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The effect of singularization on the Euler characteristic

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Abstract. In this work, singular surfaces are obtained from smooth orientable closed surfaces by applying three basic simple loop operations, *collapsing operation*, *zipping operation* and *double loop identification*, each of which produces different singular surfaces. A formula that provides the Euler characteristic of the singularized surface is proved. Also, we introduce a new definition of genus for singularized surfaces which generalizes the classical definition of genus in the smooth case. A theorem relating the Euler characteristic to the genus of the singularized surface is proved.

Keywords: Euler characteristic, singular surface, singular operations.

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1 Introduction

Singularities appear in several fields of study as a sign of qualitative change. We may experience them in Calculus, representing maximum or minimum points of a function; in Dynamical Systems, as stationary solutions that characterize the behaviour of solutions in their vicinity; or in Physics, where they can appear on larger scales, for instance, when a massive star undergoes a gravitational collapse after exhausting its internal nuclear fuel, which can lead to the birth of black holes or naked singularities, the latter being discussed as potential particle accelerators, acting like cosmic super-colliders. The formation of these so called space-time singularities is a more general phenomena in which general theory of relativity plays an important role [3]. And the most appealing example of such singularity is perhaps the Big Bang.

One aspect of the theory that studies singularities is the interplay between blowdowns and blowups as shown in Figure 1.1 which depicts a continuous deformation of a torus onto a pinched torus. In this scenario, both maps are given intuitively by the initial and final stages of this deformation, as one goes back and forth in time, blowup and blowdown, respectively. The deformation gives rise to a family of surfaces called a smoothing of the singular surface, which in this case is produced by a vanishing cycle.

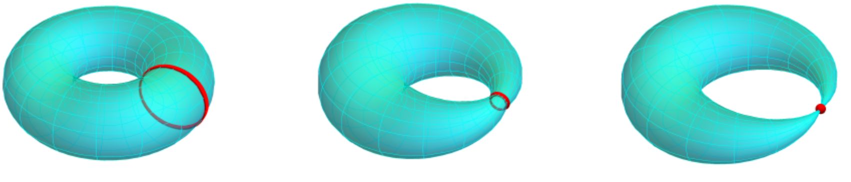


Figure 1.1: Continuous deformation of the torus (left-most) onto the pinched torus (right-most)

One would like to study how these deformations affect the topology of the spaces. This is achieved by using a fascinating tool called the Euler characteristic formula.

Certainly among the most fruitful and beautiful formulas in the history of Mathematics, is the Euler formula for a polyhedron P given by $\mathcal{X}(P) = V - E + F = 2$, where V is the number of vertices, E the number of edges and F the number of faces. See [5]. This beloved formula has entertained mathematicians such as Euler, Descartes, Cauchy and Lhuilier who gave its final form $\mathcal{X}(S) = 2 - 2g$, for what is now known as a smooth closed connected orientable surface S of genus g. Remarkably, this Euler characteristic determines precisely the closed surface up to homeomorphism.

One can say that an entire field, Algebraic Topology was inaugurated by Henri Poincaré, inspired by this formula. To state it as simply as possible, let K be a simplicial complex. Poincaré considered vector spaces C_i (over Z_2), of i-chains on K, where the sum is defined as the union minus the intersection of the i-chains. He wanted to measure the presence of special i-chains, called i-cycles that were not the border of an $i+1$-chain. This was accomplished by taking the quotient space of the space of i-cycles of K, $Z_i(K)$ by all the i-cycles that are boundaries of $i+1$-chains, $B_i(K)$. Thus, this quotient space is called the i-th homology of K and denoted by $H_i(K) = \dfrac{Z_i(K)}{B_i(K)}$. The rank of $H_i(K)$ is the *i-th Betti number* of K, $\beta_i(K)$. The Euler characteristic is defined as the alternating sum of the Betti numbers of K.

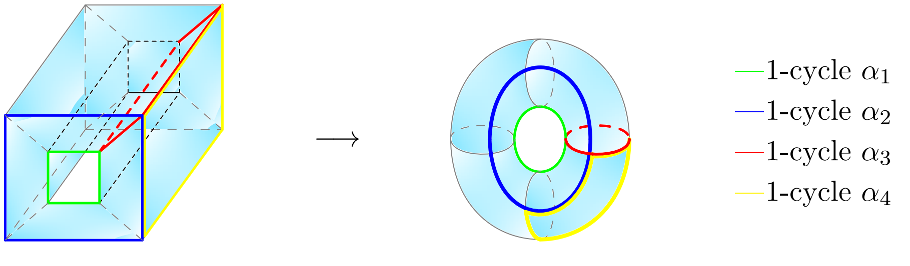


Figure 1.2: Homology on a torus

In our 2-dimensional context, 1-cycles which belong to $Z_1(P)$ and are not in $B_1(P)$, play a very important role. See Figure 1.2. Two 1-cycles are equivalent, more precisely homologous, if they form the boundary of a 2-chain. For instance, in Figure 1.2, α_1, α_2, α_3 and α_4 all belong to $Z_1(P)$. However, α_1 and α_2 are homologous and α_4 is also in $B_1(P)$. Hence, $\{\alpha_1, \alpha_3\}$ is a basis of $H_1(P) = \dfrac{Z_1(P)}{B_1(P)}$ and consequently, $\beta_1(P) = \text{rank} H_1(P) = 2$. Clearly, all 0-cycles are homologous if P is connected. Hence, $\beta_0(P) = \text{rank} H_0(P) = 1$. Also, there is only one 2-cycle, P itself, that forms the basis of $H_2(P)$. Hence, $\beta_2(P) = \text{rank} H_2(P) = 1$. The Euler characteristic of P is $\mathcal{X}(P) = \beta_0(P) - \beta_1(P) + \beta_2(P) = 0$.

