8\alpha r_1 Q_1(u_n) \leqslant 16\alpha \frac{\theta}{\theta - 4} r_1 c + o_n(1) < 4\pi.$$

By invoking the Trudinger-Moser inequality, the boundedness of the first integral in (4.7) is done and the proof is complete. □

Proposition 4.6. *The functional J satisfies the Palais-Smale condition at level $c \in \mathbb{R}$ for all $c < \frac{(\theta-4)\pi}{2\alpha_0\theta}$.*

Proof. Let (u_n, v_n) be a Palais-Smale sequence for J at the level c, with $c < \frac{(\theta-4)\pi}{2\alpha_0\theta}$. By Lemma 4.3, (u_n, v_n) is bounded in W. So, up to a subsequence, $(u_n, v_n) \rightharpoonup (u, v) \in W$. Now, by applying Lemma 4.5, we get

$$\begin{aligned} &\int_{\mathbb{R}^2} h(x, f(u_n), f(v_n)) f'(u_n)(u_n - u) \to 0 \quad \text{and} \\ &\int_{\mathbb{R}^2} g(x, f(u_n), f(v_n)) f'(v_n)(v_n - v) \to 0. \end{aligned} \tag{4.9}$$

Since f^2 is convex, $Q(u,v)$ is also convex and therefore

$$\begin{aligned}\frac{1}{2}Q(u,v) - \frac{1}{2}Q(u_n,v_n) &\geqslant \frac{1}{2}Q'(u_n,v_n).(u-u_n,v-v_n)\\ = J'(u_n,v_n).(u-u_n,v-v_n) &+ \int_{\mathbb{R}^2} h(x,f(u_n),f(v_n))f'(u_n)(u_n-u)\\ &+ \int_{\mathbb{R}^2} g(x,f(u_n),f(v_n))f'(v_n)(v_n-v).\end{aligned}$$

Thus, from the convergences (4.9) and $\|J'(u_n,v_n)\| \to 0$ we deduce that

$$\begin{aligned}&\int_{\mathbb{R}^2} \left[|\nabla u|^2 + |\nabla v|^2\right] + \int_{\mathbb{R}^2} \left[V_1(x)f^2(u) + V_2(x)f^2(v)\right]\\ &\geqslant \liminf_{n\to\infty} \int_{\mathbb{R}^2} [|\nabla u_n|^2 + |\nabla v_n|^2]\\ &\qquad + \liminf_{n\to\infty} \int_{\mathbb{R}^2} \left[V_1(x)f^2(u_n) + V_2(x)f^2(v_n)\right].\end{aligned} \tag{4.10}$$

On the other hand, Fatou's Lemma implies that

$$\begin{aligned}&\int_{\mathbb{R}^2} (V_1(x)f^2(u) + V_2(x)f^2(v))\\ &\qquad \leqslant \liminf_{n\to\infty} \int_{\mathbb{R}^2} (V_1(x)f^2(u_n) + V_2(x)f^2(v_n)),\end{aligned} \tag{4.11}$$

and since the functional $\Phi(u) := \int_{\mathbb{R}^2} |\nabla u|^2$ is weakly lower semicontinuous, we obtain

$$\int_{\mathbb{R}^2} (|\nabla u|^2 + |\nabla v|^2) \leqslant \liminf_{n\to\infty} \int_{\mathbb{R}^2} (|\nabla u_n|^2 + |\nabla v_n|^2). \tag{4.12}$$

By (4.10), (4.11) and (4.12), up to subsequences, we can deduce that

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} (|\nabla u_n|^2 + |\nabla v_n|^2) = \int_{\mathbb{R}^2} (|\nabla u|^2 + |\nabla v|^2)$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} (V_1(x)f^2(u_n) + V_2(x)f^2(v_n)) = \int_{\mathbb{R}^2} (V_1(x)f^2(u) + V_2(x)f^2(v)).$$

