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A generalized de Rham Theorem for Intersection Spaces

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Abstract. This note contains a de Rham Theorem for intersection spaces of depth one stratified pseudomanifolds with nonisolated singularities and product link bundles. The main tools are a partially smooth model for the singular chain complex of an intersection space and the Künneth formula.

Keywords: Stratified Spaces, Intersection Cohomology, Intersection Spaces, Differential Forms, de Rham Theorem.

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1 Introduction

The goal of this note is to show, that the de Rham model $\Omega I^{\bullet}_{\bar{p}}$, which was introduced in [5], computes the reduced cohomology of the intersection spaces in a nonisolated singularity setting. This de Rham type theorem was proven in [5, Theorem 9.11] for spaces with isolated singularities. One should be precise here, because there are two different notions of de Rham

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Theorems - they always relate the cohomology of some complex of differential forms to the dual space of the homology of some geometric chain complex (e.g. the singular chain complex). But the isomorphism can either be an isomorphism of vector spaces (i.e. an isomorphism in the category of vector spaces) or an isomorphism of rings. For spaces with isolated singularities, Banagl proved that the cohomology of $\Omega I^{\bullet}_{\bar{p}}(X)$ is isomorphic to the dual space of the reduced homology $\widetilde{H}_*(I^{\bar{p}}X)$ of the intersection space as a vector space. This result was generalized by F. Schlöder and the author in [14]. There, we showed, that the the cohomology ring of $\Omega I^{\bullet}_{\bar{p}}(X)$ and the (reduced) singular cohomology ring of $I^{\bar{p}}X$ are isomorphic.

In this note, Banagl's de Rham Theorem for intersection space cohomology is generalized to depth one Thom-Mather stratified pseudomanifolds with trivializable link bundles for the singular strata.

Notation and Remarks In this paper, all (co)homology groups are taken with real coefficients. For a real vector space V, we denote its dual space by $V^\dagger := \mathrm{Hom}(V, \mathbb{R})$. For background information on intersection spaces, consult the survey article [7], Banagl's book on intersection spaces [3] and his article on the de Rham model [5]. The contents of this note were covered in the author's diploma thesis, which was written under the supervision of Prof. Markus Banagl at the University of Heidelberg.

2 The Main Result

The main result of this note is a de Rham Theorem for intersection spaces of depth one Thom-Mather stratified pseudomanifold with product link bundles. To get a feeling for the stratified spaces of that type, consider the following.

$$X = M \cup_{\partial M = \Sigma \times L} (\Sigma \times \mathrm{cone}(L))$$

In this decomposition, M, Σ, L are smooth manifolds of positive dimension, in the case of M with boundary $\partial M = \Sigma \times L$. Σ is the singular set of X and the singular strata are its connected components. Note, that in general, the links of the different path components of the singular set Σ

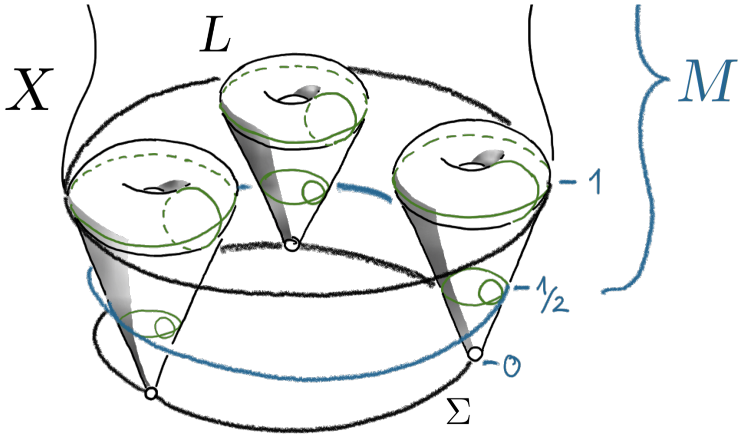

Figure 2.1: Illustration of a depth one Thom-Mather stratified space with trivial link bundle and a two torus as link.

can be different. The term *nonisolated singular set* is reflected by the positive dimensionality of Σ. L is the link of the singular set and M is a (compact) homotopy model for the regular part $X \setminus \Sigma$ of X.

A more formal description of the singular spaces considered in this article is the following. Consider Thom-Mather stratified pseudomanifolds X of depth one with singular set Σ, such that for every connected component $S \subset \Sigma$, there is a neighbourhood $U = \rho^{-1}[0, \epsilon) \subset X$ of $S \subset X$ that is stratified isomorphic to $S \times \operatorname{cone}(L)$ for a smooth manifold L. These are called depth one Thom-Mather stratified pseudomanifolds with product link bundles. The map ρ measures the distance to the singular set. Stratified isomorphic means that the homeomorphism restricts to diffeomorphisms $\rho^{-1}(t) \to S \times L$ for all $t \neq 0$. All the results of this note hold in this setting, but to keep the notation simpler, pretend that all the spaces have the shape $X = M \cup_{\partial M} (\Sigma \times \operatorname{cone}(L))$ and that the singular set Σ is connected.