It can be shown that for a smooth connected surface S with g handles, $\beta_0(S) = \beta_2(S) = 1$ and that each handle contributes with two 1-cycles to the basis of $H_1(S)$. Hence, $\beta_1(S) = 2g$ and $\mathcal{X}(S) = \beta_0(S) - \beta_1(S) + \beta_2(S) = 2 - 2g$. See [1] for more details.

Two important properties of the Euler characteristic that will be used henceforth are:

i) (Homotopy invariance) Let A and B be two homotopically equivalent spaces. Then, one has:

$$\mathcal{X}(A) = \mathcal{X}(B).$$

ii) (Inclusion-exclusion principle) Let A and B be any two closed sets. Then, the following equality holds:

$$\mathcal{X}(A \cup B) = \mathcal{X}(A) + \mathcal{X}(B) - \mathcal{X}(A \cap B).$$

In what follows, the overarching idea is to understand the topology of a singular manifold by studying a family of smooth manifolds that degenerate to it.

A very well-known and elementary example of passing from a smooth surface to a singular surface, is the family of surfaces obtained from the inverse images of the function $f : \mathbb{R}^3 \to \mathbb{R}$, given by $f(x, y, z) = x^2 +$

$y^2 - z^2$. Note that $f^{-1}(1)$ is a smooth surface, a one-sheet hyperboloid, while $f^{-1}(0)$ is a singular surface, more specifically a double cone. By considering the surfaces $f^{-1}(t)$ obtained by varying t continuously from $t = 1$ to $t = 0$, one can see a circle whose radius is decreasing until the circle degenerates into a point at the level curve $z = 0$. This contraction is responsible for the birth of the singular cone point and consequently of the singular surface.

One can visualize a similar situation in a polyhedron setting. See Figure 1.3.

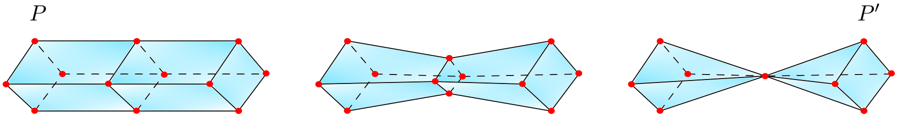


Figure 1.3: Polyhedron P (left-most) collapsing to polyhedron P' (right-most).

It is quite interesting to see the effect that this degeneracy has on the Euler characteristic . The collapsing of the middle one cycle in P to a vertex has the net effect of removing three vertices and four edges from the formula $\mathcal{X}(P) = V - E + F$, where V, E, and F are, respectively, the number of vertices, edges, and faces of the polyhedron P. See Figure 1.3.

$$\mathcal{X}(P') = V' - E' + F' = (V - 3) - (E - 4) + F = V - E + F + 1 = \mathcal{X}(P) + 1.$$

In this work, we will consider this contraction and refer to it as a *collapsing operation*. Two other operations on the images of loops on smooth surfaces are considered: *zipping* and *double loop identification* both of which produce singular surfaces.

Let M be a compact connected orientable surface. The surface M is of type (g, b) if it has genus g and b boundary components and is denoted by $M_{g,b}$. If $b = 0$, the surface is denoted by M_g and called a *closed* surface. A *loop* in M is a smooth map $\alpha : S^1 \to M$, which is identified to its image in M. All loops will be orientation preserving. A loop is *simple* if α is injective, that is, α has no self-intersection. A loop is *trivial* in M if it is homotopic to a point. Two loops are *cobordant* if the two loops bound a subsurface.

Suppose that the smooth closed surface M is embedded in $\mathbb{R}^3$, thus orientable. This embedding of M partitions $\mathbb{R}^3$ into a bounded region $\mathbb{I}$ and an unbounded region $\mathbb{O}$ so that $\mathbb{I} \cap \mathbb{O} = M$ and such that $\mathbb{I} \cup \mathbb{O} = \mathbb{R}^3$.

Definition 1.1. A simple loop $\alpha : S^1 \to M$ is called a **handle loop** if it is trivial in the homology of $\mathbb{I}$ and non-trivial in the homology of $\mathbb{O}$. A **tunnel loop** is trivial in the homology of $\mathbb{O}$ and non-trivial in the homology of $\mathbb{I}$. Whenever $(M - \alpha)$ is not connected, α is called a **separating loop**. In this case, we say that α splits M_g in two disjoint subsurfaces of genus k and $g - k$.

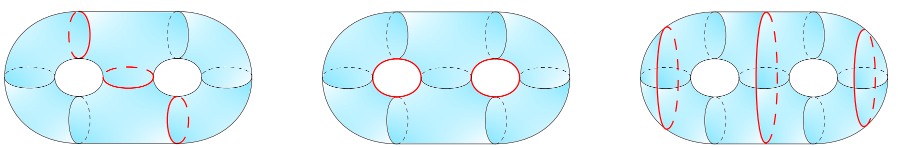


Figure 1.4: Examples of handle (left-most), tunnel (center) and separating loops (right-most) in red.

1.1 Simple loop operations

Now we define the operations that can be performed on simple loops to create singular surfaces.

Definition 1.2. Let $\alpha : S^1 \to M$ and $\beta : S^1 \to M'$ be simple loops each of which are either separating, handle or tunnel; and M and M' are smooth closed orientable surfaces, possibly the same. Define the following operations, which will be called **simple loop operations**:

1. **Collapsing** of α: consider a disk D, up to homeomorphism, bounded by α such that it is contractible to a point p in the complement of M in $\mathbb{R}^3$. The collapsing of α is the retraction of D to p.

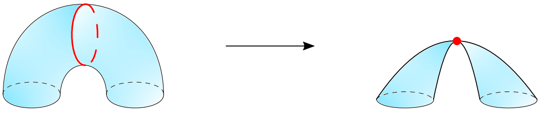


Figure 1.5: Example of collapsing

2. **Zipping** of α: consider a disk D, up to homeomorphism, bounded by α such that it is contractible to a curve d joining two distinct points on α in the complement of M in $\mathbb{R}^3$. The zipping of α is the retraction of D to d.

