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Ground state and nodal solutions for a class of fractional Schrödinger equations involving exponential growth

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Dedicated to Professor Pedro Ubilla
on the occasion of his 60th birthday

Abstract. In this paper, we investigate the existence of ground state and nodal solutions for a class of nonlinear scalar field equations defined on the whole real line involving the $1/2$-Laplacian operator and nonlinearities with subcritical and critical growth in the sense of the Trudinger-Moser inequality. By using the constraint variational method and the quantitative deformation lemma, we obtain two nonzero solutions: one is a ground state solution and the other one is a least energy nodal solution. Moreover, we show that the energy of this nodal solution is strictly larger than twice the ground state energy.

Keywords: Fractional Laplacian, Ground state solution, Nodal solution, Trudinger-Moser inequality.

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1 Introduction

In this work we consider the following fractional Schrödinger equation:

$$(-\Delta)^{\frac{1}{2}}u + V(x)u = K(x)f(u) \quad \text{in} \quad \mathbb{R}, \tag{1.1}$$

where $(-\Delta)^{\frac{1}{2}}$ is the 1/2-Laplacian operator, $V, K : \mathbb{R} \to \mathbb{R}$ are functions satisfying appropriate conditions which will be introduced later and $f : \mathbb{R} \to \mathbb{R}$ has exponential growth in the sense of the Trudinger-Moser embedding due to Ozawa [34]. Our goal in this work is to show that under appropriate conditions problem (1.1) has a ground state and a nodal solution, which are distinct. Throughout this paper, the fractional Laplacian $(-\Delta)^{\frac{1}{2}}$ of a function $u \in \mathcal{S}$ is defined by

$$(-\Delta)^{\frac{1}{2}}u(x) = -\frac{1}{2\pi}\int_{\mathbb{R}} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^2}\,\mathrm{d}y,$$

where $\mathcal{S}$ denotes the Schwartz space of the rapidly decreasing C^∞ functions in $\mathbb{R}$.

A physical motivation for studying this kind of problem is that Eq. (1.1) describes the behavior of the so-called standing wave solutions $\varphi(x,t) = e^{-\frac{iE}{\hbar}t}u(x)$ of the following time-dependent fractional Schrödinger equation:

$$i\hbar\frac{\partial\varphi}{\partial t} = -\frac{\hbar^2}{2m}(-\Delta)^{\frac{1}{2}}\varphi + W(x)\varphi - g(x,|\varphi|)\varphi, \;\; (t,x) \in \mathbb{R}\times\mathbb{R},$$

where $V(x) = \frac{2m}{\hbar}(W(x) - E)$, $K(x)f(u) = g(x,|u|)u$ and u is a solution of (1.1). This equation was introduced by Laskin [29, 30]. We would also like to quote to the reader the paper [19] and some of its references for other applications, and to underline the role played by the potential $V(x)$, we suggest to the reader the papers [13, 22]. In the specific case of applications involving the fractional Laplacian on the real line we would like to highlight applications in dynamical systems and crystal dislocation theory (cf. [14, 17, 18]). As a consequence, the study of nonlinear fractional equations has attracted the attention of many researchers and topics like existence, regularity, symmetry, uniqueness and stability were studied, see for instance, [10, 20, 27] and references contained therein.

We are interested in looking for solutions when the nonlinearity $f(t)$ has the maximal growth which allows to treat problem (1.1) variationally in the Sobolev space $H^{1/2}(\mathbb{R})$. In order to better understanding the critical growth on $f(t)$, let us to recall some well-known facts involving the limiting Sobolev embedding theorem in one-dimension. Let $H^s(\mathbb{R})$ be the Sobolev space with $s \in (0, 1/2)$. The Sobolev embedding states that $H^s(\mathbb{R}) \hookrightarrow L^{2_s^*}(\mathbb{R})$, where $2_s^* := 2/(1-2s)$ (the critical Sobolev exponent), and when $s = 1/2$ we have $H^{1/2}(\mathbb{R}) \hookrightarrow L^q(\mathbb{R})$ for any $q \in [2, +\infty)$, but $H^{1/2}(\mathbb{R})$ is not continuously embedded in $L^\infty(\mathbb{R})$ (see [19, 34]). Thus, if $s \in (0, 1/2)$ then the maximal growth on the nonlinearity $f(t)$, which lets us to work with (1.1) by considering a variational approach in $H^s(\mathbb{R})$, it is given by $|u|^{2_s^*-1}$ as $|u| \to +\infty$. On the other hand, in the limiting case $s = 1/2$, motivated by Ozawa [34], the maximal growth on $f(t)$, which allows us to study (1.1) by applying a variational framework involving the space $H^{1/2}(\mathbb{R})$, is given by $e^{\alpha u^2}$ as $|u| \to +\infty$, for some $\alpha > 0$. Precisely, we say that $f(t)$ has *exponential critical growth* if there exists $\alpha_0 > 0$ such that

$$\lim_{|t|\to+\infty} f(t)e^{-\alpha|t|^2} = \begin{cases} 0, & \text{for all} \quad \alpha > \alpha_0, \\ +\infty, & \text{for all} \quad \alpha < \alpha_0, \end{cases} \tag{1.2}$$

and we say that $f(t)$ has *exponential subcritical growth* if

$$\lim_{|t|\to+\infty} f(t)e^{-\alpha|t|^2} = 0, \quad \text{for all} \quad \alpha > 0. \tag{1.3}$$

Based on this notion of criticality, many papers have been developed in order to study issues related to the existence of solutions for problems involving the fractional Laplacian operator and nonlinearities with exponential growth. For example, by exploiting the Trudinger-Moser embedding due to Ozawa [34] and the Mountain Pass Theorem, J. M. do Ó, Miyagaki and Squassina [21] proved the existence of ground state solutions for the following class of nonlinear scalar field equations:

$$\begin{cases} (-\Delta)^{\frac{1}{2}}u + u = f(u) \quad \text{in} \quad \mathbb{R}, \\ u(x) \to 0, \quad \text{as} \quad |x| \to \infty, \end{cases}$$

when $f(t)$ is $o(|t|)$ at the origin and behaves like $e^{\alpha t^2}$ as $|t| \to +\infty$, for some $\alpha > 0$. In [15], Souza and Araújo considered a perturbation of this problem by a general potential $V(x)$, namely,

$$\begin{cases} (-\Delta)^{\frac{1}{2}}u + V(x)u = f(u) \quad \text{in} \quad \mathbb{R}, \\ u(x) \to 0, \quad \text{as} \quad |x| \to \infty, \end{cases}$$

where $V(x)$ is a nonnegative function which is asymptotically periodic at infinity. See also [2, 12, 16, 20, 25] for others investigations.

We would like to point out the recent work due to Miyagaki and Pucci [31], who considered a nonlocal Kirchhoff problem of the form

$$-M(\|u\|)((-\Delta)^{\frac{1}{2}}u + V(x)u) = K(x)f(u) \quad \text{in} \quad \mathbb{R}, \tag{1.4}$$

where $M : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous Kirchhoff function and $\|u\|$ is defined as in (1.7), V and K are continuous positive potentials satisfying the conditions introduced in [20] and f is a nonlinearity with exponential growth and is allowed to be critical or subcritical with respect to the Trudinger-Moser inequality established by Ozawa [34]. In this work the authors obtain the existence of nontrivial solutions to (1.4) by applying suitable variational methods needed to overcome the lack of compactness due to the unboundedness of the domain and to the Trudinger-Moser embedding.

For problems considering a bounded interval of the real line, we would like to mention Iannizzotto and Squassina [27], who proved the existence and multiplicity of solutions for the class of one-dimensional nonlocal equations

$$\begin{cases} (-\Delta)^{\frac{1}{2}}u = f(u) \quad \text{in} \quad (a,b), \\ u = 0 \quad \text{in} \quad \mathbb{R} \setminus (a,b), \end{cases}$$

when f has exponential subcritcal growth or critical growth. In [26], Giacomoni, Mishra and Sreenadh considered the problem

$$\begin{cases} (-\Delta)^{\frac{1}{2}}u = \lambda g(x)|u|^{q-2}u + u^p e^{u^\beta} \quad \text{in} \quad (a,b), \\ u > 0 \ \text{in} \ (a,b) \quad \text{and} \quad u = 0 \ \text{in} \ \mathbb{R} \setminus (a,b), \end{cases}$$

where $1 < q < 2$, $p > 1$, $0 \leq \beta \leq 2$, $\lambda > 0$ and the function $g \in L^{\frac{p+q+\beta}{p+q+\beta-1}}(a,b)$. They showed the existence of mountain-pass solution when the nonlinearity is concave near to the origin and has exponential growth at infinity. Furthermore, they showed the existence of multiple solutions for a suitable range of λ by analysing the fibering maps and the corresponding Nehari manifold.

In [32], Perera and Squassina, by using a suitable topological argument based on cohomological linking and by exploiting the Trudinger-Moser inequality, the existence of multiple solutions was extended for a problem involving the nonlinear $N/s-$fractional Laplacian operator and the critical exponential growth.

We point out that none of the previous works treated the existence of a sign-changing solution (nodal solution). Motivated by these facts, one of our goals in the present paper is proving the existence of the least energy nodal solutions for problem (1.1) when the nonlinearity has exponential growth. We ended this subsection by mentioning that for problems involving fractional equations, critical nonlinearities and domains Ω of $\mathbb{R}^N$, with $N > 2s$, there is a large literature and we refer to [23, 33, 35, 36, 37], and to the references therein. For the existence of sign-changing solutions we quote [1, 11], which served as inspiration for the development of this work.

1.1 Assumptions

In order to reach our goals, we assume the following assumptions on the functions V and K:

(V_1) $V : \mathbb{R} \to [0, +\infty)$, $K : \mathbb{R} \to (0, +\infty)$ are continuous and $K \in L^\infty(\mathbb{R})$;

(V_2) there exist $b_0, R_0 > 0$ such that

$$V(x) \geq b_0, \quad \text{for } |x| \geq R_0;$$

Since problem (1.1) is set on the whole real line, we face a loss of compactness. Here, motivated by Miyagaki and Pucci [31], in order to overcome this difficulty, we assume the following assumption on K:

(K_1) if $\{A_n\}$ is a sequence of Borel sets of $\mathbb{R}$ with $\sup_{n\in\mathbb{N}} |A_n| \le R$, for some $R > 0$, then

$$\lim_{r\to\infty} \int_{A_n\cap B_r^c(0)} K(x)\,\mathrm{d}x = 0,$$

uniformly with respect to $n \in \mathbb{N}$.

On the nonlinearity f, we assume the following assumptions:

(f_1) $f \in C^1(\mathbb{R})$ and there exist $C_0, t_0 > 0$ such that

$$|f(t)| \le C_0 \left(e^{\pi t^2} - 1\right), \quad \text{for all} \quad |t| \ge t_0;$$

(f_2) $\lim_{t\to 0} \dfrac{f(t)}{t} = 0;$

(f_3) there exists $\theta > 2$ such that

$$0 < \theta F(t) := \theta \int_0^t f(s)\mathrm{d}s \le tf(t), \quad \text{for all} \quad t \in \mathbb{R}\setminus\{0\};$$

(f_4) the function $\dfrac{f(t)}{|t|}$ is strictly increasing for $t \ne 0$;

(f_5) there exist constants $p > 2$ and $C_p > 0$ such that

$$sgn(t)f(t) \ge C_p|t|^{p-1}, \quad \text{for all} \quad t \in \mathbb{R}.$$

We point out that from (f_1) we can consider nonlinearities with exponential critical growth in the sense of (1.2) and with exponential subcritical growth in the sense of (1.3).