Intersection spaces of singular pseudomanifolds were introduced in [3]. The construction for spaces with isolated singularities as well as results on deformation invariance of the resulting homology theory are surveyed in [7]. Let us recall Banagl's construction for singular spaces of the form $X = M \cup_{\partial M} (\Sigma \times \mathrm{cone}(L))$, which can be found in [3, Section 2.9]. The main ingredient is the spatial homology truncation of the link L of $\Sigma \subset X$, introduced in [3, Chapter 1]. The spatial homology truncation of a CW-complex L in degree $k \in \mathbb{N}$ is a pair $(L_{<k}, f)$ of a CW-complex $L_{<k}$ together with a map $f : L_{<k} \to L$ such that

$$f_* : H_r(L_{<k}) \xrightarrow{\cong} H_r(L) \text{ for } r < k \quad \text{and} \quad H_r(L_{<k}) = 0 \text{ for } r \geq k.$$

Spatial homology truncations exist for all simply-connected CW-complexes of $\dim \neq 2$ by [3, Proposition 1.1.6] and can be constructed by hand in most other cases. The process lacks functorial properties in general, though.

The idea of the intersection spaces is the following: One replaces the tube $\Sigma \times \mathrm{cone}(L)$ of the singular set $\Sigma \subset X$ by the cone $\mathrm{cone}(Sigma \times L_{<k})$. By doing so, one forgets about the homology information of the links in low dimensions, namely in dimensions $< k$. The truncation values that are of interest are fixed by a so called perversity function $\bar{p}$ in the sense of Goresky-MacPherson, see [10, 11]. That is a function $\bar{p} : \{2, 3, 4, \ldots\} \to \mathbb{Z}$ with the properties $\bar{p}(2) = 0$ and $\bar{p}(t) \leq \bar{p}(t+1) \leq \bar{p}(t) + 1$ for all t. The cutoff value is $k = c - 1 - \bar{p}(c)$, where $c = \mathrm{codim}(\Sigma)$ is the codimension of Σ in X.

Definition 2.1 (Intersection Space)**.** The intersection space $I^{\bar{p}}X$ of $X = M \cup_{\partial M} (\Sigma \times \mathrm{cone}L)$ with respect to the perversity $\bar{p}$ is the mapping cone of the map

$$g : \Sigma \times L_{<k} \xrightarrow{\mathrm{id} \times f} \Sigma \times L = \partial M \hookrightarrow M.$$

In other words,

$$I^{\bar{p}}X := \mathrm{cone}(g) = M \cup_{\mathrm{id} \times f} \mathrm{cone}(\Sigma \times L_{<k}).$$

The complex of differential forms, which gives a de Rham model for the cohomology of the intersection space consits of forms ω on the regular part $X \setminus \Sigma = X \setminus \Sigma$, such that the pullback $t^*\omega \in \Omega^\bullet(T \setminus \Sigma)$ satisfies a cotruncation condition (see below), where $T = \Sigma \times \mathrm{cone}(L)$ is the tube

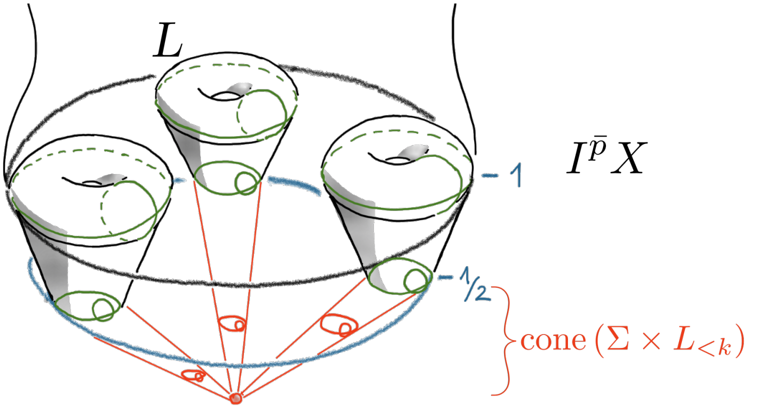

Figure 2.2: Illustration of an intersection space associated with the singular space from Figure 2.1; the homology truncation of the link is the 1-skeleton

of Σ and $t : \Sigma \times (L \times (0,1)) \cong T \setminus \Sigma \hookrightarrow X \setminus \Sigma$ is the embedding of the nonsingular tube.

In order to describe this cotruncation condition for $t^*\omega$, one first introduces the geometric cotruncation of the complex $\Omega^\bullet(L)$ in degree $k = c - 1 - \bar{p}(c)$. Fix some Riemannian metric g_L on L and let $d^* : \Omega^r(L) \to \Omega^{r-1}(L)$ be the adjoint of the differential d with respect to the induced inner product on $\Omega^\bullet(L)$. Define

$$\tau_{\geq k}\Omega^\bullet L := \ldots \to 0 \to \ker d^* \to \Omega^{k+1}(L) \xrightarrow{d} \ldots \leq \Omega^\bullet(L)$$

By the real Hodge Decomposition Theorem, the subcomplex $\tau_{\geq k}\Omega^\bullet(L) \leq \Omega^\bullet(L)$ has vanishing cohomology in degrees $< k$ and the computes the de Rham cohomology of L in degrees $\geq k$.

Cotruncated forms on L can be multiplied with forms on Σ to give so called fiberwisely cotruncated forms on $\Sigma \times L$. The definition of said complex $\mathrm{ft}_{\geq k}\Omega^\bullet_{\mathcal{MS}}$ works in the more general setting where the product space $\Sigma \times L$ is replaced by the total space E of a geometrically flat fiber bundle $p : E \to B$ with fiber L. See [5, Section 5.4] for the definition of such bundles.