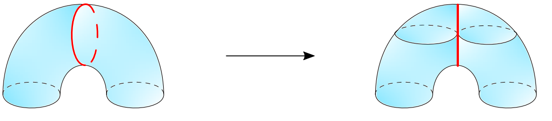


Figure 1.6: Example of zipping

3. **Double loop identification** of α and β: the loops α and β are identified, via some orientation preserving homeomorphism $h : \alpha \to \beta$;

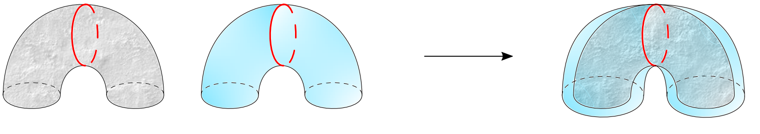


Figure 1.7: Example of double loop identification

Once a specific loop operation is performed, the resulting singularity or singular set can be easily identified and vice-versa. Typically, the operations that are chosen in a singularization process, see Definition 1.4, are based on the type of non-manifold set components that one wants the singular surface to possess.

In this manner, note that by applying the collapsing operation one obtains surfaces of revolution such as an *eight surface* (figure 1.8) and a *horn torus* (figure 1.9), the latter appears as a cyclical model of the

Universe.

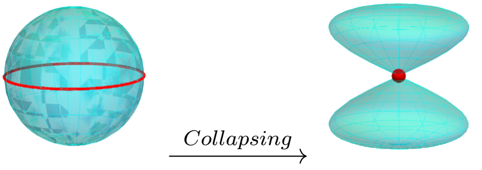


Figure 1.8: Eight Surface

Figure 1.9: Horn Torus

On the other hand, the zipping operation appears in the family of surfaces parametrized by $\phi(u,v) = ((a+b*cos(v))*cos(u),(a+b*cos(v))*sin(u), b*sin(v)*cos(ku))$, for $a > b > 0$. The number of *folds* present in a surface of this family varies for different values of k:

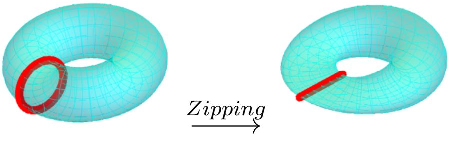


Figure 1.10: $\phi(u,v)$ with $k = 0.5$

The tori chain in Figure 1.12 illustrates a singular surface obtained by double loop identifications.

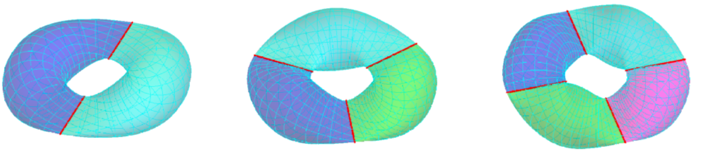


Figure 1.11: $\phi(u, v)$ with $k = 1,\ k = 1.5,\ k = 2$ (resp.)

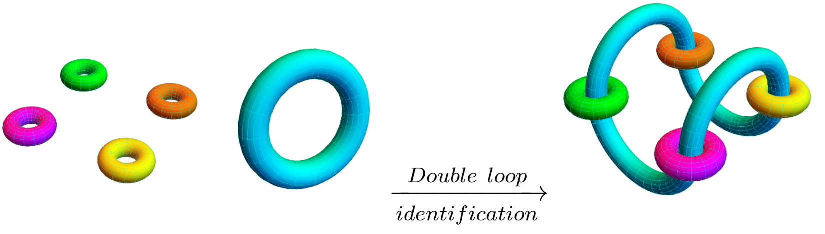


Figure 1.12: Tori chain

Definition 1.3. Given a smooth, closed, connected, orientable surface M_g, let $\mathcal{L}(M_g)$ be a collection of disjoint handle, tunnel, or separating loops in M_g with a simple loop operation assigned to each one. This definition can be extended to a disjoint union of surfaces, $\sqcup_{i=1}^{n} M_{g_i}$.

For a finite collection of smooth connected orientable closed surfaces, $\{M_{g_i} \mid i = 1, \ldots, n\}$, define $G = \sum_{i=1}^{n} g_i$ as the *genus* of $\sqcup_{i=1}^{n} M_{g_i}$.

Definition 1.4. A **singularization** of a finite collection $\{M_{g_i} \mid i = 1, \ldots, n\}$ of smooth, closed, connected, orientable surfaces will be attained from $\mathcal{L}(\sqcup_{i=1}^{n} M_{g_i})$ by performing the simple loop operations assigned therein, and denoted by $S(\sqcup_{i=1}^{n} M_{g_i})$.

Examples of such singularization can be seen in Section 2.

We also note that the resulting singular surface $S(\sqcup_{i=1}^{n} M_{g_i})$ may not be connected.

2 Effect of Loop Operations on the Euler Characteristic

In this section we will study the effect that the singularization of a smooth surface M_g (respectively, a disjoint union of smooth surfaces $\sqcup_{i=1}^n M_{g_i}$) has on the Euler characteristic of M_g (respectively, $\sqcup_{i=1}^n M_{g_i}$).

The next theorem describes the effect on the original smooth surface's Euler characteristic after the simple loop operations are performed. In other words, Theorem 2.1 describes how $\mathcal{X}(S)$ can be computed from $\mathcal{X}(M_g)$. A collapsing or zipping operation adds one to the Euler characteristic $\mathcal{X}(M_g)$, whereas a double loop identification leaves it unchanged.

Theorem 2.1. *Let $S(\sqcup_{i=1}^n M_{g_i})$ be a singularized surface obtained from the collection $\mathcal{L}(\sqcup_{i=1}^n M_{g_i})$. Then the Euler characteristic of $S(\sqcup_{i=1}^n M_{g_i})$ is independent of the number of double loop identifications performed and is equal to:*

$$\mathcal{X}(S(\sqcup_{i=1}^n M_{g_i})) \;=\; \left(\sum_{i=1}^{n} \mathcal{X}(M_{g_i})\right) + C + Z \;=\; 2n - 2G + C + Z,$$

where C is the number of collapsing operations, Z is the number of zipping operations and $G = \sum_{i=1}^{n} g_i$.