Thus, (iv) of Lemma 3.1 implies that

$$\inf_{\xi>0} \frac{1}{\xi}\left[1 + \int_{\mathbb{R}^2} V_1(x)f^2(\xi(u_n-u)) + V_2(x)f^2(\xi(v_n-v))\right] \to 0,$$

which shows that $(u_n,v_n) \to (u,v)$ in W and the proof is finished. □

5 Minimax estimate

In this section, by exploiting condition (H_4), we are going to estimate the minimax level

$$c^* := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C([0,1],E)) \times C([0,1],E); \gamma(0) = (0,0) \ \text{ and } \ J(\gamma(1)) < 0\}.$$

More precisely, we have the following lemma:

Lemma 5.1. *Suppose that (H_4) is satisfied. Then,*

$$c^* < \frac{(\theta-2)\pi}{4\alpha_0\theta}.$$

Proof. First, let $\varphi_0 \in C^{\infty}_{0,rad}(\mathbb{R}^2)\backslash\{0\}$ be such that $0 \leqslant \varphi_0 \leqslant 1$, $\operatorname{supp}(\varphi_0) \subset B_2$, $\varphi_0 \equiv 1$ in B_1 and $|\nabla\varphi_0| \leqslant 1$. If $\xi \geqslant \xi_1$ then by (H_4) and (2.10) we have

$$\begin{aligned} J(\varphi_0,\varphi_0) &< \frac{1}{2}\int_{B_2}(2|\nabla\varphi_0|^2 + V_1(x)\varphi_0^2 + V_2(x)\varphi_0^2) - \xi_1\int_{B_2} f^q(\varphi_0) \\ &\leqslant 4\pi + (M_1+M_2)2\pi - \frac{\xi_1\pi}{2^{\frac{q}{2}}} = 0. \end{aligned}$$

In particular,

$$\frac{1}{2}\left(\int_{B_2}(2|\nabla\varphi_0|^2 + V_1(x)\varphi_0^2 + V_2(x)\varphi_0^2)\right) \leqslant \frac{\xi\pi}{2^{q/2}}.$$

From this inequality and by the definition of φ_0, a simple computation shows that

$$\begin{aligned} c^* &\leqslant \max_{t\in[0,1]} J(t\varphi_0, t\varphi_0) \\ &\leqslant \max_{t\in[0,1]}\left\{\left[\frac{t^2}{2}\left(\int_{B_2}[2|\nabla\varphi_0|^2 + V_1(x)\varphi_0^2 + V_2(x)\varphi_0^2]\right)\right] - \xi\int_{B_2} f^q(t\varphi_0)\right\} \\ &< \max_{t\in[0,1]}\left[\frac{\xi_1\pi t^2}{2^{q/2}} - \frac{\xi\pi t^q}{2^{q/2}}\right] \\ &\leqslant \frac{\pi}{2^{q/2}}\max_{t\in[0,1]}[\xi_1 t^2 - \xi t^q]. \end{aligned}$$

Calculating this maximum, we obtain

$$\frac{\pi}{2^{q/2}}\max_{t\geqslant 0}[\xi_1 t^2-\xi t^q]=\frac{\pi}{2^{q/2}}\frac{\xi_1(q-2)}{q}\left(\frac{2\xi_1}{\xi q}\right)^{\frac{2}{q-2}}.$$

Hence, if

$$\xi\geqslant\frac{2\xi_1}{q}\left[\frac{\xi_1(q-2)4\alpha_0\theta}{2^{q/2}q(\theta-4)}\right]^{\frac{q-2}{2}}$$

then we conclude that

$$c^*<\frac{(\theta-4)\pi}{4\alpha_0\theta}.$$

□

6 Proof of Theorem 1.1

In this section, we prove our first main result. By Lemma 4.2 and Proposition 4.6, we show that $J(u,v)$ has the mountain-pass geometry and satisfies the condition Palais-Smale at level c, for each $c<(\theta-2)\pi/(4\alpha_0\theta)$. Once the mountain-pass level $c^*\in(0,(\theta-2)\pi/4\alpha_0\theta)$, then by invoking a version of the Mountain-Pass Theorem (see [1]) it follows that J has a critical point (z_0,w_0) in W such that $J(z_0,w_0)=c^*$. Therefore, (u_0,v_0), with $u_0=f(z_0)$ and $v_0=f(w_0)$, is a nontrivial solution of System (1.1).