Definition 2.2 (Fiberwisely cotruncated forms on product spaces)**.** For

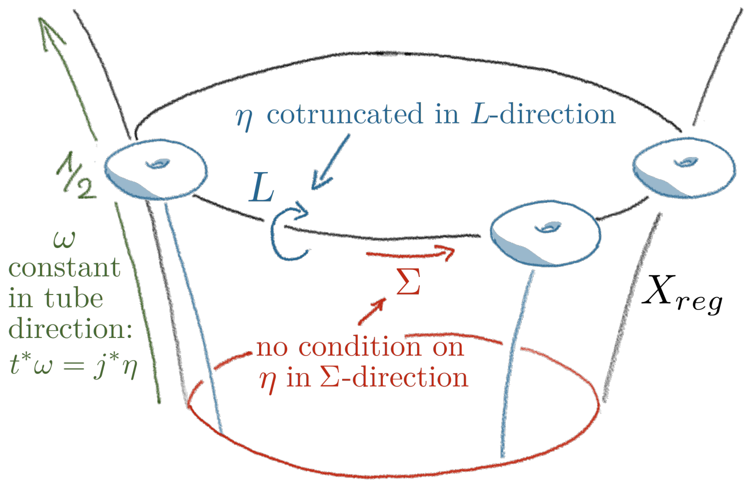

Figure 2.3: Illustration of the conditions on forms $\omega \in \Omega I^\bullet_{\bar{p}}(X)$

any $k \in \mathbb{Z}$ the complex of fiberwisely cotruncated forms on the product $\Sigma \times L$ is definied as

$$\mathrm{ft}_{\geq k}\Omega^\bullet_{\mathcal{MS}}(\Sigma \times L) := \langle \{\pi_1^*\eta \wedge \pi_2^*\gamma : \; \eta \in \Omega^\bullet(\Sigma), \gamma \in \tau_{\geq k}\Omega^\bullet(L)\}\rangle \, .$$

The compact smooth Riemannian manifold L is called *fiber* in this context.

Note, that $\mathrm{ft}_{\geq k}\Omega^\bullet_{\mathcal{MS}}(\Sigma \times L) \leq \Omega^\bullet(\Sigma \times L)$ is a subcomplex. To define the $\Omega I^\bullet_{\bar{p}}$ forms on the singular space X with stratification depth one and trivial link bundle $\Sigma \times L$ for the singular set $\Sigma \subset X$, recall the following notation. Let $X = M \cup_{\partial M} (\Sigma \times \mathrm{cone}(L))$ and let $T = (\Sigma \times \mathrm{cone}(L))$ denote the tube of $\Sigma \subset X$. We denote by $t : T \setminus \Sigma = \Sigma \times L \times (0,1) \hookrightarrow X \setminus \Sigma$ the smooth embedding of the tube into the regular part of X and by $j : \Sigma \times L \hookrightarrow T$ the embedding at $\frac{1}{2}$. Then, for any perversity function $\bar{p}$, the $\Omega I^\bullet_{\bar{p}}$-complex is defined as follows.

Definition 2.3 ($\Omega I^\bullet_{\bar{p}}$-forms)**.**

$$\Omega I^\bullet_{\bar{p}}(X) := \{\omega \in \Omega^\bullet(X \setminus \Sigma) : t^*\omega = j^*\eta, \; \eta \in \mathrm{ft}_{\geq k}\Omega^\bullet_{\mathcal{MS}}(\Sigma \times L)\} \, .$$

The main statement of this note is, that this complex $\Omega I^\bullet_{\bar{p}}(X)$ gives a de Rham model for the (reduced) cohomology of the intersection spaces for depth one Thom-Mather stratified pseudomanifolds. It is an isomorphism in the category of vector spaces and comes from integrating forms over smooth chains.

Theorem 2.4 (de Rham Theorem for Intersection Space Cohomology). *For a depth one Thom-Mather stratified pseudomanifold X with product link bundles, integration of differential forms $\omega \in \Omega I^{\bullet}_{\bar{p}}(X)$ over partially smooth chains on $I^{\bar{p}}X$ induces isomorphisms of vector spaces*

$$H^{\bullet}\big(\Omega I^{\bullet}_{\bar{p}}(X)\big) \xrightarrow{\cong} H_{\bullet}\big(I^{\bar{p}}X\big)^{\dagger}. \tag{2.1}$$

To prove the theorem, one needs the following tools.

1. A chain complex model for the homology of the intersection space, which consists of partially smooth chains.
2. Suitable de Rham morphisms, that enable us to use a Five-Lemma argument to prove our main statement.
3. In particular, the previous point includes a de Rham morphism that integrates fiberwisely cotruncated multiplicative forms on the product $\Sigma \times L$. Therefore, one needs the Künneth map to integrate product forms over product chains.

The technical preliminaries are collected in the following sections. The interplay of the Künneth map with smoothing of singular chains is revisited first. Afterwards, the partially smooth chain model for the intersection space homology is constructed, generalizing the construction from [5, Section 9]. Last, the different de Rham morphisms and their relations with each other are discussed.

3 Smoothing of chains on product spaces

Differential forms on smooth manifolds M can be integrated over smooth singular chains. These chains form a complex $S^{\infty}_{\bullet}(M)$, which is a subcomplex of the singular chain complex. The subcomplex inclusion ι_M :

$S^\infty_\bullet(M) \hookrightarrow S_\bullet(M)$ induces an isomorphism on cohomology. The inverse is induced by the smoothing operator $s_M : S_\bullet(M) \to S^\infty_\bullet(M)$. The construction of this operator can be found in [13, Chapter 16].