1.2 The main results

For a better understanding of the main results of this work, we will introduce some notations and definitions. First, we denote by $H^{1/2}(\mathbb{R})$ the fractional Sobolev space defined by

$$H^{1/2}(\mathbb{R}) = \left\{u \in L^2(\mathbb{R}) : \int_{\mathbb{R}^2} \frac{(u(x) - u(y))^2}{|x-y|^2}\mathrm{d}x\mathrm{d}y < \infty\right\}$$

endowed with the norm

$$\|u\|_{1/2,2} := \left(\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{|u(x)-u(y)|^2}{|x-y|^2} \mathrm{d}x\mathrm{d}y + \|u\|_2^2 \right)^{1/2}.$$

In order to apply variational methods to study (1.1) in $H^{1/2}(\mathbb{R})$, it is natural to work in the subspace of $H^{1/2}(\mathbb{R})$ defined as

$$X := \left\{ u \in H^{1/2}(\mathbb{R}) : \int_{\mathbb{R}} V(x)u^2 \mathrm{d}x < \infty \right\}. \tag{1.5}$$

From $(V_1)-(V_2)$ (see Lemma 2.1 and Proposition 2.2), we show that X is a Hilbert space when endowed with the inner product

$$\langle u, v \rangle := \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^2} \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}} V(x)uv\mathrm{d}x \tag{1.6}$$

and the corresponding norm

$$\|u\| := \left(\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{|u(x)-u(y)|^2}{|x-y|^2} \mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}} V(x)|u|^2 \mathrm{d}x \right)^{1/2}. \tag{1.7}$$

In this context, we say $u \in X$ is a weak solution of (1.1) if

$$\langle u, v \rangle - \int_{\mathbb{R}} K(x)f(u)v\mathrm{d}x = 0, \quad \text{for all} \quad v \in X,$$

and if u is a weak solution of (1.1) such that $u^+ \not\equiv 0$ and $u^- \not\equiv 0$, we say that u is a sign-changing solution (nodal solution), where $u^+(x) = \max\{u(x), 0\}$ and $u^-(x) = \min\{u(x), 0\}$.

In Section 2, we will show that the energy functional

$$I(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} K(x)F(u)\mathrm{d}x \tag{1.8}$$

is well defined and belongs to $C^1(X, \mathbb{R})$ with

$$I'(u)v = \langle u, v \rangle - \int_{\mathbb{R}} K(x)f(u)v\mathrm{d}x, \text{ for } u, v \in X,$$

and consequently its critical points are weak solutions of (1.1).

In order to find nodal solutions for problem (1.1) by applying an appropriate minimization argument, first we will introduce the Nehari manifold

$$\mathcal{N} = \{u \in X \setminus \{0\} : I'(u)u = 0\}. \tag{1.9}$$

Now, we define the nodal set as

$$\mathcal{N}_{nod} = \{u \in X : u^+ \not\equiv 0, u^- \not\equiv 0 \text{ and } I'(u)u^+ = I'(u)u^- = 0\}; \tag{1.10}$$

the ground state level

$$c := \inf_{u \in \mathcal{N}} I(u); \tag{1.11}$$

and the nodal level

$$c^* := \inf_{u \in \mathcal{N}_{nod}} I(u). \tag{1.12}$$

We recall that a necessary condition for $u \in X$ to be a critical point of I is that $I'(u)u = 0$. Thus, the Nehari manifold is a natural constraint for the set of nontrivial solutions.

Note that since $\mathcal{N}_{nod} \subset \mathcal{N}$ we have $c \leq c^*$. We say that a nonzero critical point $w \in X$ of I is the least energy solution if w achieves the infimum c. Since we are looking for nodal solutions, one of our goals will be to show that the minimum c^* is reached by a critical point of I. Notice that the set $\mathcal{N}_{nod}$ contains all sign-changing solutions of (1.1). The function that achieves c^* is called the least energy nodal solution.

We point out that in order to find a critical point on $\mathcal{N}_{nod}$ our approach is a little different from the usual and is taken from [3, 4, 7, 11]. In particular, we do not need to make customary assumptions which imply that $I \in C^2$ and $I''(u)(u,u) < 0$ on $\mathcal{N}_{nod}$.

Now we can state our main results.

Theorem 1.1. *Suppose that* $(V_1)-(V_2)$, (K_1) *and* $(f_1)-(f_5)$ *are satisfied. Then problem* (1.1) *possesses a least energy nodal solution, provided that*

$$C_p > \left[\frac{2\theta\kappa c_p^*}{\theta - 2}\right]^{(p-2)/2}, \tag{1.13}$$

where

$$c_p^* := \inf_{u \in \mathcal{M}_{nod}} I_p(u),$$

$$\mathcal{M}_{nod} := \{u \in X : u^+ \not\equiv 0, u^- \not\equiv 0 \ \ and \ \ I_p'(u)u^+ = I_p'(u)u^- = 0\}$$

and

$$I_p(u) := \frac{1}{2}\|u\|^2 - \frac{1}{p}\int_{\mathbb{R}} K(x)|u|^p \mathrm{d}x,$$

and $\kappa > 0$ *is the constant given in* (2.6).

Another goal of this paper is to prove that the energy of any sign-changing solution of (1.1) is strictly larger than twice the ground state energy. This property is so-called energy doubling by Weth [40].

Theorem 1.2. *Suppose that* $(V_1)-(V_2)$, (K_1), $(f_1)-(f_5)$ *and* (1.13) *are satisfied. Then problem* (1.1) *has a least energy solution and*

$$I(w) > 2c, \tag{1.14}$$

where w *is the least energy sign-changing solution obtained in Theorem* 1.1*. In particular,* c *is achieved either by a positive or a negative function.*

Remark 1.3. Note that if we assume that the function f is odd, then, using Theorem 1.2, it follows that problem (1.1) has at least one negative solution, one positive solution, and one nodal solution.

Remark 1.4. Using the regularity results due to Servadei and Valdinoci [37], we have that weak solutions of problem (1.1) belong to $C(\mathbb{R})$.

Remark 1.5. In this paper, we deal with nonlinearities that have exponential growth in the sense of the Trudinger-Moser inequality proved by Ozawa [31]. This inequality is valid in general for functions in the Sobolev space $W^{s,p}(\mathbb{R}^N)$ whenever $p > 1$, $N \geq 1$ and $s = p/N$. However, in our arguments it is fundamental the Hilbert structure of the space X, see Lemma 2.6. Thus, we are restricted to the case $p = 2$. Besides that, since we are interested in the fractional case $0 < s < 1$ our approach is also restricted to the one-dimensional case $N = 1$. We have that these restrictions imply $s = 1/2$.

It is interesting to note that in the last decades the existence and multiplicity of positive and nodal solutions of elliptic problems have been widely investigated, see for example, [3, 4, 5, 6, 7, 8, 24, 41] and references therein. Specially, some results on nodal solutions of nonlinear elliptic equations involving different operators have been obtained by combining minimax method with invariant sets of descending flow, such as Laplacian operator [5, 7, 8], $p-$Laplacian operator [6] and Schrödinger operator [3, 4, 24]. In the special case of the stationary equation of Schrödinger

$$-\Delta u + V(x)u = f(u) \quad \text{in} \quad \mathbb{R}^N, \tag{1.15}$$

there are several ways in the literature to obtain a sign-changing solution (see [3, 4, 5, 8, 24, 41]). However, the methods used in these works heavily rely on the following two decompositions:

$$J(u) = J(u^+) + J(u^-), \tag{1.16}$$

$$J'(u)u^+ = J'(u^+)u^+ \text{ and } J'(u)u^- = J'(u^-)u^-, \tag{1.17}$$

where J is the energy functional associated to (1.15) given by

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\mathrm{d}x - \int_{\mathbb{R}^N} F(u)\mathrm{d}x.$$

In the case of problem (1.1), the energy functional associated does not have the same decompositions as (1.16) and (1.17). Indeed, since $\langle u^+, u^- \rangle > 0$ when $u^+ \not\equiv 0$ and $u^- \not\equiv 0$, a straightforward computation shows that (see Lemma 2.6)

$$I(u) > I(u^+) + I(u^-),$$

$$I'(u)u^+ > I'(u^+)u^+ \text{ and } I'(u)u^- > I'(u^-)u^-.$$

Therefore, the methods used to obtain sign-changing solutions for the local problem (1.15) seem not to be applicable to problem (1.1). Furthermore, a second well-known difficulty for the class of problems (1.1) is the loss of compactness due to the critical growth on the nonlinearity f.

In order to overcome these difficulties, we define the constrained set $\mathcal{N}_{nod}$ (see (1.10)) and consider a minimization problem of I on $\mathcal{N}_{nod}$. Borrowing ideas from [11], we prove $\mathcal{N}_{nod} \neq \emptyset$ via modified Miranda's Theorem (see Lemma 3.5 and Lemma 3.6). Combining the ideas developed in [3, 4, 7, 11], we prove that the minimizer of the constrained problem is also a sign-changing solution via the Quantitative Deformation Lemma and Degree Theory (see Section 3).

In this paper, the symbols C, C_i, $i = 1, 2, \dots$, will be used to denote various positive constants and B_R denotes the open ball centered at the origin with radius R.

2 Preliminaries

We begin this section by presenting an equality that will be widely used throughout this work. From [19, Proposition 3.6], it is well-known that for all $u \in H^{1/2}(\mathbb{R})$, it holds

$$\|(-\Delta)^{\frac{1}{4}} u\|_2^2 = \frac{1}{2\pi}[u]_{1/2,2}^2 = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{|u(x)-u(y)|^2}{|x-y|^2} \mathrm{d}x\mathrm{d}y. \tag{2.1}$$

With this in mind, we prove the following result:

Lemma 2.1. *Assume that* $(V_1)-(V_2)$ *are satisfied. Then,*

$$\lambda_1 := \inf_{\substack{u\in X\\ \|u\|_2=1}} \left(\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{|u(x)-u(y)|^2}{|x-y|^2}\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}} V(x)u^2 \mathrm{d}x \right) > 0.$$

Proof. Suppose, by contradiction, that $\lambda_1 = 0$. Hence, there exists $(u_n) \subset X$ such that, as $n \to \infty$, we have

$$\|u_n\|_2^2 = 1 \quad \text{and} \quad \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{|u_n(x)-u_n(y)|^2}{|x-y|^2}\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}} V(x)u_n^2\mathrm{d}x \to 0. \tag{2.2}$$

From [34], for any $2 \leq q < +\infty$ and $v \in H^{1/2}(\mathbb{R})$, there exists $M > 0$ such that

$$\|v\|_q \leq M q^{1/2} \|(-\Delta)^{1/4} v\|_2^{1-2/q} \|v\|_2^{2/q}. \tag{2.3}$$

Combining (2.1), (2.2) and (2.3), for each $q>2$, we obtain

$$\|u_n\|_q \le Mq^{1/2}\|(-\Delta)^{1/4}u_n\|_2^{1-2/q} \to 0, \quad \text{as} \quad n\to\infty.$$

Now, note that choosing $t>1$ such that $2t=q$ and by using the Hölder inequality, we get

$$\|u_n\|^2_{L^2(B_{R_0})} \le |B_{R_0}|^{\frac{1}{t'}}\|u_n\|^2_{L^q(B_{R_0})} \to 0, \quad \text{as} \quad n\to\infty. \tag{2.4}$$

On the other hand, by (V_2) and (2.2), we have

$$\int_{B^c_{R_0}} u_n^2 \mathrm{d}x \le \frac{1}{b_0}\int_{B^c_{R_0}} V(x)u_n^2\mathrm{d}x \to 0, \quad \text{as} \quad n\to\infty. \tag{2.5}$$

But, (2.4) and (2.5) imply that

$$1 = \|u_n\|^2_{L^2(B_{R_0})} + \|u_n\|^2_{L^2(B^c_{R_0})} \to 0,$$

as $n\to\infty$, which is a contradiction. Thus, we have completed the proof of the lemma. □

From Lemma 2.1, we reach the following result:

Corollary 2.2. *Assume that $(V_1)-(V_2)$ are satisfied. Then the embedding $X \hookrightarrow H^{1/2}(\mathbb{R})$ is continuous and there exists $\kappa>0$ such that*

$$\frac{1}{\kappa} := \inf_{\substack{u\in X\\ u\neq 0}} \frac{\|u\|^2}{\|u\|^2_{1/2,2}}. \tag{2.6}$$

In particular, X is a Hilbert space with the inner product (1.6) and the embedding $X\hookrightarrow L^q(\mathbb{R})$ is continuous and locally compact for all $q\in[2,+\infty)$.