7 Proof of Theorem 1.2

In order to prove Theorem 1.2, we suppose that condition (H_5) is satisfied. Moreover, let

$$b=\inf_{(u,v)\in S}J(u,v)\ \text{ and }\ S=\{(u,v)\in W;(u,v)\neq(0,0)\ \text{ and }\ J'(u,v)=0\}.$$

Let us show that the minimax level c^* of J satisfies $c^*\leq b$. Let (u,v) be in S and define $\xi:(0,\infty)\to\mathbb{R}$ by $\xi(t)=J(tu,tv)$. We have that ξ is differentiable and

$$\begin{aligned}\xi'(t) &= J'(tu,tv).(u,v)\\ &= t\int_{\mathbb{R}^2}(|\nabla u|^2+|\nabla v|^2)+\int_{\mathbb{R}^2}(V_1(x)f(tu)f^{'}(tu)u+V_2(x)f(tv)f^{'}(tv)v)\\ &\quad -\int_{\mathbb{R}^2}\left[h(x,f(tu),f(tv))f^{'}(tu)u+g(x,f(tu),f(tv))f^{'}(tv)v\right].\end{aligned}$$

Since $J'(u,v).(u,v)=0$, we have

$$\begin{aligned}\int_{\mathbb{R}^2}|\nabla u|^2+|\nabla v|^2) &= -\int_{\mathbb{R}^2}(V_1(x)f(u)f'(u)u+V_2(x)f(v)f'(v)v)\\ &\quad+\int_{\mathbb{R}^2}[h(x,f(u),f(v))f'(u)u+g(x,f(u),f(v))f'(v)v]\end{aligned}$$

and therefore we can write

$$\begin{aligned}\xi'(t) &= t\int_{\mathbb{R}^2}V_1(x)\left[\frac{f(tu)f'(tu)}{tu}-\frac{f(u)f'(u)}{u}\right]u^2\\ &\quad+\int_{\mathbb{R}^2}V_2(x)\left[\frac{f(tv)f'(tv)}{tv}-\frac{f(v)f'(v)}{v}\right]v^2\\ &\quad+t\left[\int_{\mathbb{R}^2}\frac{h(x,f(u),f(v))}{f^3(u)}\frac{f^3(u)f^{'}(u)}{u}-\frac{h(x,f(tu),f(tv))}{f^3(tu)}\frac{f^3(tu)f^{'}(tu)}{tu}\right]u^2\\ &\quad+t\left[\int_{\mathbb{R}^2}\frac{g(x,f(u),f(v))}{f^3(v)}\frac{f^3(v)f^{'}(v)}{v}-\frac{g(x,f(tu),f(tv))}{f^3(tv)}\frac{f^3(tv)f^{'}(tv)}{tv}\right]v^2.\end{aligned}$$

Using the properties of f, we can show that $f(s)f^{'}(s)s^{-1}$ is decreasing for $s>0$ (see Corollary 2.3 in [10]). Hence, by hypothesis (H_5), $\xi^{'}(t)>0$ for $0<t<1$, $\xi^{'}(t)<0$ for $t>1$ and $\xi^{'}(1)=0$. This guarantees that

$$J(u,v)=\max_{t\geqslant 0}J(tu,tv).$$

Now, set $\gamma_0:[0,1]\to W$, $\gamma_0(t)=(\gamma_1,\gamma_2)$ with $\gamma_1(t)=tt_0u$ and $\gamma_2(t)=tt_0v$, where t_0 is such that $J(t_0u,t_0v)<0$. We have $\gamma_0\in\Gamma$ and therefore

$$c^*\leqslant\max_{t\geqslant 0}J(\gamma_0(t))\leqslant\max_{t\geqslant 0}J(tu,tv)=J(u,v).$$

Since $(u, v) \in S$ is arbitrary, we have $c^* \leqslant b$ and the proof is complete.

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