On product manifolds $\Sigma \times L$, product chains also compute the whole homology by the Künneth Theorem. The Künneth morphism

$$\kappa : H_\bullet(\Sigma) \otimes H_\bullet(L) \xrightarrow{\cong} H_\bullet(\Sigma \times L)$$

is decomposed as $\kappa = P_* \circ \mu$.

$$\mu : H_\bullet(\Sigma) \otimes H_\bullet(L) \xrightarrow{\cong} H_\bullet\left(S_\bullet(\Sigma) \otimes S_\bullet(L)\right),$$
$$[x] \otimes [y] \mapsto [x \otimes y]$$

denotes the algebraic Künneth morphism and $P : S_\bullet(\Sigma) \otimes S_\bullet(L) \to S_\bullet(\Sigma \times L)$ a natural Eilenberg-Zilber chain transformation, that can be explicitly defined on simplices by the formula

$$P(\sigma \otimes \tau) := \sum_\lambda \operatorname{sgn}(\lambda)\left((\sigma \times \tau) \circ \lambda\right).$$

In this formula, $\sigma : \Delta_p \to \Sigma$ is a p-simplex and $\tau : \Delta_q \to L$ is a q-simplex and the sum is taken over all (p, q)-shuffles λ, which naturally give a triangulation of $\Delta_p \times \Delta_q$ by $(p + q)$-simplices. See [15, Chapter 9.7] and [12, Chapter VIII] for more details. The Künneth map can be defined in the same way on the homologies of the smooth singular chain complexes,

$$\kappa^\infty : H^\infty_\bullet(\Sigma) \otimes H^\infty_\bullet(L) \xrightarrow{\cong} H^\infty_\bullet(\Sigma \times L).$$

Lemma 3.1 (Smoothing of product chains)**.** *Let κ, κ^∞ be the Künneth isomorphisms introduced above and let s_Σ and s_L be smoothing operators on Σ and L. Then, there is a special smoothing operator $s_{\Sigma\times L}$ on $\Sigma \times L$ such that*

$$\kappa^\infty \circ ((s_\Sigma)_* \otimes (s_L)_*) = (s_{\Sigma\times L})_* \circ \kappa.$$

Proof. The construction of the smoothing operators $s_- : S_\bullet(-) \to S^\infty_\bullet(-)$ in [13, Chapter 16] is based on the construction of homotopies between simplices and smooth simplices, which interact nicely with face maps, see [13, Lemma 16.7]. In the product space setting $\Sigma \times L$, one can take the following product type homotopies for product simplices $(\sigma \times \tau) \circ \lambda$, where $\sigma : \Delta_p \to \Sigma$, $\tau : \Delta_q \to L$ are simplices and $\lambda := (\lambda_1, \lambda_2) : \Delta_{p+q} \hookrightarrow \Delta_p \times \Delta_q$ is a (smooth) embedding.

$$\begin{aligned} H_{\sigma\times\tau} &: \Delta_{p+q} \times I \to \Sigma \times L, \\ &H_{\sigma\times\tau}(x,t) := (H_\sigma(\lambda_1(x),t), H_\tau(\lambda_2(x),t)) \,. \end{aligned}$$

There, H_σ and H_τ are homotopies for σ and τ, which satisfiy the conditions of [13, Lemma 16.7]. Together with the definition $s_{\Sigma\times L}(\sigma)(x) := H_\sigma(x,1)$ for the smoothing operator, this construction implies the following property for product simplices:

$$s_{\Sigma\times L}\left((\sigma \times \tau) \circ \lambda\right) = \left((s_\Sigma \sigma) \times (s_L \tau)\right) \circ \lambda$$

Using this property, one can calculate the following commutativity of the Eilenberg-Zilber chain transformation P and the smoothing operators. Again, σ is a simplex on Σ and τ is a simplex on L.

$$\begin{aligned} P \circ (s_\Sigma \otimes s_L)(\sigma \otimes \tau) &= \sum_\lambda \operatorname{sgn}(\lambda)\left((s_\Sigma \sigma \times s_L \tau) \circ \lambda\right) \\ &= \sum_\lambda \operatorname{sgn}(\lambda) s_{\Sigma\times L}\left((\sigma \times \tau) \circ \lambda\right) = s_{\Sigma\times L} P(\sigma \otimes \tau). \end{aligned}$$

By definition, the algebraic Künneth maps satisfy

$$\mu^\infty \circ (s_\Sigma \otimes s_L) = \left((s_\Sigma)_* \otimes (s_L)_*\right) \circ \mu$$

and thus, the statement of the lemma is justified. □

From now on, the short hand notation $a \times b \in S_\bullet(\Sigma \times L)$ is used for the chains, which are the Eilenberg-Zilber transformation of a product $a \otimes b \in S_\bullet(\Sigma) \otimes S_\bullet(L)$. Precisely, $a \times b = P(a \otimes b)$. In the proof of the main statement of this note, one needs to integrate multiplicative differential forms on $\Sigma \times L$ over (smooth) product chains. An application of the Integral Theorem of Fubini and Tonelli results in the following formula.

Lemma 3.2 (Integration of product forms over product chains)**.** *Let* $a \in S_p^\infty(\Sigma)$, $b \in S_q^\infty(L)$, $\eta \in \Omega^p(\Sigma)$ *and* $\gamma \in \Omega^q(L)$. *Let further* $\pi_1 : \Sigma \times L \to \Sigma$, $\pi_2 : \Sigma \times L \to L$ *denote the first and second factor projections. Then, the following formula holds.*

$$\int_{a \times b} \pi_1^* \eta \wedge \pi_2^* \gamma = \int_a \eta \cdot \int_b \gamma.$$

Proof. Assume without loss of generalization, that $a = \sigma$, $b = \tau$ are simplices. Note, that the shuffles $\lambda : \Delta_{p+q} \to \Delta_p \times \Delta_q$, triangulate $\Delta_p \times \Delta_q$ and thus

$$\int_{\Delta_p \times \Delta_q} \ldots = \sum_\lambda \operatorname{sgn}(\lambda) \int_\lambda \ldots$$

The dots emphasize that this formula holds for any integrand.