Now, given $r\ge 1$, we define weighted Banach space

$$L^r_K := \left\{u:\mathbb{R}\to\mathbb{R} : u \text{ is measurable and } \int_{\mathbb{R}} K(x)|u|^r\mathrm{d}x < \infty\right\}$$

endowed with the norm

$$\|u\|_{L^r_K} := \left(\int_{\mathbb{R}} K(x)|u|^r\mathrm{d}x\right)^{\frac{1}{r}}.$$

Note that, since $K \in L^\infty(\mathbb{R})$, the embedding $H^{1/2}(\mathbb{R}) \hookrightarrow L^q_K$ is continuous for all $q \geq 2$. In [20, Proposition 2.2], the authors show that this injection is compact for all $q > 2$. As a consequence, we have:

Corollary 2.3. *X is continuously embedded in L^2_K and compactly embedded into L^q_K for all $q \in (2, +\infty)$.*

One of the main tools to study problems involving exponential growth in the fractional Sobolev spaces is the so-called fractional Trudinger-Moser inequality due to Ozawa [34]. Combining the results in [15, 28, 34, 38], the Trudinger-Moser inequality due to Ozawa has been refined and can be stated as follows.

Lemma 2.4. *For any $u \in H^{1/2}(\mathbb{R})$ and $\alpha \geq 0$, the integral*

$$\int_{\mathbb{R}} \left(e^{\alpha u^2} - 1 \right) \mathrm{d}x < \infty. \tag{2.7}$$

Furthermore, if $0 \leq \alpha \leq \pi$, it holds

$$\sup_{\{u \in H^{1/2}(\mathbb{R}) \,:\, \|u\|_{1/2,2} \leq 1\}} \int_{\mathbb{R}} \left(e^{\alpha u^2} - 1 \right) \mathrm{d}x < \infty \tag{2.8}$$

and if $0 \leq \alpha < \pi$, there exists $C_\alpha > 0$ such that

$$\int_{\mathbb{R}} \left(e^{\alpha u^2} - 1 \right) \mathrm{d}x \leq C_\alpha \|u\|_2^2, \tag{2.9}$$

whenever $u \in H^{1/2}(\mathbb{R})$ and $\|(-\Delta)^{\frac{1}{4}} u\|_2 \leq 1$.

As an application of this inequality, we get the following convergence result:

Lemma 2.5. *Let $\alpha > 0$ and $(u_n) \subset H^{1/2}(\mathbb{R})$ be such that $u_n \to u$ strongly in $H^{1/2}(\mathbb{R})$. Then*

$$\lim_{n \to +\infty} \int_{\mathbb{R}} \left(e^{\alpha u_n^2} - 1 \right) \mathrm{d}x = \int_{\mathbb{R}} \left(e^{\alpha u^2} - 1 \right) \mathrm{d}x.$$

Proof. The proof of this lemma follows directly from the Mean Value Theorem, the Hölder inequality and Lemma 2.4, and we will omit it. □

Now, note that by Lemma 2.4, Lemma 2.5 and the hypotheses on f and V, we obtain that the energy functional $I : X \to \mathbb{R}$ associated to problem (1.1) given by

$$I(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} K(x)F(u)\mathrm{d}x$$

is well defined and belongs to $C^1(X, \mathbb{R})$ with

$$I'(u)v = \langle u, v\rangle - \int_{\mathbb{R}} K(x)f(u)v\mathrm{d}x, \quad \text{for} \quad u, v \in X$$

and consequently critical points of I are precisely the weak solutions of (1.1).

Before presenting our next result we would like to mention that due to the characteristics of the Gagliardo semi-norm $[u]^2_{1/2,2}$, the energy functional I does not possess certain properties that are typically satisfied for the energy functional of equations of type

$$-\Delta u + V(x)u = f(u) \quad \text{in} \quad \mathbb{R}^N. \tag{2.10}$$

In theses cases, the following equalities are satisfied

$$J(u) = J(u^+) + J(u^-), \tag{2.11}$$

and

$$J'(u)u^+ = J'(u^+)u^+ \text{ and } J'(u)u^- = J'(u^-)u^-, \tag{2.12}$$

where J is the energy functional of (2.10), which is given by

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + V(x)u^2)\mathrm{d}x - \int_{\mathbb{R}^N} F(u)\mathrm{d}x.$$

The above decompositions are applied in several ways in the literature to obtain sign-changing solutions for problem (2.10) (see [3, 4, 5, 8, 24, 41]). However, these methods can not be directly apply to problem (1.1). Here, inspired by [11], we have the following result:

Lemma 2.6. *Let $u \in X$ be such that $u^+ \not\equiv 0$ and $u^- \not\equiv 0$. Then,*

(i) $\langle u^+, u^- \rangle > 0$*;*

(ii) $I(u) > I(u^+) + I(u^-)$*;*

(iii) $I'(u)u^+ > I'(u^+)u^+$ *and* $I'(u)u^- > I'(u^-)u^-$*.*

Proof. By density (see [19, Theorem 2.4]), we can assume that u is continuous. Defining

$$\Omega_+ = \{x \in \mathbb{R} : u(x) \geq 0\} \quad \text{and} \quad \Omega_- = \{x \in \mathbb{R} : u(x) \leq 0\},$$

we get

$$\begin{aligned} 2\pi\langle u^+, u^- \rangle = & \int_{\Omega_+ \times \Omega_+} \frac{(u^+(x) - u^+(y))(u^-(x) - u^-(y))}{|x-y|^2} \mathrm{d}x\mathrm{d}y \\ & + \int_{\Omega_+ \times \Omega_-} \frac{(u^+(x) - u^+(y))(u^-(x) - u^-(y))}{|x-y|^2} \mathrm{d}x\mathrm{d}y \\ & + \int_{\Omega_- \times \Omega_+} \frac{(u^+(x) - u^+(y))(u^-(x) - u^-(y))}{|x-y|^2} \mathrm{d}x\mathrm{d}y \\ & + \int_{\Omega_- \times \Omega_-} \frac{(u^+(x) - u^+(y))(u^-(x) - u^-(y))}{|x-y|^2} \mathrm{d}x\mathrm{d}y. \end{aligned}$$

Hence, since $u^+ = 0$ in Ω_- and $u^- = 0$ in Ω_+, we reach

$$\begin{aligned} 2\pi\langle u^+, u^- \rangle = & \int_{\Omega_+ \times \Omega_-} \frac{u^+(y)(-u^-(x))}{|x-y|^2} \mathrm{d}x\mathrm{d}y \\ & + \int_{\Omega_- \times \Omega_+} \frac{u^+(x)(-u^-(y))}{|x-y|^2} \mathrm{d}x\mathrm{d}y > 0, \end{aligned}$$

which implies the item (i). Now, since $I(u) = \langle u^+, u^- \rangle + I(u^+) + I(u^-)$, $I'(u)u^+ = \langle u^+, u^- \rangle + I'(u^+)u^+$ and $I'(u)u^- = \langle u^+, u^- \rangle + I'(u^-)u^-$, the proof of (ii) and (iii) follows from item (i). □

Corollary 2.7. *If* $u \in X$ *then*

$$\|u\|^2 \geq \|u^+\|^2 + \|u^-\|^2.$$

Proof. By Lemma 2.6, we have

$$\|u\|^2 = \|u^+\|^2 + 2\langle u^+, u^- \rangle + \|u^-\|^2 \geq \|u^+\|^2 + \|u^-\|^2$$

which implies the desired inequality. □

3 Constrained minimization problem

In order to prove some properties of $\mathcal{N}_{nod}$ and $\mathcal{N}$, we observe that by $(f_1)-(f_2)$, given $\varepsilon > 0$ and $q \geq 1$, there is a positive constant C_ε such that

$$|f(t)| \leq \varepsilon|t| + C_\varepsilon|t|^{q-1}(e^{\pi t^2}-1), \quad \text{for all} \quad t \in \mathbb{R} \tag{3.1}$$

and, by virtue of (f_3),

$$|F(t)| \leq \varepsilon|t|^2 + C_\varepsilon|t|^{q}(e^{\pi t^2}-1), \quad \text{for all} \quad t \in \mathbb{R}. \tag{3.2}$$

Moreover, by (f_5), we have

$$|f(t)| \geq C_p|t|^{p-1}, \quad \text{for all} \quad t \in \mathbb{R} \tag{3.3}$$

and

$$F(t) \geq \frac{C_p}{p}|t|^p, \quad \text{for all} \quad t \in \mathbb{R}. \tag{3.4}$$

Lemma 3.1. *Assume that $(V_1)-(V_2)$ and $(f_1)-(f_5)$ are satisfied. Then, given $u \in X \setminus \{0\}$, there is an unique $t = t(u) > 0$ such that $tu \in \mathcal{N}$. In addition, the number t satisfies*

$$I(tu) = \max_{s\geq 0} I(su). \tag{3.5}$$

Proof. Given $u \in X \setminus \{0\}$, we define $h(s) := I(su)$ for $s \geq 0$. By (3.4) and since $p > 2$, we obtain

$$h(s) \leq \frac{s^2}{2}\|u\|^2 - \frac{C_p s^p}{p}\int_{\mathbb{R}} K(x)|u|^p \mathrm{d}x \to -\infty, \quad \text{as} \quad s \to \infty. \tag{3.6}$$

On the other hand, choosing $q > 2$, by using (3.2) and that $K(x) \leq C$, we get

$$h(s) \geq \frac{s^2}{2}\|u\|^2 - C\int_{\mathbb{R}}(\varepsilon s^2|u|^2 + C_\varepsilon s^q|u|^q(e^{\pi s^2u^2}-1))\mathrm{d}x. \tag{3.7}$$

If $s \in [0,1]$, we have $(e^{\pi s^2u^2}-1) \leq (e^{\pi u^2}-1)$. Hence, by Proposition 2.2, we get

$$h(s) \geq s^2\left(\frac{1}{2} - C_1\varepsilon\right)\|u\|^2 - C_{2,\varepsilon}s^q\int_{\mathbb{R}}|u|^q(e^{\pi u^2}-1)\mathrm{d}x > 0 \tag{3.8}$$

for $s>0$ small enough. Thus, from (3.6) and (3.8), there exists $t=t(u)>0$ such that $I(tu)=\max_{s\geq 0} I(su)$ and, consequently, $tu\in\mathcal{N}$. Now, if $s>0$ is such that $su\in\mathcal{N}$, we have

$$s^2\|u\|^2=\int_{\mathbb{R}} f(su)su\,\mathrm{d}x$$

and since

$$t^2\|u\|^2=\int_{\mathbb{R}} f(tu)tu\,\mathrm{d}x,$$

it follows that

$$\int_{\mathbb{R}}\left(\frac{f(tu)}{tu}-\frac{f(su)}{su}\right)u^2\mathrm{d}x=0. \tag{3.9}$$

By (f_4) and since $u\neq 0$, we get from (3.9) that $t=s$. Thus, we have completed the proof. □

Lemma 3.2. *Assume that* $(V_1)-(V_2)$ *and* $(f_1)-(f_5)$ *are satisfied. Then, there exists* $m_0>0$ *such that* $\|u\|^2\geq m_0$ *for all* $u\in\mathcal{N}$.