The proof of Theorem 2.1 will follow from a series of lemmas, each of which will prove the effect on the Euler characteristic of $\sqcup_{i=1}^n M_{g_i}$ after performing a specific type of operations, i.e. collapsing, zipping or double loop identification, on a collection of loops $\mathcal{L}(\sqcup_{i=1}^n M_{g_i})$.

Lemma 2.2 (Collapsing and Zipping)**.** *Let $S(M_g)$ be a singularized surface originating from M_g by performing C collapsings and Z zippings. Then*

$$\mathcal{X}(S(M_g)) = 2 - 2g + C + Z.$$

Proof. The idea behind the proof is to show that by performing a collapsing operation, as well as, a zipping operation on a loop in $\mathcal{L}(M_g)$ the effect on the Euler characteristic, $\mathcal{X}(M_g)$, will be an increase by one.

a) First, note that collapsing a loop $\alpha \in \mathcal{L}(M_g)$, where α is:

   i) a **separating simple loop**, increases $\beta_2(M_g)$ by one;

   ii) a **tunnel loop**, decreases $\beta_1(M_g)$ by one;

   iii) a **handle loop**, decreases $\beta_1(M_g)$ by one, unless α is cobordant to another handle loop $\beta \in \mathcal{L}(M_g)$ being collapsed. Recall that if α and β are cobordant, then there is a subsurface $N_{k,2} \subset M_g$, for some $k \in \{1, \ldots, g\}$, such that $\partial(N_{k,2}) = \alpha \cup \beta$. Thus, collapsing α and β transforms $N_{k,2}$ in a closed subsurface, so that one of them decreases $\beta_1(M_g)$ by one, and the other increases $\beta_2(M_g)$ by one. Something similar occurs if n cobordant handle loops in $\mathcal{L}(M_g)$ are collapsed.

   Since the Euler characteristic of M_g is given by the alternating sum:

$$\mathcal{X}(M_g) = \beta_0(M_g) - \beta_1(M_g) + \beta_2(M_g),$$

   the net effect of decreasing $\beta_1(M_g)$ by one as well as increasing $\beta_2(M_g)$ by one is the increase of $\mathcal{X}(M_g)$ by one. So, if $S(M_g)$ is obtained by C collapsing operations, it follows that:

$$\mathcal{X}(S(M_g)) \quad = \quad \mathcal{X}(M_g) + C \quad = \quad 2 - 2g + C;$$

b) Now, note that zipping a loop $\alpha \in \mathcal{L}(M_g)$ is homotopically equivalent to collapsing α. The homotopy simply contracts a segment to a point. In other words, the following diagram is commutative:

$$\begin{array}{ccc} M_g & \xrightarrow{\textit{Z zippings}} & S(M_g) \\ & \searrow^{\textit{Z collapsings}} & \downarrow \textit{Homotopy} \\ & & \widetilde{S}(M_g) \end{array} \tag{2.1}$$

   Consequently, by item a) and Euler characteristic's invariance under homotopy, if $S(M_g)$ originates from M_g by performing Z zippings, we have that $\mathcal{X}(S(M_g)) = 2 - 2g + Z$.

c) Finally, once the loops in $\mathcal{L}(M_g)$ are all disjoint, if a total of C collapsings and Z zippings are performed in the singularization of M_g, it follows from a) and b) that $\mathcal{X}(S(M_g)) = 2 - 2g + C + Z$.

□

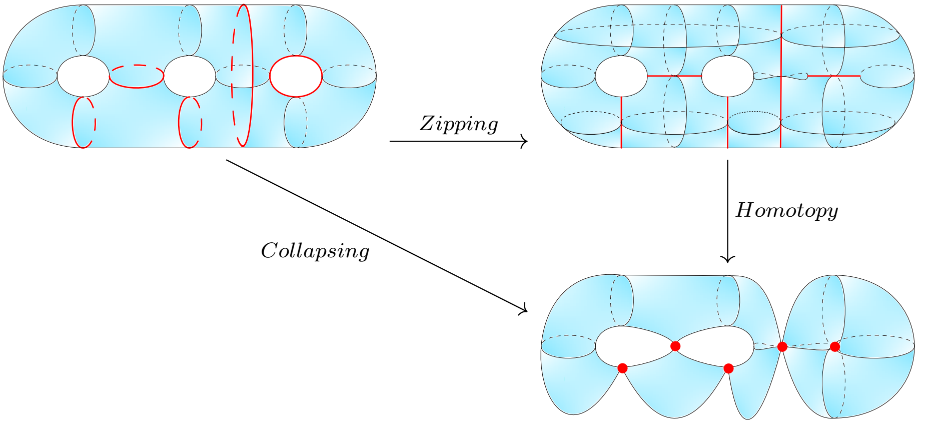


Figure 2.1: Singularization via Collapsing and Zipping Operations

In Figure 2.1, a singularization via zipping and collapsing operations of a genus $g = 3$ surface, a 3-torus, is presented. Note that the surfaces are homotopy equivalent. Hence the Euler characteristic for both singularized surfaces are the same, that is, $\mathcal{X}(\widetilde{S}(M_3)) = 2 - 2 \times 3 + 5 = 1$.

It is easy to see that Lemma 2.2 generalizes whenever the singular surface is obtained from a collection $\{M_{g_i} \mid i = 1, \ldots, n\}$ of smooth surfaces by performing C collapsing operations and Z zipping operations on $\mathcal{L}(\sqcup_{i=1}^{n} M_{g_i})$. In this case,

$$\mathcal{X}(S(\sqcup_{i=1}^{n} M_{g_i})) \;=\; 2n - 2G + C + Z,$$

where $G = \sum_{i=1}^{n} g_i$.