If σ and τ are smooth simplices, then for any shuffle λ the simplex $(\sigma \times \tau) \circ \lambda$ is also smooth. Gathering these facts, the formula of the lemma can be derived as follows. Denote by $\widetilde{\pi}_1 : \Delta_p \times \Delta_q \to \Delta_p$ and $\widetilde{\pi}_2 : \Delta_p \times \Delta_q \to \Delta_q$ the first and second factor projections on the standard simplices to avoid confusions with the projections on $\Sigma \times L$.

$$\begin{aligned}
\int_{\sigma \times \tau} \pi_1^* \eta \wedge \pi_2^* \gamma &= \sum_\lambda \operatorname{sgn}(\lambda) \int_{(\sigma \times \tau) \circ \lambda} \pi_1^* \eta \wedge \pi_2^* \gamma \\
&= \sum_\lambda \operatorname{sgn}(\lambda) \int_\lambda \widetilde{\pi}_1^* \sigma^* \eta \wedge \widetilde{\pi}_2^* \tau^* \gamma = \int_{\Delta_p \times \Delta_q} \widetilde{\pi}_1^* \sigma^* \eta \wedge \widetilde{\pi}_2^* \tau^* \gamma \\
&= \int_{\Delta_p} \sigma^* \eta \cdot \int_{\Delta_q} \tau^* \gamma = \int_\sigma \eta \cdot \int_\tau \gamma.
\end{aligned}$$

In the last line, the Fubini and Tonelli Integral Theorem was applied together with the fact that the pullbacks with respect to the first and second factor projections are constant in the other factor. □

4 Partial smooth models for intersection spaces

To establish a de Rham theorem for intersection space homology, one needs (smooth) singular chains on a smooth manifold, so that differential forms can be integrated over those chains. The smooth manifold will be the compact manifold M. Below, the the *partially smooth chain complex* $S_\bullet^\propto(g)$ is itnroduced. This chain model is a variation of the mapping cone of the map $\Sigma \times L_{<k} \to M$ from Definition 2.1 and computes the homology of $I^{\bar{p}}X$. The definition goes back to [5, Section 9.1], where it was introduced for pseudomanifolds with isolated singularities. In the definition, splittings $Z_k = B_k \oplus H'_k$ of the singular cycles on $\Sigma \times L_{<k}$ are needed to define a map $q : H_\bullet(\Sigma \times L_{<k}) \to S_k(\Sigma \times L_{<k})$, which maps a homology class to some representative. To be precise, q is defined as the following composition.

$$H_k(\Sigma \times L_{<k}) = \frac{Z_k}{B_k} = \frac{B_k \oplus H'_k}{B_k} \cong H'_k \hookrightarrow Z_k \hookrightarrow S_k(\Sigma \times L_{<k}).$$

Note, that the partially smooth chain complex depends on the choice of these splittings $Z_k = B_k \oplus H'_k$.

Definition 4.1 (Partially smooth chain complex)**.**

$$\begin{aligned} S_k^\propto(g) &:= H_{k-1}(\Sigma \times L_{<k}) \oplus S_k^\infty(M), \\ \partial(x, v) &:= (0, \partial v - s_M g_* q(x)) \, . \end{aligned}$$

The arguments of [5, Section 9.1] extend to the current setting. Thus, $H_\bullet\left(S_\bullet^\infty(g)\right) \cong H_\bullet\left(\mathrm{cone}(g)\right)$. In other words, $S_\bullet^\infty(g)$ computes the reduced homology of the intersection space, since that is the topological mapping cone of g.

5 De Rham maps

This section is built upon standard de Rham maps $\Phi_Z : H^\bullet_{dR}(Z) \xrightarrow{\cong} H_\bullet(Z)^\dagger$, for smooth manifolds Z. These are used in the context $Z \in \{X \setminus \Sigma, \Sigma, L, \Sigma \times L\}$. The de Rham map is defined using the smoothing operator $s_Z : S_\bullet^\infty(Z) \to S_\bullet(Z)$ by integrating differential k-forms over smooth singular simplices. Let $\omega \in \Omega^k(Z)$ be a closed k-form and $\sum n_\sigma \sigma \in S_k(Z)$ a singular cycle. Then,

$$\Phi_Z\left([\omega]\right)\left(\sum n_\sigma \sigma\right) := \sum n_\sigma \int_{\Delta_k} (s_Z \sigma)^* \omega. \tag{5.1}$$

To get a de Rham map $H^\bullet(\Omega I^\bullet_{\bar{p}}(X)) \to H_\bullet\left(S^{\propto}_\bullet(g)\right)^\dagger$, it has to be shown that integrating forms $\omega \in \Omega I^\bullet_{\bar{p}}(X)$ over the second component of partially smooth chains $(x, v) \in S^{\propto}_\bullet(g)$ induces a map on (co)homology. In the definition, the embedding $\alpha : M \hookrightarrow X \setminus \Sigma$ of the compact manifold M with boundary $\partial M = \Sigma \times L$ into the regular part $X \setminus \Sigma$ of X is used. This embedding is a homotopy equivalence and M is a deformation retract of $X \setminus \Sigma$.