Proof. In order to obtain a contradiction, suppose that there exists $(u_n)\subset\mathcal{N}$ such that $\|u_n\|\to 0$ as $n\to\infty$. By definition, we know that

$$\|u_n\|^2=\int_{\mathbb{R}} K(x)f(u_n)u_n\mathrm{d}x. \tag{3.10}$$

Since $K(x)\leq C$, utilizing (3.1) with $q>2$, we get

$$\begin{aligned}\|u_n\|^2&\leq\int_{\mathbb{R}} K(x)|f(u_n)u_n|\mathrm{d}x\\&\leq\varepsilon C\int_{\mathbb{R}}|u_n|^2\mathrm{d}x+C_\varepsilon\int_{\mathbb{R}}|u_n|^q(e^{\pi u_n^2}-1)\mathrm{d}x.\end{aligned} \tag{3.11}$$

Now, from Lemma 2.4, by using the Hölder inequality and the assumptions $\|u_n\|\to 0$, we obtain

$$\begin{aligned}\int_{\mathbb{R}}|u_n|^q(e^{\pi u_n^2}-1)\mathrm{d}x&\leq C\|u_n\|_{2q}^q\left(\int_{\mathbb{R}}(e^{2\pi\|u_n\|^2\left(\frac{u_n}{\|u_n\|}\right)^2}-1)\mathrm{d}x\right)^{\frac{1}{2}}\\&\leq C_\pi\|u_n\|_{2q}^q\end{aligned} \tag{3.12}$$

for $n \in \mathbb{N}$ sufficiently large. From Proposition 2.2, there exist $C_1, C_2 > 0$ such that $\|u_n\|_{2q}^q \leq C_1 \|u_n\|^q$ and $\|u_n\|_2^2 \leq C_2 \|u_n\|^2$. Hence, choosing $\varepsilon > 0$ and utilizing (3.10), (3.11) and (3.12), we have $0 < C_0 \leq \|u_n\|^{q-2}$, for $n \in \mathbb{N}$ sufficiently large. But, as $q > 2$, this contradicts the assumption $\|u_n\| \to 0$ and the proof of the lemma is complete. □

Corollary 3.3. *Assume that $(V_1) - (V_2)$ and $(f_1) - (f_5)$ are satisfied. Then, there exists $\delta_0 > 0$ such that $I(u) \geq \delta_0$ for all $u \in \mathcal{N}$. In particular,*

$$0 < \delta_0 \leq c \leq c^*.$$

Proof. Since $I'(u)u = 0$, by Lemma 3.2 and (f_3), we have

$$\begin{aligned} I(u) = I(u) - \frac{1}{\theta} I'(u)u &= \left(\frac{1}{2} - \frac{1}{\theta}\right) \|u\|^2 + \frac{1}{\theta} \int_{\mathbb{R}} [f(u)u - \theta F(u)] \, \mathrm{d}x \\ &\geq \left(\frac{1}{2} - \frac{1}{\theta}\right) \|u\|^2 \geq \left(\frac{1}{2} - \frac{1}{\theta}\right) m_0 := \delta_0, \end{aligned}$$

which is the desired inequality. □

Lemma 3.4. *Assume that $(V_1) - (V_2)$ and $(f_1) - (f_5)$ are satisfied. Then, there exists $m_0' > 0$ such that $\|u^+\|^2 \geq m_0'$ and $\|u^-\|^2 \geq m_0'$ for all $u \in \mathcal{N}_{nod}$.*

Proof. The proof is similar to Lemma 3.2. Hence, it is sufficient to prove a similar estimate to (3.11) for u^+ and u^-. Since $u \in \mathcal{N}_{nod}$ we have $u^+ \neq 0$ and $\langle u, u^+ \rangle = \int_{\mathbb{R}} K(x) f(u^+) u^+ \mathrm{d}x$. Now, by Lemma 2.6, we have $\langle u^+, u^+ \rangle < \langle u, u^+ \rangle$. Thus, by using (3.1) we obtain

$$\|u^+\|^2 \leq \int_{\mathbb{R}} K(x) f(u^+) u^+ \mathrm{d}x \leq \varepsilon C \int_{\mathbb{R}} |u^+|^2 \mathrm{d}x + C_\varepsilon \int_{\mathbb{R}} |u^+|^q (e^{\pi |u^+|^2} - 1) \mathrm{d}x.$$

Similarly,

$$\|u^-\|^2 \leq \int_{\mathbb{R}} K(x) f(u^-) u^- \mathrm{d}x \leq \varepsilon C \int_{\mathbb{R}} |u^-|^2 \mathrm{d}x + C_\varepsilon \int_{\mathbb{R}} |u^-|^q (e^{\pi |u^-|^2} - 1) \mathrm{d}x.$$

This completes the proof. □

Now, we recall the so-called Poincaré-Miranda Theorem (see [39]).

Lemma 3.5. *Let* $h : P \subset \mathbb{R}^N \longrightarrow \mathbb{R}^N$ *be a continuous function, where* $P = \prod_{i=1}^N [a_i, b_i]$ *is a* N*-dimensional block in* $\mathbb{R}^N$*, with* $a_i \neq b_i$*, for* $i = 1, \ldots, N$*. Let* $P_i^- = \{x \in P : x_i = a_i\}$ *and* $P_i^+ = \{x \in P : x_i = b_i\}$*. Assume that the coordinates functions of* h *satisfy:*

(i) $h_i(x) \geq 0$*, for all* $x \in P_i^-$*,*

(ii) $h_i(x) \leq 0$*, for all* $x \in P_i^+$*.*

Then there exists $x_0 \in P$ *such that* $h(x_0) = 0$*.*

As an application of Lemma 3.5, we shall show that $\mathcal{N}_{nod} \neq \emptyset$.

Lemma 3.6. *Assume that* $(V_1) - (V_2)$ *and* $(f_1) - (f_5)$ *are satisfied. Then, given* $u \in X$ *with* $u^+ \not\equiv 0$ *and* $u^- \not\equiv 0$*, there exists an unique pair* (t, s) *of positive numbers such that* $tu^+ + su^- \in \mathcal{N}_{nod}$*.*

Proof. Let $u \in X$ such that $u^+ \not\equiv 0$ and $u^- \not\equiv 0$. We define the continuous vector field $g : (0, \infty) \times (0, \infty) \to \mathbb{R}^2$ by

$$g(t, s) = \left(I'(tu^+ + su^-)tu^+, I'(tu^+ + su^-)su^-\right).$$

Firstly, we want to find $(t, s) \in (0, \infty) \times (0, \infty)$ such that $g(t, s) = (0, 0)$. The first step is to show that for t and s sufficiently small the coordinates functions are positive. Given $\varepsilon > 0$ and $q > 2$, by (3.1) and $K(x) \leq C$, we get

$$\begin{aligned} I'(tu^+ + su^-)tu^+ &= t^2\|u^+\|^2 + ts\langle u^+, u^-\rangle - \int_{\mathbb{R}} K(x)f(tu^+)tu^+ \mathrm{d}x \\ &\geq t^2\|u^+\|^2 + ts\langle u^+, u^-\rangle \\ &\quad - \varepsilon C t^2 \int_{\mathbb{R}} |u^+|^2 \mathrm{d}x - C_\varepsilon C t^q \int_{\mathbb{R}} |u^+|^q (e^{\pi t^2 |u^+|^2} - 1)\mathrm{d}x. \end{aligned}$$

Hence, if $t \in [0, 1]$, by using Proposition 2.2, there exists $C_1 > 0$ such that

$$\begin{aligned} I'(tu^+ + su^-)tu^+ \geq t^2\|u^+\|^2 + ts\langle u^+, u^-\rangle - \varepsilon C_1 C t^2 \|u^+\|^2 \\ - C_\varepsilon C t^q \int_{\mathbb{R}} |u^+|^q (e^{\pi |u^+|^2} - 1)\mathrm{d}x. \end{aligned}$$

By Lemma 2.6 we have $\langle u^+, u^-\rangle > 0$. Then there exists $r > 0$ small enough such that

$$I'(ru^+ + su^-)ru^+ > 0, \quad \text{for all} \quad s > 0.$$

Analogously, there exists $r > 0$ large enough such that

$$I'(tu^+ + ru^-)ru^- > 0, \quad \text{for all} \quad t > 0.$$

Now, we shall show that, for t and s large enough, the coordinates functions are negative. Indeed, by (f_3) and (3.4), we have

$$\int_{\mathbb{R}} K(x)f(tu^+)tu^+ \mathrm{d}x \geq \theta \int_{\mathbb{R}} K(x)F(tu^+)\mathrm{d}x \geq \frac{\theta C_p t^p}{p} \int_{\mathbb{R}} K(x)|u^+|^p \mathrm{d}x.$$

Thus,

$$\begin{aligned} I'(tu^+ + su^-)tu^+ &= t^2\|u^+\|^2 + ts\langle u^+, u^-\rangle - \int_{\mathbb{R}} K(x)f(tu^+)tu^+ \mathrm{d}x \\ &\leq t^2\|u^+\|^2 + ts\langle u^+, u^-\rangle - \frac{\theta C_p t^p}{p}\|u^+\|^p_{L^p_K}. \end{aligned}$$

Since $p > 2$, there exists $R > r$ large enough such that

$$I'(Ru^+ + su^-)Ru^+ < 0, \text{ for all } 0 \leq s \leq R.$$

Analogously, there exists $R > r$ small enough such that

$$I'(tu^+ + Ru^-)Ru^- < 0, \text{ for all } 0 \leq t \leq R.$$

Hence, considering the block $P = [r, R] \times [r, R]$ and applying Lemma 3.5, there exists $(t, s) \in [r, R]\times[r, R]$ such that $g(t, s) = (0, 0)$ and consequently, we have $tu^+ + su^- \in \mathcal{N}_{nod}$.

Finally, we shall prove the uniqueness of the pair (t, s). First, we assume that $u = u^+ + u^- \in \mathcal{N}_{nod}$ and $(t, s) \in (0, \infty) \times (0, \infty)$ is such that $tu^+ + su^- \in \mathcal{N}_{nod}$. In this case, we need to show that $(t, s) = (1, 1)$. Note that

$$\|u^+\|^2 + \langle u^+, u^-\rangle = \int_{\mathbb{R}} K(x)f(u^+)u^+ \mathrm{d}x \tag{3.13}$$

$$\|u^-\|^2 + \langle u^+, u^-\rangle = \int_{\mathbb{R}} K(x) f(u^-) u^- \mathrm{d}x \tag{3.14}$$

$$t^2\|u^+\|^2 + ts\langle u^+, u^-\rangle = \int_{\mathbb{R}} K(x) f(tu^+) tu^+ \mathrm{d}x \tag{3.15}$$

$$s^2\|u^-\|^2 + ts\langle u^+, u^-\rangle = \int_{\mathbb{R}} K(x) f(su^-) su^- \mathrm{d}x. \tag{3.16}$$

We can assume, without loss of generality, that $t \leq s$. Then, by using $\langle u^+, u^-\rangle > 0$ and (3.15), we have

$$\|u^+\|^2 + \langle u^+, u^-\rangle \leq \int_{\mathbb{R}} K(x) \frac{f(tu^+)}{t} u^+ \mathrm{d}x.$$

It follows from (3.13) that

$$\int_{\mathbb{R}} K(x) \left(\frac{f(tu^+)}{tu^+} - \frac{f(u^+)}{u^+} \right) (u^+)^2 \mathrm{d}x \geq 0.$$

Hence, by (f_4) and since $u^+ \neq 0$ we obtain $t \geq 1$. On the other hand, since $t/s \leq 1$ and $\langle u^+, u^-\rangle > 0$, we get

$$\|u^-\|^2 + \langle u^+, u^-\rangle \geq \int_{\mathbb{R}} K(x) \frac{f(su^+)}{s} u^- \mathrm{d}x.$$

This together with (3.14) implies

$$\int_{\mathbb{R}} K(x) \left(\frac{f(su^-)}{su^-} - \frac{f(u^-)}{u^-} \right) (u^-)^2 \mathrm{d}x \leq 0$$

and consequently $s \leq 1$. Thus, we conclude that $t = s = 1$.

For the general case, we suppose that u does not necessarily belong to $\mathcal{N}_{nod}$. Let $(t, s), (t', s') \in (0, \infty) \times (0, \infty)$ such that $tu^+ + su^-$ and $t'u^+ + s'u^-$ belongs to $\mathcal{N}_{nod}$. We define $v = v^+ + v^-$, where $v^+ = tu^+$ and $v^- = su^-$. Then, we have that $v \in \mathcal{N}_{nod}$ and

$$\frac{t'}{t} v^+ + \frac{s'}{s} v^- = t'u^+ + s'u^- \in \mathcal{N}_{nod}.$$

Hence, by the first case, we reach $t'/t = 1$ and $s'/s = 1$, which completes the proof. □

Now, we shall present two technical lemmas that will be used in the next section.

Lemma 3.7. *Assume that* $(V_1)-(V_2)$ *and* $(f_1)-(f_5)$ *are satisfied. Let* $u \in X$ *be a function such that* $u^+ \not\equiv 0$, $u^- \not\equiv 0$, $I'(u)u^+ \leq 0$ *and* $I'(u)u^- \leq 0$. *Then the unique pair* (t,s) *given in Lemma* 3.6 *satisfies* $0 < t, s \leq 1$.