The next couple of lemmas, Lemmas 2.3 and 2.5, will prove that double loop identification on $\mathcal{L}(\sqcup_{i=1}^{n} M_{g_i})$ does not change the Euler characteristic given by $\sum_{i=1}^{n} \mathcal{X}(M_{g_i})$.

Lemma 2.3 (Double loop identification)**.** *Let $S(\sqcup_{i=1}^{n} M_{g_i})$ be a singular surface obtained from double loop identifications on $\mathcal{L}(\sqcup_{i=1}^{n} M_{g_i})$ where each double loop identification is performed on a pair of loops lying on different surface components. Then the Euler characteristic of $S(\sqcup_{i=1}^{n} M_{g_i})$ remains the same, that is,*

$$\mathcal{X}(S(\sqcup_{i=1}^{n} M_{g_i})) \ = \ \sum_{i=1}^{n} \mathcal{X}(M_{g_i}) \ = \ 2n - 2G.$$

Proof. Since the singular surface $S(\sqcup_{i=1}^{n} M_{g_i})$ is a union of smooth surfaces $\sqcup_{i=1}^{n} M_{g_i}$, intersecting along loops in $\mathcal{L}(\sqcup_{i=1}^{n} M_{g_i})$, it follows from the inclusion-exclusion principle that:

$$\mathcal{X}(S(\sqcup_{i=1}^{n} M_{g_i})) \ = \ \sum_{i=1}^{n} \mathcal{X}(M_{g_i}) - \sum_{i=1}^{D} \mathcal{X}(S^1), \tag{2.2}$$

where D is the number of double loop identifications. Since $\mathcal{X}(S^1) = 0$, the Euler characteristic of $S(\sqcup_{i=1}^{n} M_{g_i})$ does not depend on the number of double loop identifications performed and is given by:

$$\mathcal{X}(S(\sqcup_{i=1}^{n} M_{g_i})) \ = \ \sum_{i=1}^{n} \mathcal{X}(M_{g_i}) \ = \sum_{i=1}^{n} 2 - 2g_i \ = \ 2n - 2G.$$

□

Example 2.4. In Figure 2.2, an example of singularization via double loop identifications on disjoint surfaces is presented. The collection $\{M_2^1, M_2^2, M_1\}$ contains two bi-tori M_2^1 and M_2^2 and a torus M_1, as well as, a family of loops identifications $\mathcal{L}$. The singular surface $S(\sqcup_{i=1}^{3} M_{g_i})$, according to Lemma 2.3, has Euler characteristic equal to:

$$\mathcal{X}(S(\sqcup_{i=1}^{3} M_{g_i})) \ = \ 2n - 2G \ = \ 2 \times 3 - 2 \times 5 \ = \ -4.$$

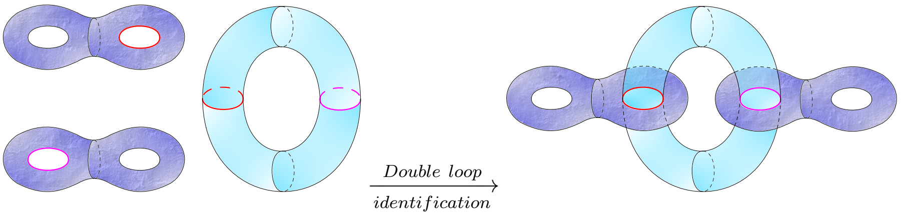


Figure 2.2: Surface chain

The following lemma will prove that the invariance of the Euler characteristic under double loop identifications still holds when the loops are chosen on the same smooth surface.

Lemma 2.5 (Double loop identification). *Let α, $\beta : S^1 \to M_g$ be two disjoint simple loops and $S(M_g)$ the singular surface obtained by a double loop identification of α and β. Then*

$$\mathcal{X}(S(M_g)) = \mathcal{X}(M_g)$$

Proof. Either the two loops α and β are cobordant, or they are not. We consider both cases. Technically, this distinction is not needed, however the cobordant case presents a more topological description of the singular surface produced by the operation.

i) **$\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are non-cobordant**:

The quotient space $M_g/\alpha \sim \beta$ given by a double loop identification of α and β is homotopically equivalent to gluing a cylinder C_α^β on M_g, where α and β are each glued to an end circle of C_α^β. The homotopy is the contraction of the cylinder to a circle.
Hence it follows from the homotopy invariance of the Euler characteristic and the inclusion-exclusion principle that:

$$\mathcal{X}(S(M_g)) \quad = \quad \mathcal{X}(M_g) + \mathcal{X}(C_\alpha^\beta) - \mathcal{X}(S^1 \cup S^1) \quad = \quad \mathcal{X}(M_g)$$

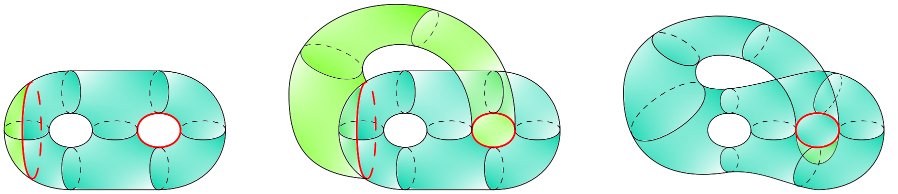


Figure 2.3: Double loop identification of non-cobordant loops

ii) **α and β are cobordant**:

Since α and β are cobordant, there exists a connected subsurface $N \subset M_g$ such that $\partial(N) = \alpha \cup \beta$. Hence $M_g = N \cup N^c$, where $N^c \subset M_g$ is the closure of $M_g - N$, with $\partial(N^c) = \alpha \cup \beta$. Note that N^c need not be connected and $N \cap N^c = \alpha \cup \beta$. Also note that the quotient spaces $N/\alpha \sim \beta$ and $N^c/\alpha \sim \beta$ are surfaces (since each curve is single-sided in N and N^c).