Proposition 5.1. *For any perversity function $\bar{p}$, the assignment*

$$\Phi_{\bar{p}}\left[\omega\right]\left[(x, v)\right] := \int_{\alpha_* v} \omega$$

gives a well defined map $\Phi_{\bar{p}} : H^\bullet(\Omega I^\bullet_{\bar{p}}(X)) \to H_\bullet\left(S^{\propto}_\bullet(g)\right)^\dagger$.

Proof. Left to the reader. Compare to [5, Proposition 9.8] and make use of the Künneth Theorem to represent homology classes on $\Sigma \times L_{<k}$. □

To prove that $\Phi_{\bar{p}}$ is an isomorphism, it will be compared to two other de Rham maps: The almost standard de Rham map Φ_{reg} on $X \setminus \Sigma$ and a product de Rham map $\Phi_{<k}$ on $\Sigma \times L$, which will be expained below. In the definition of Φ_{reg}, the subcomplex $\Omega^\bullet_{\partial \mathcal{MS}}(X \setminus \Sigma) \leq \Omega^\bullet(X \setminus \Sigma)$ occurs. It is defined as follows.

$$\Omega^\bullet_{\partial \mathcal{MS}}(X \setminus \Sigma) := \left\{\omega \in \Omega^\bullet(X \setminus \Sigma) : t^*\omega = j^*\eta,\ \eta \in \Omega^\bullet_{\mathcal{MS}}(\Sigma \times L)\right\}.$$

In this definition, $\Omega^\bullet_{\mathcal{MS}}(\Sigma \times L) := \langle\{\pi_1^*\eta \wedge \pi_2^*\gamma :\ \eta \in \Omega^\bullet(\Sigma), \gamma \in \Omega^\bullet(L)\}\rangle$ is the complex of product forms on $\Sigma \times L$. The subcomplex inclusion of $\Omega^\bullet_{\partial \mathcal{MS}}(X \setminus \Sigma)$ into $\Omega^\bullet(X \setminus \Sigma)$ gives rise to an isomorphism on cohomology by [5, Proposition 6.3].

Definition 5.2 (The almost standard de Rham map on $X \setminus \Sigma$)**.** The almost standard de Rham map Φ_{reg} on $X \setminus \Sigma$ is defined as the composition

$$H^\bullet(\Omega^\bullet_{\partial \mathcal{MS}}(X \setminus \Sigma)) \cong H^\bullet_{dR}(X \setminus \Sigma) \xrightarrow[\Phi_{X \setminus \Sigma}]{\cong} H^\infty_\bullet(X \setminus \Sigma)^\dagger \xrightarrow[{\alpha_*}^\dagger]{\cong} H^\infty_\bullet(M)^\dagger.$$

By construction, Φ_{reg} is an isomorphism and it fits into the following commutative square.

Lemma 5.3. *Let $i : \Omega I^{\bullet}_{\bar{p}}(X) \hookrightarrow \Omega^{\bullet}_{\partial\mathcal{MS}}(X \setminus \Sigma)$ and $inc : S^{\infty}_{\bullet}(M) \hookrightarrow S^{\propto}_{\bullet}(g)$ denote the subcomplex inclusions. Then, the following square commutes.*

$$\begin{tikzcd} H^{\bullet}\left(\Omega I^{\bullet}_{\bar{p}}(X)\right) \arrow[r, "i^*"] \arrow[d, "\Phi_{\bar{p}}"] & H^{\bullet}(\Omega^{\bullet}_{\partial\mathcal{MS}}(X \setminus \Sigma)) \arrow[d, "\Phi_{reg}", "\cong"'] \\ H_{\bullet}\left(S^{\propto}_{\bullet}(g)\right)^{\dagger} \arrow[r, "(inc_*)^{\dagger}"] & H^{\infty}_{\bullet}(M)^{\dagger} \end{tikzcd}$$

Proof. This is a direct consequence of the definitions of $\Phi_{\bar{p}}$ and Φ_{reg}. $\square$

Before the aforementioned map $\Phi_{<k}$ is introduced, its definition shall be motivated by the role it has to play in the proof of the main theorem. The center of that proof is a Five-Lemma argument. The partially smooth chain complex $S^{\propto}_{\bullet}(g)$, which models the homology of the intersection space, fits into the following short exact sequence

$$0 \to S^{\infty}_{\bullet}(M) \to S^{\propto}_{\bullet}(g) \to H_{\bullet-1}(\Sigma \times L_{<k}) \to 0 \tag{5.2}$$

The dual space of the homology of the first two complexes in this sequence is paired with the cohomologies of the two complexes $\Omega I^{\bullet}_{\bar{p}}(X) \hookrightarrow \Omega^{\bullet}_{\partial\mathcal{MS}}(X \setminus \Sigma)$ by $\Phi_{\bar{p}}$ and Φ_{reg}. To apply a Five-Lemma argument, one needs to pair the cohomology of the cokernel $\Omega^{\bullet}_{\partial\mathcal{MS}}(X \setminus \Sigma)/\Omega I^{\bullet}_{\bar{p}}(X)$ of this subcomplex inclusion with $H_{\bullet-1}(\Sigma \times L_{<k})$. The following lemma provides a much nicer description of $\Omega^{\bullet}_{\partial\mathcal{MS}}(X \setminus \Sigma)/\Omega I^{\bullet}_{\bar{p}}(X)$. Using the truncated complex

$$\tau_{<k}\Omega^{\bullet}(L) := \ldots \to \Omega^{k-1}(L) \to \operatorname{im}\partial \to 0 \to \ldots,$$

one defines the complex of fiberwisely truncated forms on $\Sigma \times L$ as follows.