Proof. We can assume, without loss of generality, that $s \geq t > 0$ and $tu^+ + su^- \in \mathcal{N}_{nod}$. Now, since $I'(u)u^- \leq 0$ and $I'(tu^+ + su^-)su^- = 0$, we have

$$\|u^-\|^2 + \langle u^+, u^- \rangle \leq \int_{\mathbb{R}} K(x) f(u^-) u^- \mathrm{d}x$$

and

$$\|u^-\|^2 + \frac{t}{s}\langle u^+, u^- \rangle = \int_{\mathbb{R}} K(x) \frac{f(su^-)}{s} u^- \mathrm{d}x.$$

By Lemma 2.6 we get

$$\begin{aligned}
&\int_{\mathbb{R}} K(x) \left(\frac{f(u^-)}{u^-} - \frac{f(su^-)}{su^-} \right) (u^-)^2 \mathrm{d}x \\
&\qquad = \int_{\mathbb{R}} K(x) f(u^-) u^- \mathrm{d}x - \|u^-\|^2 - \frac{t}{s}\langle u^+, u^- \rangle \\
&\qquad \geq \|u^-\|^2 + \langle u^+, u^- \rangle - \|u^-\|^2 - \frac{t}{s}\langle u^+, u^- \rangle \\
&\qquad \geq \left(1 - \frac{t}{s}\right) \langle u^+, u^- \rangle \geq 0.
\end{aligned}$$

From this estimate, (f_4) and $u^- \not\equiv 0$, we obtain that $s \leq 1$. Thus, we finish the proof. □

Lemma 3.8. *Assume that* $(V_1)-(V_2)$ *and* $(f_1)-(f_5)$ *are satisfied. Let* $u \in X$ *be a function such that* $u^+ \not\equiv 0$, $u^- \not\equiv 0$ *and* (t,s) *be the unique pair of positive numbers given in Lemma* 3.6. *Then* (t,s) *is the unique maximum point of the function* $\phi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ *defined by* $\phi(\alpha, \beta) = I(\alpha u^+ + \beta u^-)$.

Proof. In the demonstration of Lemma 3.6, we saw that (t, s) is the unique critical point of ϕ in $(0, \infty) \times (0, \infty)$. Note that, by using (3.4), we get

$$\begin{aligned}\phi(\alpha, \beta) &\leq \frac{1}{2}\left\|\alpha u^+ + \beta u^-\right\|^2 - \frac{C_p}{p}\int_{\mathbb{R}} K(x)|\alpha u^+ + \beta u^-|^p \mathrm{d}x \\ &= \frac{(\alpha+\beta)^2}{2}\left\|(\frac{\alpha}{\alpha+\beta})u^+ + (\frac{\beta}{\alpha+\beta})u^-\right\|^2 \\ &- \frac{C_p}{p}(\alpha+\beta)^p\left\|(\frac{\alpha}{\alpha+\beta})u^+ + (\frac{\beta}{\alpha+\beta})u^-\right\|^p_{L^p_K}.\end{aligned}$$

Hence, since $p > 2$, $\phi(\alpha, \beta) \to -\infty$ as $|(\alpha, \beta)| \to \infty$. In particular, there exists $R > 0$ such that $\phi(\alpha, \beta) < \phi(t, s)$ for all $(\alpha, \beta) \in (0, \infty) \times (0, \infty) \setminus \overline{B_R}$, where $\overline{B_R}$ is the closure of the ball of radius R in $\mathbb{R}^2$. In order to finalize the proof, we shall show that the maximum of ϕ does not occur in the boundary of $\mathbb{R}_+ \times \mathbb{R}_+$. Suppose, by contradiction, that $(0, \beta)$ is a maximum point of ϕ, given $\alpha \geq 0$, we have that

$$\phi(\alpha, \beta) = \frac{\alpha^2}{2}\|u^+\|^2 + \alpha\beta\langle u^+, u^-\rangle - \int_{\mathbb{R}} K(x)F(\alpha u^+)\mathrm{d}x + \phi(0, \beta).$$

Arguing similarly to Lemma 3.1, we get

$$\frac{\alpha^2}{2}\|u^+\|^2 + \alpha\beta\langle u^+, u^-\rangle - \int_{\mathbb{R}} K(x)F(\alpha u^+)\mathrm{d}x > 0$$

for $\alpha > 0$ small enough. But this contradicts the assumption that $(0, \beta)$ is a maximum point of ϕ. The case $(\alpha, 0)$ is similar to the first one and we omit it. □

Now, we shall prove an upper bound for the nodal level c^* defined in (1.12).

Lemma 3.9. *Assume that $(V_1) - (V_2)$ and $(f_1) - (f_5)$ hold and C_p satisfies (1.13). If θ is the constant given by (f_3) and κ is given in (2.6), then*

$$c^* < \frac{\theta - 2}{2\theta\kappa}. \tag{3.17}$$

Proof. From Theorem 6.8 (see Appendix), there exists $w \in \mathcal{M}_{nod}$ such that $I_p(w) = c_p^*$, $I_p'(w)w^+ = 0$ and $I_p'(w)w^- = 0$. Consequently,

$$\frac{1}{2}\|w\|^2 - \frac{1}{p}\|w\|_{L_K^p}^p = c_p^*, \tag{3.18}$$

$$\|w^\pm\|^2 = \|w^\pm\|_{L_K^p}^p - \langle w^+, w^- \rangle \tag{3.19}$$

$$\|w\|^2 = \|w\|_{L_K^p}^p. \tag{3.20}$$

Hence, by (3.18) and (3.20), we get

$$\left(\frac{1}{2} - \frac{1}{p}\right)\|w\|_{L_K^p}^p = c_p^*. \tag{3.21}$$

Since $w^+ \not\equiv 0$ and $w^- \not\equiv 0$, by Lemma 3.6, there exist $t, s > 0$ such that $tw^+ + sw^- \in \mathcal{N}_{nod}$. Consequently, we obtain

$$\begin{aligned} c^* \le I(tw^+ + sw^-) &= \frac{t^2}{2}\|w^+\|^2 + ts\langle w^+, w^- \rangle + \frac{s^2}{2}\|w^-\|^2 \\ &\quad - \int_{\mathbb{R}} K(x)F(tw^+)\mathrm{d}x - \int_{\mathbb{R}} K(x)F(sw^-)\mathrm{d}x. \end{aligned}$$

This together with (3.4) implies

$$c^* \le \frac{t^2}{2}\|w^+\|^2 + ts\langle w^+, w^- \rangle + \frac{s^2}{2}\|w^-\|^2 - \frac{C_p t^p}{p}\|w^+\|_{L_K^p}^p - \frac{C_p s^p}{p}\|w^-\|_{L_K^p}^p.$$

By (3.19) and Lemma 2.6, we have

$$\begin{aligned} c^* &\le \frac{t^2}{2}(\|w^+\|_{L_K^p}^p - \langle w^+, w^- \rangle) + ts\langle w^+, w^- \rangle + \frac{s^2}{2}(\|w^-\|_{L_K^p}^p - \langle w^+, w^- \rangle) \\ &\quad - \frac{C_p t^p}{p}\|w^+\|_{L_K^p}^p - \frac{C_p s^p}{p}\|w^-\|_{L_K^p}^p \\ &= \left(\frac{t^2}{2} - \frac{C_p t^p}{p}\right)\|w^+\|_{L_K^p}^p + \left(\frac{s^2}{2} - \frac{C_p s^p}{p}\right)\|w^-\|_{L_K^p}^p \\ &\quad - \frac{1}{2}(t-s)^2\langle w^+, w^- \rangle \\ &\le \max_{\xi \ge 0}\left(\frac{\xi^2}{2} - \frac{C_p \xi^p}{p}\right)\|w\|_{L_K^p}^p. \end{aligned}$$

On the other hand, it is easy to see that

$$\max_{\xi \geq 0} \left(\frac{\xi^2}{2} - \frac{C_p \xi^p}{p} \right) = C_p^{\frac{2}{2-p}} \left(\frac{1}{2} - \frac{1}{p} \right).$$

Hence, by (3.21) it follows that

$$c^* \leq C_p^{\frac{2}{2-p}} \left(\frac{1}{2} - \frac{1}{p} \right) \|w\|_{L_K^p}^p = C_p^{\frac{2}{2-p}} c_p^*.$$

Therefore, by the definition of C_p given in (1.13), we obtain (3.17). □

The next step will be to obtain a minimizing sequence for the nodal level c^* with a special behavior. For this, for $\lambda > 0$, we begin by defining the set

$$\widetilde{S}_\lambda = \{u \in \mathcal{N}_{nod} \,:\, I(u) < c^* + \lambda\}.$$

Lemma 3.10. *Assume that* $(V_1)-(V_2)$ *and* $(f_1)-(f_5)$ *hold and* C_p *satisfies* (1.13). *For* $\lambda > 0$ *small enough, there exists* $m_\lambda \in (0, \frac{1}{\kappa})$ *such that*

$$0 < m_0' \leq \|u^+\|^2, \|u^-\|^2 < \|u\|^2 \leq m_\lambda,$$

for any $u \in \widetilde{S}_\lambda$.

Proof. Let $u \in \widetilde{S}_\lambda$. By Lemma 3.4 and by using $\langle u^+, u^- \rangle > 0$, we have $m_0' \leq \|u^+\|^2, \|u^-\|^2 < \|u\|^2$. On the other hand, by (f_3) and since $I'(u)u = 0$, we obtain

$$\begin{aligned} c^* + \lambda > I(u) &= I(u) - \frac{1}{\theta} I'(u)u \\ &= \left(\frac{1}{2} - \frac{1}{\theta} \right) \|u\|^2 + \frac{1}{\theta} \int_{\mathbb{R}} K(x) \left[f(u)u - \theta F(u) \right] \mathrm{d}x \geq \left(\frac{1}{2} - \frac{1}{\theta} \right) \|u\|^2. \end{aligned}$$

By Lemma 3.9, we can take $\lambda > 0$ such that $c^* + \lambda < \left(\frac{\theta - 2}{2\theta\kappa} \right)$. Consequently, it follows that

$$\|u\|^2 \leq \frac{2\theta}{\theta - 2}(c^* + \lambda) =: m_\lambda < \frac{1}{\kappa},$$

for all $u \in \widetilde{S}_\lambda$. This concludes the proof of the lemma. □

Lemma 3.11. *Assume that* $(V_1)-(V_2)$, (K_1) *and* $(f_1)-(f_5)$ *are satisfied. Let* $(u_n) \subset H^{1/2}(\mathbb{R})$ *be a sequence such that* $u_n \rightharpoonup u$ *weakly in* $H^{1/2}(\mathbb{R})$ *and* $b := \sup_{n\in\mathbb{N}} \|u_n\|^2_{1/2,2} < 1$. *Then, up to a subsequence, one has*

$$\lim_{n\to+\infty} \int_{\mathbb{R}} K(x)f(u_n)u_n \mathrm{d}x = \int_{\mathbb{R}} K(x)f(u)u\mathrm{d}x; \tag{3.22}$$

$$\lim_{n\to+\infty} \int_{\mathbb{R}} K(x)f(u_n^+)u_n^+ \mathrm{d}x = \int_{\mathbb{R}} K(x)f(u^+)u^+\mathrm{d}x; \tag{3.23}$$

$$\lim_{n\to+\infty} \int_{\mathbb{R}} K(x)f(u_n^-)u_n^- \mathrm{d}x = \int_{\mathbb{R}} K(x)f(u^-)u^-\mathrm{d}x; \tag{3.24}$$

$$\lim_{n\to+\infty} \int_{\mathbb{R}} K(x)F(u_n) \mathrm{d}x = \int_{\mathbb{R}} K(x)F(u)\mathrm{d}x; \tag{3.25}$$

$$\lim_{n\to+\infty} \int_{\mathbb{R}} K(x)f(u_n)v \mathrm{d}x = \int_{\mathbb{R}} K(x)f(u)v\mathrm{d}x, \forall v \in H^{1/2}(\mathbb{R}). \tag{3.26}$$

Proof. We will prove only (3.22), since the proofs of (3.24)-(3.26) are similar and we will omit them. Let $\pi < \alpha < \pi/b^2$. Then, by using (f_1) and (f_2), we have

$$\lim_{|t|\to\infty} \frac{f(t)t}{e^{\alpha t^2}-1} = 0 \quad \text{and} \quad \lim_{|t|\to 0} \frac{f(t)t}{t^2} = 0. \tag{3.27}$$