Thus, a double loop identification of α and β on M_g is equivalent to considering the surfaces $(N/\alpha \sim \beta)$ and $(N^c/\alpha \sim \beta)$ intersecting along $\alpha \sim \beta$. It follows that:

$$(N/\alpha \sim \beta) = M_k \qquad \text{and} \qquad (N^c/\alpha \sim \beta) = M_{g-k+1},$$

for some $k \in \{1, \ldots, g+1\}$, with $M_k \cap M_{g-k+1}$ homeomorphic to S^1.

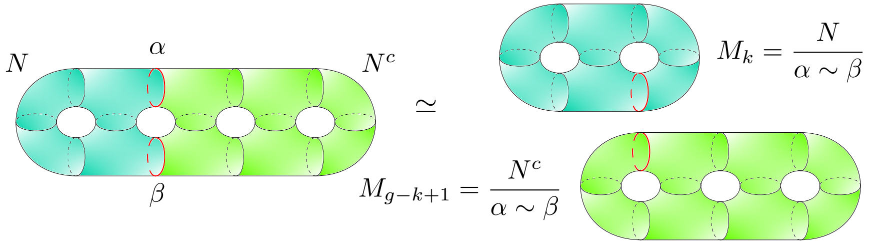


Figure 2.4: Double loop identification of cobordant loops

Thus, the result follows from the inclusion-exclusion principle for the

Euler characteristic:

$$\begin{aligned}\mathcal{X}(S(M_g)) &= \mathcal{X}(M_k) + \mathcal{X}(M_{g-k+1}) - \mathcal{X}(S^1) \\ &= (2-2k) \ + \ (2-2(g-k+1)) \\ &= 2-2g \ = \ \mathcal{X}(M_g)\end{aligned}$$

□

Now, by using Lemmas 2.2, 2.3 and 2.5 the proof of Theorem 2.1 follows.

Theorem 2.1. By the invariance of the Euler characteristic for double loop identifications proven in Lemmas 2.3 and 2.5, and by applying the inclusion-exclusion principle it follows that:

$$\mathcal{X}(S(\sqcup_{i=1}^{n} M_{g_i})) \ = \ \sum_{i=1}^{n} \mathcal{X}(S(M_{g_i})),$$

where $S(M_{g_i})$ is the singular surface obtained from the collection of M_{g_i} by performing the collapsing and zipping operations on the loops in $\mathcal{L}(M_{g_i})$.

According to Lemma 2.2, the equality holds:

$$\mathcal{X}(S(M_{g_i})) \ = \ 2-2g_i + C_i + Z_i,$$

where C_i and Z_i are the numbers of loop collapsing operations and loop zipping operations, respectively, that are performed on M_{g_i}.

Thus, by adding $\mathcal{X}(M_{g_i})$ for $i \in \{1,\ldots,n\}$, the proof follows:

$$\begin{aligned}\mathcal{X}(S(\sqcup_{i=1}^{n} M_{g_i})) &= \sum_{i=1}^{n} \mathcal{X}(S(M_{g_i})) \\ &= \sum_{i=1}^{n} 2-2g_i + C_i + Z_i \ = \ 2n-2G+C+Z\end{aligned}$$

□

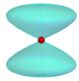
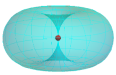
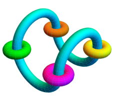
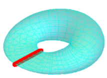
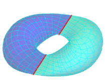
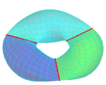
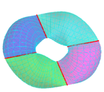
Singular surface	Smooth data (n, G)	Collapsings (C)	Zippings (Z)	Euler characteristic $(\mathcal{X} = 2n - 2G + C + Z)$
	$(1, 0)$	1	0	$\mathcal{X} = 3$
	$(1, 1)$	1	0	$\mathcal{X} = 1$
	$(5, 5)$	0	0	$\mathcal{X} = 0$
	$(1, 1)$	0	1	$\mathcal{X} = 1$
	$(1, 1)$	0	2	$\mathcal{X} = 2$
	$(1, 1)$	0	3	$\mathcal{X} = 3$
	$(1, 1)$	0	4	$\mathcal{X} = 4$

Table 2.1: Computation of the Euler characteristic

In Table 2.1 we compute according to Theorem 2.1 the Euler characteristics of the singular surfaces presented as examples in this article.

Example 2.6. In Figure 2.5, we consider a collection of three spheres

$\{M_0^1, M_0^2, M_0^3\}$ and on it a collection of loops

$$\mathcal{L}(\sqcup_{i=1}^3 M_0^i) = \{(M_0^1, \ell_1, C), (M_0^1, \ell_2; M_0^2, \ell_3, D), (M_0^3, \ell_4, C), (M_0^3, \ell_5, C), (M_0^3, \ell_6, C)(M_0^3, \ell_7, C), (M_0^3, \ell_8, Z)\}.$$

After all loop operations are performed, the singularized manifold, given by $S(\sqcup_{i=1}^3 M_{g_i})$, has two connected components and the Euler characteristic on each connected component is:

$$\mathcal{X}(S(\sqcup_{i=1}^2 M_0^i)) = 2 \times 2 - 2 \times (0) + 1 + 0 = 5$$

and

$$\mathcal{X}(S(M_0^3)) = 2 \times 1 - 2 \times 0 + 4 + 1 = 7.$$

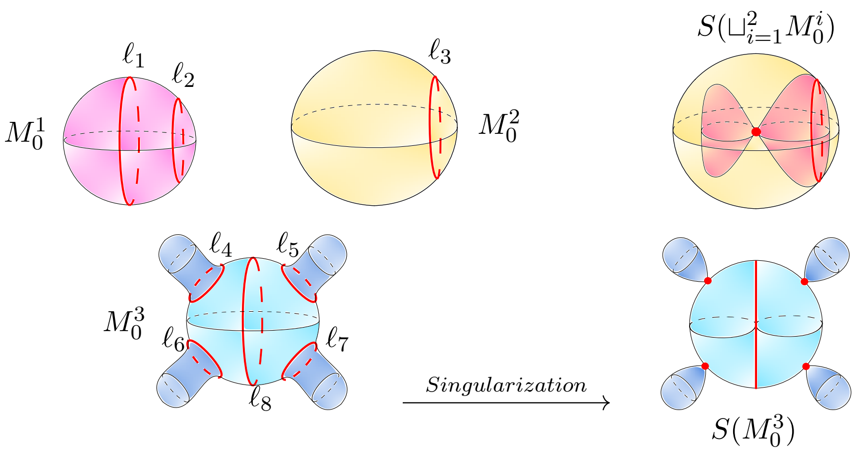


Figure 2.5: Example of singularization of three spheres

Definition 2.7. The **genus** g^S of a singularized connected surface $S(\sqcup_{i=1}^n M_{g_i})$ is the maximal number of disjoint simple closed curves that can be removed from its nonsingular part without disconnecting $S(\sqcup_{i=1}^n M_{g_i})$.

Note that the above definition generalizes the classical definition of genus for a smooth surface. Furthermore, the restriction on the removal of simple closed curves to the nonsingular part of the singularized surface, avoids the problem of infinitely many simple closed curves intersecting a one dimensional singular set without disconnecting it.

The next lemma shows that the genus of a connected singularized surface $S(\sqcup_{i=1}^n M_{g_i})$ depends on the genus of the surfaces M_{g_i} and the number of double loop identifications performed. Moreover, it is invariant with respect to collapsing and zipping.

Lemma 2.8. *Let $S(\sqcup_{i=1}^n M_{g_i})$ be a connected singularized surface obtained from the collection $\{M_{g_i} \mid i = 1, \dots n\}$ of smooth surfaces by a singularization process. Then*

$$g^S = \left(\sum_{i=1}^n g_i\right) + [D - (n-1)],$$

where D is the number of double loop identification in $\mathcal{L}(\sqcup_{i=1}^n M_{g_i})$.

Proof. In order for a singularized surface $S(\sqcup_{i=1}^n M_{g_i})$ to be connected, the minimum number D of double loop identifications in $\mathcal{L}(\sqcup_{i=1}^n M_{g_i})$ needed to achieve this is $n-1$. Thus, define $k = D - (n-1)$ as the **number of exceeding double loop identifications in the singularization**. The proof will follow by induction on k.

a) First, suppose $k = 0$, that is, $D = n - 1$;

   In the nonsingular case, there is a one-to-one correspondence between the genus g_i and the number of handles on the surface M_{g_i}. So for each handle, one can pick either one of its handle loops or one of its tunnel loops to be removed, and the number of handles gives the maximum number g_i of disjoint simple closed curves that can be removed from M_{g_i} without disconnecting it.

   In the singular case, one can proceed in a similar fashion, since the genus, g^S, is equivalent to the maximum number of simple closed curves in $\sqcup_{i=1}^n M_{g_i}$, not intersecting loops in $\mathcal{L}(\sqcup_{i=1}^n M_{g_i})$, which can be removed from $S(\sqcup_{i=1}^n M_{g_i})$ without disconnecting it. Given a handle in $\sqcup_{i=1}^{n} M_{g_i}$, if one of its handle (resp. tunnel) loops is in $\mathcal{L}(\sqcup_{i=1}^{n} M_{g_i})$, one chooses a handle (resp. tunnel) loop to be removed. Otherwise, one can choose between removing a handle or a tunnel loop.

Proceeding as above for each handle in $\sqcup_{i=1}^{n} M_{g_i}$, one removes a total of $\sum_{i=1}^{n} g_i$ simple closed curves in the nonsingular part of $S(\sqcup_{i=1}^{n} M_{g_i})$ without disconnecting it. Thus, it follows that $g^S \geq \sum_{i=1}^{n} g_i$.

Suppose by contradiction that $g^S > \sum_{i=1}^{n} g_i$. Then there is one more simple closed curve α in the nonsingular part of $S(\sqcup_{i=1}^{n} M_{g_i})$ that can be removed without disconnecting it. However, α must lie in M_{g_i} for some $i = 1, \ldots, n$. This means that $g_i + 1$ simple closed curves are being removed from M_{g_i}. Hence M_{g_i} becomes disconnected. Now we have $n+1$ disjoint connected components in $\sqcup_{i=1}^{n} M_{g_i}$ and only $D = n-1$ double loop operations in $\mathcal{L}(\sqcup_{i=1}^{n} M_{g_i})$; this is clearly not enough to obtain a connected space. Thus $S(\sqcup_{i=1}^{n} M_{g_i})$ is not connected after the removal of α which is a contradiction.

Therefore, $g^S = \sum_{i=1}^{n} g_i$.

b) Suppose $k \geq 1$ and the formula holds in the case that there are $k-1$ exceeding double loop identifications in $\mathcal{L}(\sqcup_{i=1}^{n} M_{g_i})$. That is,

$$g^S = \sum_{i=1}^{n} g_i + [k-1].$$

We prove that the formula still holds if a k-th exceeding double loop identification is performed.