$$\mathrm{ft}_{<k}\Omega^{\bullet}_{\mathcal{MS}}(\Sigma \times L) := \left\langle \{\pi_1^*\eta \wedge \pi_2^*\gamma : \ \eta \in \Omega^{\bullet}(\Sigma), \gamma \in \tau_{<k}\Omega^{\bullet}(L)\} \right\rangle.$$

Note, that the complexes of fiberwisely (co)truncated forms are isomorphic to tensor products. $\mathrm{ft}_{<k}\Omega^{\bullet}_{\mathcal{MS}}(\Sigma \times L) \cong \Omega^{\bullet}(\Sigma) \otimes \tau_{<k}\Omega^{\bullet}(L)$ and $\mathrm{ft}_{\geq k}\Omega^{\bullet}_{\mathcal{MS}}(\Sigma \times L) \cong \Omega^{\bullet}(\Sigma) \otimes \tau_{\geq k}\Omega^{\bullet}(L)$. Thus, the orthogonal direct sum decomposition $\Omega^{\bullet}(L) = \tau_{<k}\Omega^{\bullet}(L) \oplus \tau_{\geq k}\Omega^{\bullet}(L)$, which is a consequence of

the Hodge Decomposition Theorem, introduces the following direct sum decomposition of $\Omega^{\bullet}_{\mathcal{MS}}(\Sigma \times L)$.

$$\Omega^{\bullet}_{\mathcal{MS}}(\Sigma \times L) = \mathrm{ft}_{<k}\Omega^{\bullet}_{\mathcal{MS}}(\Sigma \times L) \oplus \mathrm{ft}_{\geq k}\Omega^{\bullet}_{\mathcal{MS}}(\Sigma \times L).$$

This decomposition is used later, in Lemma 5.7.

Lemma 5.4. *Let $J = t \circ j : \Sigma \times L \hookrightarrow X_{reg} = X \setminus \Sigma$ be the embedding of $\Sigma \times L$, the total space of the link bundle, into the nonsingular part $X \setminus \Sigma$ of X (at 1/2). Then, pullback under J induces the following isomorphism.*

$$J^* : \frac{\Omega^{\bullet}_{\partial\mathcal{MS}}(X \setminus \Sigma)}{\Omega I^{\bullet}_{\bar{p}}(X)} \xrightarrow{\cong} \frac{\Omega^{\bullet}_{\mathcal{MS}}(\Sigma \times L)}{ft_{\geq k}\Omega^{\bullet}_{\mathcal{MS}}(\Sigma \times L)} \cong ft_{<k}\Omega^{\bullet}_{\mathcal{MS}}(\Sigma \times L).$$

Thus, there is a short exact sequence

$$0 \to \Omega I^{\bullet}_{\bar{p}}(X) \to \Omega^{\bullet}_{\partial\mathcal{MS}}(X \setminus \Sigma) \to ft_{<k}\Omega^{\bullet}_{\mathcal{MS}}(\Sigma \times L) \to 0 \tag{5.3}$$

Proof. The result about J^* can be found on the pp. 36-37 of [5]. The existence of the short exact sequence is a direct consequence. $\square$

Based on that result, one needs to introduce a de Rham map

$$\Phi_{<k} : H^{\bullet}(\mathrm{ft}_{<k}\Omega^{\bullet}_{\mathcal{MS}}(\Sigma \times L)) \to H_{\bullet}(\Sigma \times L_{<k})^{\dagger}.$$

To define it, use the de Rham map $\Phi_{\Sigma\times L} : H^{\bullet}(\Sigma \times L) \to H_{\bullet}(\Sigma \times L)^{\dagger}$ and the homology truncation map $f : L_{<k} \to L$.

Definition 5.5. The map $\Phi_{<k}$ is defined as the following composition.

$$\begin{tikzcd}
H^{\bullet}(\mathrm{ft}_{<k}\Omega^{\bullet}_{\mathcal{MS}}(\Sigma \times L)) \arrow[r, "\Phi_{<k}"] \arrow[d] & H_{\bullet}(\Sigma \times L_{<k})^{\dagger} \\
H^{\bullet}_{dR}(\Sigma \times L) \arrow[r, "\Phi_{\Sigma\times L}", "\cong"'] & H_{\bullet}(\Sigma \times L)^{\dagger} \arrow[u, "(\mathrm{id}\times f)_*{}^{\dagger}"']
\end{tikzcd}$$

Compared to the regular de Rham cohomology of $\Sigma \times L$, the cohomology of the subcomplex $\mathrm{ft}_{<k}\Omega^{\bullet}_{\mathcal{MS}}(\Sigma \times L)$ misses the same information in the fiber direction that is absent in $H_{\bullet}(\Sigma \times L_{<k})$. This leads to the following statement.