Hence, given $q > 2$ and $\varepsilon > 0$, there exists $0 < t_0(\varepsilon) < t_1(\varepsilon)$ and $C_\varepsilon > 0$ such that, for all $t, x \in \mathbb{R}$,

$$K(x)|f(t)t| \leq \varepsilon C(|t|^2 + e^{\alpha t^2} - 1) + C_\varepsilon K(x)\chi_{[t_0(\varepsilon),t_1(\varepsilon)]}(|t|)|t|^q, \tag{3.28}$$

Now, from the continuous embedding $H^{1/2}(\mathbb{R}) \hookrightarrow L^s(\mathbb{R})$ for $s \geq 2$, and Lemma 2.4, we can find $M > 0$ such that, for all $n \in \mathbb{N}$,

$$\int_{\mathbb{R}} |u_n|^2 \mathrm{d}x \leq M, \int_{\mathbb{R}} |u_n|^q \mathrm{d}x \leq M \quad \text{and} \quad \int_{\mathbb{R}} (e^{\alpha u_n^2} - 1)\mathrm{d}x \leq M. \tag{3.29}$$

Denoting $A_n^\varepsilon = \{x \in \mathbb{R} : t_0(\varepsilon) \leq |u_n(x)| \leq t_1(\varepsilon)\}$, we get

$$t_0(\varepsilon)^2 |A_n^\varepsilon| = \int_{A_n^\varepsilon} t_0(\varepsilon)^2 \mathrm{d}x \leq \int_{\mathbb{R}} |u_n|^2 \mathrm{d}x \leq M, \quad \text{for all} \quad n \in \mathbb{N}.$$

Thus, utilizing (K_1), there exists $r(\varepsilon) > 0$ such that

$$\int_{A_n^\varepsilon \cap B^c_{r(\varepsilon)}(0)} K(x)\mathrm{d}x < \frac{\varepsilon}{C_\varepsilon t_1(\varepsilon)^q}, \quad \text{for all} \quad n \in \mathbb{N} \tag{3.30}$$

and by using (3.29) and (3.30) in (3.28), we reach

$$\int_{B^c_{r(\varepsilon)}(0)} K(x)|f(u_n)u_n|\mathrm{d}x \leq (2CM+1)\varepsilon, \quad \text{for all} \quad n \in \mathbb{N}. \tag{3.31}$$

On the other hand, using that $u_n \rightharpoonup u$ weakly in $H^{1/2}(\mathbb{R})$ and the locally compact embedding $H^{1/2}(\mathbb{R}) \hookrightarrow L^2(\mathbb{R})$, up to a subsequence, we have $u_n(x) \to u(x)$ a.e. in $\mathbb{R}$. Thus, $K(x)f(u_n(x))u_n(x) \to K(x)f(u(x))u(x)$ a.e. in $\mathbb{R}$ and according to (3.27), (3.29) and Strauss Lemma [9, Theorem A.I], one has

$$\lim_{n\to+\infty} \int_{B_{r(\varepsilon)}} K(x)f(u_n)u_n\mathrm{d}x = \int_{B_{r(\varepsilon)}} K(x)f(u)u\mathrm{d}x. \tag{3.32}$$

Combining (3.31) and (3.32), the proof of (3.22) follows. □

From now on, we will write $\widetilde{S}_\lambda$ with $\lambda > 0$ given in Lemma 3.10.

Lemma 3.12. *Assume that $(V_1)-(V_2)$, (K_1) and $(f_1)-(f_5)$ hold and C_p satisfies (1.13). Then for any $q > 2$, there exists $\delta_q > 0$ such that*

$$0 < \delta_q \leq \int_{\mathbb{R}} K(x)|u^+|^q\mathrm{d}x, \int_{\mathbb{R}} K(x)|u^-|^q\mathrm{d}x < \int_{\mathbb{R}} K(x)|u|^q\mathrm{d}x,$$

for each $u \in \widetilde{S}_\lambda$.

Proof. Let $u \in \widetilde{S}_\lambda$ and $q > 2$. We know that

$$\|u^\pm\|^2 + \langle u^+, u^-\rangle = \int_{\mathbb{R}} K(x)f(u^\pm)u^\pm\mathrm{d}x.$$

By using Lemma 2.6 and Lemma 3.4, it follows that

$$0 < m_0' \leq \|u^\pm\|^2 < \int_{\mathbb{R}} K(x)f(u^\pm)u^\pm\mathrm{d}x$$

and from (3.1), we have

$$m_0' \leq \varepsilon \int_{\mathbb{R}} K(x)|u^{\pm}|^2 \mathrm{d}x + C_\varepsilon \int_{\mathbb{R}} K(x)|u^{\pm}|(e^{\pi|u^{\pm}|^2} - 1)\mathrm{d}x.$$

Now, Corollary 2.3 and the fact that $u \in \widetilde{S}_\lambda$ imply that there exists $C_1 > 0$, independent of u, such that

$$\int_{\mathbb{R}} K(x)|u|^2 \mathrm{d}x \leq C_1.$$

Choosing $\varepsilon > 0$ such that $m_0' - \varepsilon C_1 > 0$, we obtain

$$0 < \frac{m_0' - \varepsilon C_1}{C_\varepsilon} \leq \int_{\mathbb{R}} K(x)|u^{\pm}|(e^{\pi|u^{\pm}|^2} - 1)\mathrm{d}x. \tag{3.33}$$

Let $t' > 0$ sufficiently close to 1 such that $\pi t' m_\lambda \kappa \leq \pi$, with $1/t + 1/t' = 1$ and $t > q$. Utilizing the Hölder inequality, Lemma 3.10, $K(x) \leq C$ and Lemma 2.4, we reach

$$\begin{aligned}
\int_{\mathbb{R}} K(x)|u^{\pm}|(e^{\pi|u^{\pm}|^2} - 1)\mathrm{d}x &= \int_{\mathbb{R}} K(x)^{\frac{1}{t}}|u^{\pm}|K(x)^{\frac{1}{t'}}(e^{\pi|u^{\pm}|^2} - 1)\mathrm{d}x \\
&\leq \left(\int_{\mathbb{R}} K(x)|u^{\pm}|^t \mathrm{d}x\right)^{\frac{1}{t}} \left(\int_{\mathbb{R}} K(x)(e^{\pi t'\|u^{\pm}\|^2_{1/2,2}\left(\frac{|u^{\pm}|}{\|u\|_{1/2,2}}\right)^2} - 1)\mathrm{d}x\right)^{\frac{1}{t'}} \\
&\leq C^{\frac{1}{t'}} \left(\int_{\mathbb{R}} K(x)|u^{\pm}|^t \mathrm{d}x\right)^{\frac{1}{t}} \left(\int_{\mathbb{R}} (e^{\pi t' m_\lambda \kappa\left(\frac{|u^{\pm}|}{\|u^{\pm}\|_{1/2,2}}\right)^2} - 1)\mathrm{d}x\right)^{\frac{1}{t'}} \\
&\leq C\|u^{\pm}\|_{L^t_K}.
\end{aligned}$$

This last inequality and (3.33) imply that

$$0 < \frac{m_0' - \varepsilon C_1}{C_\varepsilon} \leq C\|u^{\pm}\|_{L^t_K}. \tag{3.34}$$

Now, we suppose, by contradiction, that there exists $(u_n) \subset \widetilde{S}_\lambda$ such that $\|u_n^{\pm}\|_{L^q_K} \to 0$ as $n \to \infty$. From Lemma 3.10 we obtain that $(u_n^{\pm})$ is bounded in $L^{2t}(\mathbb{R})$. Consequently, since $q < t < 2t$, by the interpolation inequality we get that $\|u_n^{\pm}\|_{L^t_K} \to 0$ as $n \to \infty$, which is impossible in view of (3.34). Thus, we have completed the proof. □

The next technical result will be used in the proof of Lemma 3.14.

Lemma 3.13. *Assume* (f_3)–(f_4). *Then the function* $H(t) := f(t)t - 2F(t)$ *satisfies*

(i) $H(0) = 0$ *and* $H(t) > 0$, *for all* $t \neq 0$;

(ii) $H(t_0) \leq H(t_1)$ *if* $0 < t_0 \leq t_1$;

(iii) $H(t_0) \geq H(t_1)$ *if* $t_0 \leq t_1 < 0$.

Proof. Let us show (iii). First we note that $H \in C^1(\mathbb{R})$ and $H'(t) = f'(t)t - f(t)$, for all $t \in \mathbb{R}$. From (f_4), we have

$$\frac{d}{dt}\left(\frac{f(t)}{|t|}\right) \geq 0, \quad \text{for all} \quad t \in \mathbb{R} \setminus \{0\}.$$

If $t < 0$ then $f(t) - f'(t)t \geq 0$ and therefore $H'(t) \leq 0$ for all $t < 0$. Thus, $H(t)$ is decreasing for $t \leq 0$, which implies the item (iii). The proof of the item (ii) is similar. □

Next, we have all the results that will allow us to prove that the nodal level c^* is attained in a function $u \in X$ with $u^+ \not\equiv 0$ and $u^- \not\equiv 0$.

Lemma 3.14. *Assume that* $(V_1) - (V_2)$, (K_1) *and* $(f_1) - (f_5)$ *hold and* C_p *satisfies* (1.13). *Then there exists* $\widetilde{u} \in \mathcal{N}_{nod}$ *such that* $I(\widetilde{u}) = c^*$.

Proof. Let $(u_n) \subset \mathcal{N}_{nod}$ be a sequence such that $I(u_n) \to c^*$ as $n \to +\infty$. We can assume that $u_n \in \widetilde{S}_\lambda$, for all $n \in \mathbb{N}$. In particular, by Lemma 3.10, we have

$$m_0' \leq \|u_n^\pm\|^2 < \|u_n\|^2 \leq m_\lambda, \quad \text{for all} \quad n \in \mathbb{N}, \quad \text{with} \quad m_\lambda \in \left(0, \frac{1}{\kappa}\right).$$

Thus, $(u_n), (u_n^+)$ and (u_n^-) are bounded in X. Since X is a Hilbert space, up to a subsequence, there exists $u \in X$ such that $u_n^\pm \rightharpoonup u^\pm$ and $u_n \rightharpoonup u$ in X. Let $q > 2$. From Corollary 2.3, up to a subsequence, we have $u_n^\pm \to u^\pm$ in L_K^q and utilizing Lemma 3.12, there exists $\delta_q > 0$ such that

$$0 < \delta_q \leq \int_{\mathbb{R}} K(x)|u_n^\pm|^q \mathrm{d}x < \int_{\mathbb{R}} K(x)|u_n|^q \mathrm{d}x, \quad \text{for all} \quad n \in \mathbb{N}.$$

Hence $u^+ \not\equiv 0$ and $u^- \not\equiv 0$ in X. Now, from Lemma 3.6 there exist $t, s \in (0,\infty)$ such that $\widetilde{u} = tu^+ + su^- \in \mathcal{N}_{nod}$. We claim that $I'(u)u^\pm \le 0$. Since $\sup_{n\in\mathbb{N}} \|u_n\|^2 \le m_\lambda$ and $\|u_n\|_{1/2,2}^2 \le \kappa\|u_n\|^2$, we have $\sup_{n\in\mathbb{N}} \|u_n\|_{1/2,2} \in (0,1)$. Moreover, since the embedding $X \hookrightarrow L^2_{loc}(\mathbb{R})$ is compact, up to a subsequence, we can assume that $u_n^\pm(x) \to u^\pm(x)$ a.e. in $\mathbb{R}$. By the convergence (3.24) in Lemma 3.11 and by the Fatou Lemma, it follows that

$$\begin{aligned}\|u^+\|^2 + \langle u^+, u^-\rangle &\le \liminf_{n\to+\infty} \left(\|u_n^+\|^2 + \langle u_n^+, u_n^-\rangle\right)\\ &= \liminf_{n\to+\infty} \int_{\mathbb{R}} K(x) f(u_n^+)u_n^+ \mathrm{d}x = \int_{\mathbb{R}} K(x) f(u^+)u^+ \mathrm{d}x.\end{aligned}$$

Hence, $I'(u)u^+ \le 0$. Similarly, we get $I'(u)u^- \le 0$. Then, by Lemma 3.7, we obtain $0 < t, s \le 1$. In particular, $\|\widetilde{u}\|^2 \le \|u\|^2$. Now, in order to conclude the proof, note that using the convergence in Lemma 3.11 and Lemma 3.13, it holds

$$\begin{aligned}c^* \le I(\widetilde{u}) = I(\widetilde{u}) - \frac{1}{2}I'(\widetilde{u})\widetilde{u} &= \frac{1}{2}\int_{\mathbb{R}} K(x)\left(f(\widetilde{u})\widetilde{u} - 2F(\widetilde{u})\right)\mathrm{d}x\\ &= \frac{1}{2}\int_{\mathbb{R}} K(x)H(tu^+)\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}} K(x)H(su^-)\mathrm{d}x\end{aligned}$$

and therefore

$$\begin{aligned}c^* &\le \frac{1}{2}\int_{\mathbb{R}} K(x)H(u^+)\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}} K(x)H(u^-)\mathrm{d}x\\ &= \frac{1}{2}\int_{\mathbb{R}} K(x)\left(f(u)u - 2F(u)\right)\mathrm{d}x\\ &= I(u_n) - \frac{1}{2}I'(u_n)u_n + o_n(1),\\ &= I(u_n) + o_n(1) = c^*\end{aligned}$$

which concludes the proof. □

Next, we consider $D = (\frac{1}{2}, \frac{3}{2})\times(\frac{1}{2}, \frac{3}{2})$ and $g : \overline{D} \to X$ given by $g(\alpha,\beta) = \alpha\widetilde{u}^+ + \beta\widetilde{u}^-$, where $\widetilde{u}$ was obtained in Lemma 3.14. We shall prove an auxiliary result and present some notations that will be used in the proof of Theorem 1.1.