Suppose $\mathcal{L}(\sqcup_{i=1}^{n} M_{g_i})$ contains k exceeding double loop identifications and determines the singularized surface $S(\sqcup_{i=1}^{n} M_{g_i})$. Let $S'(\sqcup_{i=1}^{n} M_{g_i})$ be the singularized surface determined by $\mathcal{L}(\sqcup_{i=1}^{n} M_{g_i})$ with $k-1$ exceeding double loop identifications performed, leaving out the pair of disjoint loops (α, α') that will eventually be doubly identified. By the induction hypothesis, a total of $g^{S'} = \sum_{i=1}^{n} g_i + [k-1]$ disjoint simple closed curves can be removed from the nonsingular part of

$S'(\sqcup_{i=1}^n M_{g_i})$ without disconnecting it. The removal of any other simple closed curve γ in the nonsingular part of $S'(\sqcup_{i=1}^n M_{g_i})$ will divide it into two disjoint connected components. One can choose γ in such a way that these disjoint connected components S'_1 and S'_2 contains the loops α and α' respectively. Therefore, by performing the double loop identification on (α, α'), these two components S'_1 and S'_2 become connected forming $S(\sqcup_{i=1}^n M_{g_i})$. Hence, we have shown that $S(\sqcup_{i=1}^n M_{g_i})$ remains connected after the removal of γ, meaning that:

$$g^S = \sum_{i=1}^{n} g_i + [k-1] + 1 = \left(\sum_{i=1}^{n} g_i\right) + k = \left(\sum_{i=1}^{n} g_i\right) + [D - (n-1)],$$

concluding the proof.

□

Example 2.9. In Figure 2.6, a connected singularized surface is obtained from the disjoint union of smooth surfaces, specifically a sphere M_0, a torus M_1, and a 3-torus M_3 by performing the operations in

$$\mathcal{L}(M_0 \sqcup M_1 \sqcup M_3) = \{(M_1,\ell_1;M_3,\ell_4,D), (M_1,\ell_2;M_3,\ell_6,D), (M_0,\ell_3;M_3,\ell_5,D)\}.$$

By Lemma 2.8, $g^S = 1 + 0 + 3 + [3 - (3 - 1)] = 5$. In Figure 2.6, we show five disjoint curves missing the singularities that don't separate the singularized surface.

Theorem 2.10. *Let $S(\sqcup_{i=1}^n M_{g_i})$ be a connected singularized surface obtained from the disjoint union of smooth surfaces $\{M_{g_i} \mid i = 1, \ldots n\}$ by the singularization process determined by $\mathcal{L}(\sqcup_{i=1}^n M_{g_i})$. Then the Euler characteristic of $S(\sqcup_{i=1}^n M_{g_i})$ is given by:*

$$\mathcal{X}(S(\sqcup_{i=1}^n M_{g_i})) = 2 - 2g^S + 2D + C + Z,$$

where C, Z and D are, respectively, the number of collapses, zips and double loop identifications in $\mathcal{L}(\sqcup_{i=1}^n M_{g_i})$.

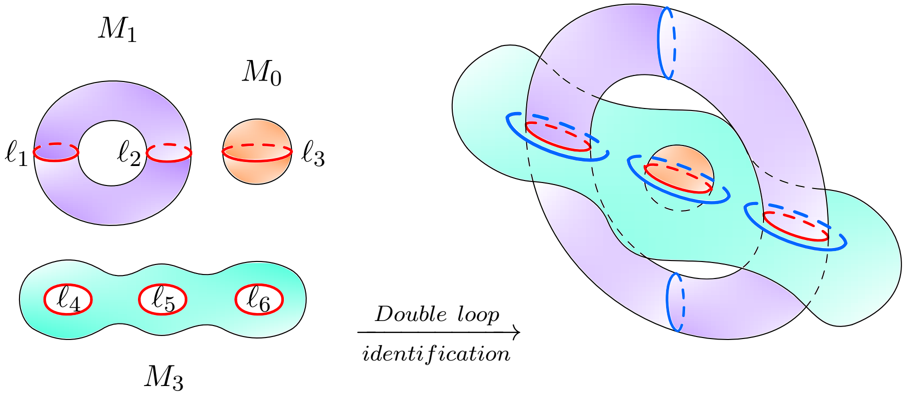


Figure 2.6: Genus 5 singularized surface

Proof. By Theorem 2.1, we have that:

$$\mathcal{X}(S(\sqcup_{i=1}^{n} M_{g_i})) = 2n - 2G + C + Z \tag{2.3}$$

where $G = \sum_{i=1}^{n} g_i$.

By Lemma 2.8,

$$G = g^S - D + (n-1) \tag{2.4}$$

Hence, the result follows by substituting (2.4) in (2.3):

$$\begin{aligned} \mathcal{X}(S(\sqcup_{i=1}^{n} M_{g_i})) &= 2n - 2[g^S - D + (n-1)] + C + Z \\ &= 2 - 2g^S + 2D + C + Z. \end{aligned}$$

□

We conclude this article by remarking that many interesting questions arise in the context of singularization. For instance, one can explore the dependency of the singularized surface to the loop operations assigned in a singularization. When are different assignments of loop operations topologically equivalent? What are the ranges of the Euler characteristic attainable in this case?

Note that, in general, the Euler characteristic is not a complete topological invariant for singularized surfaces, two singularized surfaces may have the same Euler characteristic but not be homeomorphic. Indeed, it is easy to find two singularized surfaces having the same Euler characteristic but with different singular sets. Are the singularized surfaces given here classifiable?

More complex singularization operations can be investigated. A 3-sheet cone and a triple crossing are examples of more degenerate singular sets that appear to be attainable by quotient maps similar to the collapsing and the double loop identification, respectively. Are all the singular sets produced by quotient maps of loops more degenerated cases of the singularities discussed here?

Also one can't help but wonder the effect on the Euler Characteristic of simple loop operations on a closed surface S where the family L of loops are not necessarily disjoint.

The explicit computation of the Betti numbers of a singularized surface can be posed as well, since their alternating sum yields another proof for the Euler characteristic formula presented here. Moreover, one can search if there is a relation between the Betti numbers of a singularized surface and its genus, since in the smooth case the genus of a surface is half its first Betti number.

There are many interesting and accessible questions that can be taken up from where this article left off. We entrust our reader will accept the challenge.

The results presented in this article have the potential to aid in the development of the research line concerning characteristic classes of singular varieties. For example, in [4], Reinhart uses Euler classes and Euler numbers to obtain cobordism results. Using the generalizations of characteristic classes, notions and concepts introduced by Brasselet, Libardi, Rizziolli and Saia in [2], it seems plausible that similar results may br achieved in our context.

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