Lemma 3.15. *Let* $P = \{u \in X : u(x) \geq 0\, a.e.\ x \in \mathbb{R}\}$ *and* $-P = \{u \in X : u(x) \leq 0\, a.e.\ x \in \mathbb{R}\}$*. Then* $d_0' = dist(g(\overline{D}), \Lambda) > 0$*, where* $\Lambda := P \cup (-P)$*.*

Proof. We suppose, by contradiction, that $d_0' = dist(g(\overline{D}), \Lambda) = 0$. Hence, we can find $(v_n) \subset g(\overline{D})$ and $(w_n) \subset \Lambda$ such that $\|v_n - w_n\| \to 0$ as $n \to \infty$. We can assume, without loss of generality, that $w_n \geq 0$ a.e. in $\mathbb{R}$. Since $v_n \in g(\overline{D})$, there exist $\alpha_n, \beta_n \in [\frac{1}{2}, \frac{3}{2}]$ such that $v_n = \alpha_n \widetilde{u}^+ + \beta_n \widetilde{u}^-$. By compactness of $[\frac{1}{2}, \frac{3}{2}]$, up to a subsequence, we have $\alpha_n \to a_0$ and $\beta_n \to b_0$ as $n \to \infty$. Hence

$$v_n \to a_0 \widetilde{u}^+ + b_0 \widetilde{u}^- \quad \text{in} \quad X.$$

Thus, we obtain $w_n \to a_0 \widetilde{u}^+ + b_0 \widetilde{u}^-$ in X. Now, by Proposition 2.2, we have

$$w_n(x) \to a_0 \widetilde{u}^+(x) + b_0 \widetilde{u}^-(x) \quad \text{a.e. in} \quad \mathbb{R}.$$

Since $\widetilde{u}^- \neq 0$, the convergence above produces a contradiction with the assumption that $w_n \geq 0$ a.e. in $\mathbb{R}$, which completes the proof. □

4 Proof of Theorem 1.1

By Lemma 3.14, there exists $\widetilde{u} \in \mathcal{N}_{nod}$ such that $I(\widetilde{u}) = c^*$. We shall prove that $\widetilde{u}$ is a critical point of the functional I. Suppose, by contradiction, that $I'(\widetilde{u}) \neq 0$. Thus, by the continuity of I', there exist $\lambda, \delta > 0$ with $\delta \leq \frac{d_0'}{2}$, where d_0' is given in Lemma 3.15, such that

$$\|I'(v)\| \geq \lambda, \quad \text{for all} \quad v \in B_{3\delta}(\widetilde{u}). \tag{4.1}$$

From Lemma 3.8 we have that the function $(I \circ g)(\alpha, \beta)$, for $(\alpha, \beta) \in \overline{D}$, has a strict maximum point $(1, 1)$. In particular,

$$m^* = \max_{(\alpha,\beta) \in \partial D} (I \circ g)(\alpha, \beta) < c^*.$$

Let $0 < \varepsilon < \min\{(c^* - m^*)/2, \lambda\delta/8\}$ and $S = B_\delta(\widetilde{u})$. By the choice of ε and by condition (4.1), if $v \in S_{2\delta} = B_{3\delta}(\widetilde{u})$ we have $\|I'(v)\| \geq \frac{8\varepsilon}{\delta}$. In particular,

$$\forall\, v \in I^{-1}([c^* - 2\varepsilon, c^* + 2\varepsilon]) \cap S_{2\delta}, \text{ it has to satisfy } \|I'(v)\| \geq \frac{8\varepsilon}{\delta}.$$

Hence, by the Quantitative Deformation Lemma in [41, Lemma 2.3], there exists $\eta \in C([0,1] \times X, X)$ such that

(i) $\eta(t,u) = u$, if $t = 0$ or $u \notin I^{-1}([c^* - 2\varepsilon, c^* + 2\varepsilon]) \cap S_{2\delta}$;

(ii) $\eta(1, I^{c^*+\varepsilon} \cap S) \subset I^{c^*-\varepsilon}$;

(iii) $\eta(t,\cdot)$ is a homeomorphism of X, $\forall\, t \in [0,1]$;

(iv) $\|\eta(t,u) - u\| \leq \delta$, $\forall\, u \in X$, $\forall\, t \in [0,1]$;

(v) $I(\eta(\cdot,u))$ is non increasing, $\forall\, u \in X$;

(vi) $I(\eta(t,u)) < c^*$, $\forall\, u \in I^{c^*} \cap S_\delta$, $\forall\, t \in (0,1]$.

As an application, we get

$$\max_{(\alpha,\beta)\in\overline{D}} I(\eta(1, g(\alpha,\beta))) < c^*. \tag{4.2}$$

Indeed, if $(\alpha,\beta) \in D$ with $(\alpha,\beta) \neq (1,1)$, by using Lemma 3.8 we have $I(g(\alpha,\beta)) < c^*$. Hence

$$I(\eta(1, g(\alpha,\beta))) \leq I(\eta(0, g(\alpha,\beta))) = I(g(\alpha,\beta)) < c^*.$$

If $(\alpha,\beta) = (1,1)$ then $g(1,1) = \widetilde{u} \in I^{c^*+\varepsilon} \cap S$. Thus $I(\eta(1, g(1,1))) < c^* - \varepsilon$, showing (4.2). Notice that, by definition of c^*, inequality (4.2) implies that $\eta(1, g(D)) \cap \mathcal{N}_{nod} = \emptyset$.

Now, let us define $h(\alpha,\beta) = \eta(1, g(\alpha,\beta))$. We claim that

$$h(\alpha,\beta) = g(\alpha,\beta) \quad \text{in} \quad \partial D. \tag{4.3}$$

Indeed, given $(\alpha,\beta) \in \partial D$, by the definition of m^* and by the choice of ε, we have

$$I(g(\alpha,\beta)) \leq m^* = c^* - 2\frac{(c^* - m^*)}{2} < c^* - 2\varepsilon,$$

which implies that $g(\alpha,\beta) \notin I^{-1}([c^* - 2\varepsilon, c^* + 2\varepsilon])$. Thus, by using the properties of η, we get (4.3).

Claim 4.1. It holds that $h^+(\alpha,\beta) \neq 0$ and $h^-(\alpha,\beta) \neq 0$ for all $(\alpha,\beta) \in \overline{D}$.

Indeed, let $v \in \Lambda$. By the choice of $\delta > 0$ and Lemma 3.15, we have

$$\begin{aligned}\|h(\alpha,\beta) - v\| &\geq \|g(\alpha,\beta) - v\| - \|h(\alpha,\beta) - g(\alpha,\beta)\| \\ &\geq \|g(\alpha,\beta) - v\| - \delta \geq d_0' - \frac{d_0'}{2} = \frac{d_0'}{2}.\end{aligned}$$

Consequently, $h^+(\alpha,\beta) \neq 0$ and $h^-(\alpha,\beta) \neq 0$ for all $(\alpha,\beta) \in \overline{D}$, concluding the statement.

Next, we consider the vector fields

$$\mathcal{F}(\alpha,\beta) = (I'(g(\alpha,\beta))\widetilde{u}^+, I'(g(\alpha,\beta))\widetilde{u}^-)$$

and

$$\mathcal{G}(\alpha,\beta) = (\frac{1}{\alpha}I'(h(\alpha,\beta))h(\alpha,\beta)^+, \frac{1}{\beta}I'(h(\alpha,\beta))h(\alpha,\beta)^-).$$

From (4.3), we have $\mathcal{F} = \mathcal{G}$ in ∂D. Hence, by the degree theory, we have

$$\deg(\mathcal{F}, D, (0,0)) = \deg(\mathcal{G}, D, (0,0)). \tag{4.4}$$

Claim 4.2. $\deg(\mathcal{F}, D, (0,0)) = 1$.

Indeed, consider

$$\mathcal{F}_1(\alpha,\beta) = I'(\alpha\widetilde{u}^+ + \beta\widetilde{u}^-)\widetilde{u}^+ \quad \text{and} \quad \mathcal{F}_2(\alpha,\beta) = I'(\alpha\widetilde{u}^+ + \beta\widetilde{u}^-)\widetilde{u}^-,$$

the coordinates functions of the vector field $\mathcal{F}$. Calculating the partial derivatives of $\mathcal{F}_1$ and $\mathcal{F}_2$, we get

$$\begin{cases} \dfrac{\partial \mathcal{F}_1}{\partial \alpha}(\alpha,\beta) &= \|\widetilde{u}^+\|^2 - \displaystyle\int_{\mathbb{R}} K(x) f'(\alpha\widetilde{u}^+)(\widetilde{u}^+)^2 \mathrm{d}x, \\ \dfrac{\partial \mathcal{F}_1}{\partial \beta}(\alpha,\beta) &= \dfrac{\partial \mathcal{F}_2}{\partial \alpha}(\alpha,\beta) = \langle \widetilde{u}^+, \widetilde{u}^- \rangle, \\ \dfrac{\partial \mathcal{F}_2}{\partial \beta}(\alpha,\beta) &= \|\widetilde{u}^-\|^2 - \displaystyle\int_{\mathbb{R}} K(x) f'(\beta\widetilde{u}^-)(\widetilde{u}^-)^2 \mathrm{d}x. \end{cases}$$

Now, for $(\alpha,\beta)=(1,1)$ in the above equations and since $I'(\widetilde{u})\widetilde{u}^+=0$ and $I'(\widetilde{u})\widetilde{u}^-=0$, we obtain

$$\begin{cases} \dfrac{\partial \mathcal{F}_1}{\partial\alpha}(1,1) &= -\langle \widetilde{u}^+,\widetilde{u}^-\rangle + \displaystyle\int_{\mathbb{R}} K(x)G(\widetilde{u}^+)\widetilde{u}^+\mathrm{d}x,\\ \dfrac{\partial \mathcal{F}_1}{\partial\beta}(1,1) &= \dfrac{\partial \mathcal{F}_2}{\partial\alpha}(1,1) = \langle \widetilde{u}^+,\widetilde{u}^-\rangle,\\ \dfrac{\partial \mathcal{F}_2}{\partial\beta}(1,1) &= -\langle \widetilde{u}^+,\widetilde{u}^-\rangle + \displaystyle\int_{\mathbb{R}} K(x)G(\widetilde{u}^-)\widetilde{u}^-\mathrm{d}x, \end{cases}$$

where $G(t)=f(t)-f'(t)t$, for $t\in\mathbb{R}$. By (f_4), $\widetilde{u}^+\neq 0$ and $\widetilde{u}^-\neq 0$, it is easy to see that

$$\int_{\mathbb{R}} K(x)G(\widetilde{u}^+)\widetilde{u}^+\mathrm{d}x<0 \text{ and } \int_{\mathbb{R}} K(x)G(\widetilde{u}^-)\widetilde{u}^-\mathrm{d}x<0. \tag{4.5}$$

Hence, by (4.5) and $\langle \widetilde{u}^+,\widetilde{u}^-\rangle>0$, it follows that

$$\det\begin{bmatrix} \dfrac{\partial \mathcal{F}_1}{\partial\alpha}(1,1) & \dfrac{\partial \mathcal{F}_1}{\partial\beta}(1,1)\\ \dfrac{\partial \mathcal{F}_2}{\partial\alpha}(1,1) & \dfrac{\partial \mathcal{F}_2}{\partial\beta}(1,1)\end{bmatrix}>0.$$

Since $(1,1)$ is the unique solution of $\mathcal{F}(\alpha,\beta)=(0,0)$ in D, by the definition of topological degree, we have $\deg(\mathcal{F},D,(0,0))=1$, showing Claim 4.2.

In view of Claim 4.2 and (4.4), we obtain

$$\deg(\mathcal{G},D,(0,0))=\deg(\mathcal{F},D,(0,0))=1$$

and therefore there exists $(\alpha_0,\beta_0)\in D$ such that $\mathcal{G}(\alpha_0,\beta_0)=(0,0)$, that is,

$$\begin{cases} I'(\eta(1,g(\alpha_0,\beta_0)))\eta(1,g(\alpha_0,\beta_0))^+ &=0,\\ I'(\eta(1,g(\alpha_0,\beta_0)))\eta(1,g(\alpha_0,\beta_0))^- &=0.\end{cases} \tag{4.6}$$

By Claim 4.1, one has $h(\alpha_0,\beta_0)^+\neq 0$ and $h(\alpha_0,\beta_0)^-\neq 0$. Hence, system (4.6) implies that $h(\alpha_0,\beta_0)$ belongs to $\eta(1,g(D))\cap\mathcal{N}_{nod}$ and by the definition of c^*, $I(h(\alpha_0,\beta_0))=I(\eta(1,g(\alpha_0,\beta_0))\geq c^*$, which is a contradiction according to (4.2). Therefore, we conclude that $I'(\widetilde{u})=0$ and this completes the proof of Theorem 1.1.

5 Proof of Theorem 1.2

First, we define the set $\overline{S}_\lambda = \{u \in \mathcal{N} : I(u) < c^* + \lambda\}$, where λ is given in Lemma 3.10. By using similar ideas from of the proof of Lemma 3.14, we find $\widetilde{v} \in \mathcal{N}$ such that $I(\widetilde{v}) = c$, where c is the ground state level defined in (1.11). Moreover, utilizing the same steps of the proof of Theorem 1.1, we show that the function $\widetilde{v}$ satisfies that $I'(\widetilde{v}) = 0$. Thus, $\widetilde{v}$ is a ground state solution of problem (1.1). Now, in order to prove (1.14), we consider the function $\widetilde{u}$ obtained in Theorem 1.1. Since $\widetilde{u}^+ \neq 0$ and $\widetilde{u}^- \neq 0$, by Lemma 3.1, there exists an unique pair (t_1, t_2) such that $t_1\widetilde{u}^+ \in \mathcal{N}$ and $t_2\widetilde{u}^- \in \mathcal{N}$. By Corollary 3.3, we have $c > 0$. Now, the definition of c, Lemma 2.6, Lemma 3.7 and Lemma 3.8, we conclude that

$$0 < 2c \leq I(t_1\widetilde{u}^+) + I(t_2\widetilde{u}^-) < I(t_1\widetilde{u}^+ + t_2\widetilde{u}^-) \leq I(\widetilde{u}^+ + \widetilde{u}^-) = c^*,$$

showing (1.14). In particular, the inequality above shows that can not exist a nodal ground state solution of problem (1.1). Therefore, the ground state solution $\widetilde{v}$ is positive or negative.

6 Appendix

In this section, we consider the problem

$$(-\Delta)^{1/2}u + V(x)u = K(x)|u|^{p-2}u \quad \text{in} \quad \mathbb{R}, \tag{6.1}$$

where $p > 2$, V and K are such that $(V_1) - (V_2)$ and (K_1) hold. The energy functional $I_p : X \to \mathbb{R}$ associated to (6.1) is given by

$$I_p(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p}\|u\|^p_{L^p_K}.$$

We define the Nehari manifold and nodal set associated to I_p and the respective ground state and nodal levels by

$$\mathcal{M} = \{u \in X \setminus \{0\} : I'_p(u)u = 0\}, \tag{6.2}$$

$$\mathcal{M}_{nod} = \{u \in X : u^+ \not\equiv 0, u^- \not\equiv 0, I'_p(u)u^+ = I'_p(u)u^- = 0\}, \tag{6.3}$$

$$c_p = \inf_{u \in \mathcal{M}} I_p(u), \tag{6.4}$$

$$c_p^* = \inf_{u \in \mathcal{M}_{nod}} I_p(u). \tag{6.5}$$

We will show that problem (6.1) has a nodal solution of least energy. The steps to show this are the same of Sections 2, 3 and 4. Thus, many computations will be omitted in order to avoid repetitions.

Lemma 6.1. *Given $u \in X \setminus \{0\}$, there exists an unique $t = t(u) > 0$ such that $tu \in \mathcal{M}$. In addition, t satisfies*

$$I_p(tu) = \max_{s \geq 0} I_p(su). \tag{6.6}$$

Proof. Lethttps://pt.overleaf.com/project/6119b3013b27d19d037226ac $h(s) := I_p(su) = s^2\|u\|^2/2 - s^p\|u\|^p_{L^p_K}/p$, for $s \geq 0$. Since $p > 2$, we have $h(s) > 0$ for $s > 0$ small enough and $h(s) \to -\infty$ as $s \to \infty$. Hence, there exists a $t > 0$ satisfying (6.6). In particular, $tu \in \mathcal{M}$. Moreover, $h'(t) = 0$ if and only if $t = (\|u\|^2/\|u\|^p_{L^p_K})^{1/(p-2)}$. □

Corollary 6.2. *Let $u \in X \setminus \{0\}$. Then $u \in \mathcal{M}$ if only if $I_p(u) = \max_{s \geq 0} I_p(su)$.*

Lemma 6.3. *There exist $\beta_0 > 0$ and $\ell_0 > 0$ such that $\|u\|^2 \geq \ell_0$, for all $u \in \mathcal{M}$, $\|u^+\|^2 \geq \ell_0, \|u^-\|^2 \geq \ell_0$, for all $u \in \mathcal{M}_{nod}$ and $I_p(u) \geq \beta_0$.*

Proof. The proof of this result follows by Corollary 2.3 and using the same ideas of Lemmas 3.2 and 3.4. □

The lemma above shows that the levels c_p and c_p^* are well defined and $c_p \geq c_p^* \geq \beta_0$, since $\mathcal{M}_{nod} \subset \mathcal{M}$. The proofs of the next three result follow the same ideas of Lemmas 3.6, 3.7 and 3.8, and we omit them.

Lemma 6.4. *Given $u \in X$ with $u^+ \not\equiv 0$ and $u^- \not\equiv 0$, there exists an unique pair (t, s) of positive numbers such that $tu^+ + su^- \in \mathcal{M}_{nod}$.*

Lemma 6.5. *Let $u \in X$ such that $u^+ \not\equiv 0$, $u^- \not\equiv 0$, $I_p'(u)u^+ \leq 0$ and $I_p'(u)u^- \leq 0$. Then the unique pair (t, s) given in Lemma 6.4 satisfies that $0 < t, s \leq 1$.*

Lemma 6.6. *Let $u \in X$ such that $u^+ \not\equiv 0$ and $u^- \not\equiv 0$, and (t, s) the unique pair of positive numbers given in Lemma 6.4. Then (t, s) is the unique maximum point of the function $\phi_p : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ defined by $\phi_p(\alpha, \beta) = I_p(\alpha u^+ + \beta u^-)$.*

Now, we shall show that the nodal level c_p^* is attained.

Lemma 6.7. *There exists $\bar{u} \in \mathcal{M}_{nod}$ such that $I_p(\bar{u}) = c_p^*$.*

Proof. Let $(u_n) \subset \mathcal{M}_{nod}$ be a sequence such that $I_p(u_n) \to c_p^*$. Since $u_n \in \mathcal{M}$, for all $n \in \mathbb{N}$, we have

$$c_p^* + o_n(1) = I_p(u_n) = \frac{1}{2}\|u_n\|^2 - \frac{1}{p}\|u_n\|_{L_K^p}^p = \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2.$$

Hence, (u_n) is bounded in X. Therefore, (u_n^+) and (u_n^-) are also bounded in X. Since X is a Hilbert space, up to a subsequence, there exists $u \in X$ such that $u_n^\pm \rightharpoonup u^\pm$ in X. Since $p > 2$, utilizing Proposition 2.2 and Corollary 2.3, passing to a subsequence, we can assume that $u_n^\pm \to u^\pm$ in L_K^p and $u_n^\pm(x) \to u^\pm(x)$ a.e. in $\mathbb{R}$.

We claim that $u^+ \not\equiv 0$ and $u^- \not\equiv 0$. We suppose, by contradiction, that $u^+ \equiv 0$ (similarly $u^- \equiv 0$). Since $u_n \in \mathcal{M}_{nod}$, we have $I_p'(u_n)u_n^+ = 0$. Thus,

$$\langle u_n, u_n^+ \rangle = \int_{\mathbb{R}} K(x)|u_n^+|^p \mathrm{d}x \to \int_{\mathbb{R}} K(x)|u^+|^p \mathrm{d}x = 0.$$

However, by Lemma 2.6 we have $\langle u_n, u_n^+ \rangle \geq \|u_n^+\|^2$. This implies that $\|u_n^+\|^2 \to 0$, which is a contradiction in view of Lemma 6.3. Utilizing Lemma 6.4, there exists a pair of positive numbers (t, s) such that $tu^+ + su^- \in \mathcal{M}_{nod}$. Let $\bar{u} = tu^+ + su^-$. We will show that $I_p'(u)u^+ \leq 0$ and

$I_p'(u)u^- \le 0$. Indeed, by Fatou's lemma, we have

$$\begin{aligned} \|u^+\|^2 + \langle u^+, u^- \rangle &= \|u^+\|^2 \\ &\quad + \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(u^+(x) - u^+(y))(u^-(x) - u^-(y))}{|x-y|^2} \mathrm{d}x\mathrm{d}y \\ &\le \liminf_{n\to+\infty} \left(\|u_n^+\|^2 + \langle u_n^+, u_n^- \rangle \right) \\ &= \liminf_{n\to+\infty} \int_{\mathbb{R}} K(x)|u_n^+|^p \mathrm{d}x \\ &= \int_{\mathbb{R}} K(x)|u^+|^p \mathrm{d}x = \|u^+\|^p_{L^p_K}. \end{aligned}$$

Analogously, $I_p'(u)u^- \le 0$. Hence, using Lemma 6.5, we have $0 < t, s \le 1$. In particular, $\|\bar{u}\|^2 \le \|u\|^2$. Now, by using that $\bar{u} \in \mathcal{M}_{nod}$ and Fatou's lemma, we reach

$$\begin{aligned} c_p^* \le I_p(\bar{u}) &= I_p(\bar{u}) - \frac{1}{p} I_p'(\bar{u})\bar{u} \\ &= \left(\frac{1}{2} - \frac{1}{p}\right) \|\bar{u}\|^2 \le \left(\frac{1}{2} - \frac{1}{p}\right) \|u\|^2 \\ &\le \liminf_{n\to+\infty} \left(\frac{1}{2} - \frac{1}{p}\right) \|u_n\|^2 = \liminf_{n\to+\infty} \left(\frac{1}{2}\|u_n\|^2 - \frac{1}{p}\|u_n\|^p_{L^p_K}\right) = c_p^* \end{aligned}$$

and this completes the proof. □

Now, we will present the main result of this section.

Theorem 6.8. *The function $\bar{u} \in \mathcal{M}_{nod}$ found in Lemma 6.7 is a nodal solution of least energy of problem (6.1).*

Proof. It follows by applying the same ideas used in the proof of Theorem 1.1 and we omit it. □

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