Lemma 5.6. *The map $\Phi_{<k}$ is an isomorphism.*

Proof. First, compare the de Rham isomorphism for the product $\Sigma \times L$ with the tensor product of the de Rham isomorphisms for Σ and L.

$$\begin{tikzcd}
H^\bullet_{dR}(\Sigma \times L) \arrow[d, "\int", "\cong"'] \arrow[dd, bend right=60, "\Phi_{\Sigma\times L}"'] & H^\bullet_{dR}(\Sigma) \otimes H^\bullet_{dR}(L) \arrow[l, "\cong", "\psi"'] \arrow[d, "\int \otimes \int", "\cong"'] \arrow[dd, bend left=60, "\Phi_\Sigma \otimes \Phi_L"] \\
H^\infty_\bullet(\Sigma \times L)^\dagger \arrow[r, "(\kappa^\infty)^\dagger", "\cong"'] \arrow[d, "(s_{\Sigma\times L*})^\dagger", "\cong"'] & (H^\infty_\bullet(\Sigma) \otimes H^\infty_\bullet(L))^\dagger \arrow[d, "(s_{\Sigma*}\otimes s_{L*})^\dagger", "\cong"'] \\
H_\bullet(\Sigma \times L)^\dagger \arrow[r, "\kappa^\dagger", "\cong"'] & (H_\bullet(\Sigma) \otimes H_\bullet(L))^\dagger
\end{tikzcd}$$

The horizontal map ψ in the top line is the Künneth isomorphism for de Rham cohomology, see [8, I §5]. The vertical map starting on the top left maps $[\omega]$ to the functional $\int_{\_} \omega$. The vertical map on the top right maps $[\eta] \otimes [\tau]$ to the functional, which sends $[x] \otimes [y] \in H^\infty_\bullet(\Sigma) \otimes H^\infty_\bullet(L)$ to $\int_x \eta \cdot \int_y \tau$. The first square commutes because of Lemma 3.2, the second square because of Lemma 3.1.

Now, consider the following diagram.

$$\begin{tikzcd}
H^\bullet(\mathrm{ft}_{<k}\Omega^\bullet_{\mathcal{MS}}(\Sigma \times L)) \arrow[d] & \bigoplus_{\ell=0}^{k-1} H^{n-1-\ell}_{dR}(\Sigma) \otimes H^\ell_{dR}(L) \arrow[l, "\cong", "\psi|"'] \arrow[d, "\int"] \\
H^\bullet_{dR}(\Sigma \times L) \arrow[d, "\Phi_{\Sigma\times L}", "\cong"'] & H^\bullet_{dR}(\Sigma) \otimes H^\bullet_{dR}(L) \arrow[l, "\cong", "\psi"'] \arrow[d, "\Phi_\Sigma \otimes \Phi_L", "\cong"'] \\
H_\bullet(\Sigma \times L)^\dagger \arrow[r, "\kappa^\dagger", "\cong"'] \arrow[d, "(\mathrm{id} \times f_*)^\dagger"] & (H_\bullet(\Sigma) \otimes H_\bullet(L))^\dagger \arrow[d, "(\mathrm{id} \otimes f_*)^\dagger"] \\
H_\bullet(\Sigma \times L_{<k})^\dagger \arrow[r, "\kappa^\dagger", "\cong"'] & (H_\bullet(\Sigma) \otimes H_\bullet(L_{<k}))^\dagger
\end{tikzcd}$$

The bottom square commutes because of the naturality of the Künneth isomorphism. Since the homology truncation map $f : L_{<k} \to L$ induces an isomorphism $f_* : H_\ell(L_{<k}) \to H_\ell(L)$ in degrees $\ell < k$, while $H_\ell(L_{<k}) = 0$ for $\ell \geq k$, the composition of the maps in the second column is an isomorphism. This property transfers to $\Phi_{<k}$, since it is the composition of the maps in the first column. $\square$

Since smoothing of singular chains can be arranged to be compatible with pullback under the embeddings $j : \Sigma \times L \hookrightarrow M$, $\alpha : M \hookrightarrow X \setminus \Sigma$ and hence also $J = \alpha \circ j : \Sigma \times L \hookrightarrow X \setminus \Sigma$, the de Rham maps Φ_{reg} and $\Phi_{<k}$ are compatible. In other words, the following lemma is true.

Lemma 5.7 (Compatibility of Φ_{reg} and $\Phi_{<k}$)**.** *The following square commutes.*

$$\begin{tikzcd}
H^\bullet\left(\Omega^\bullet_{\partial\mathcal{MS}}(X\setminus\Sigma)\right) \arrow[r, "J^*"] \arrow[d, "\Phi_{reg}", "\cong"'] & H^\bullet\left(\Omega^\bullet_{\mathcal{MS}}(\Sigma\times L)\right) \arrow[r, "proj^*"] & H^\bullet\left(ft_{<k}\Omega^\bullet_{\mathcal{MS}}(\Sigma\times L)\right) \arrow[d, "\Phi_{<k}", "\cong"'] \\
H^\infty_\bullet(M)^\dagger \arrow[r, "{(s_M)_*}^\dagger", "\cong"'] & H_\bullet(M)^\dagger \arrow[r, "{g_*}^\dagger"] & H_\bullet(\Sigma\times L_{<k})^\dagger
\end{tikzcd}$$

The projection in the upper row comes from the direct sum decomposition $\Omega^\bullet_{\mathcal{MS}}(\Sigma\times L) = ft_{<k}\Omega^\bullet_{\mathcal{MS}}(\Sigma\times L) \oplus ft_{\geq k}\Omega^\bullet_{\mathcal{MS}}(\Sigma\times L)$.

Proof. Note, that for a closed form $\eta \in \Omega^\bullet_{\mathcal{MS}}(\Sigma\times L)$ and a cycle $x \in S_\bullet(\Sigma\times L_{<k})$, it holds that

$$\Phi_{\Sigma\times L}\left([\eta]\right)\left((s_{\Sigma\times L})_*(\mathrm{id}\times f)_*[x]\right) = \Phi_{\Sigma\times L}\left(\mathrm{proj}^*[\eta]\right)\left((s_{\Sigma\times L})_*(\mathrm{id}\times f)_*[x]\right).$$

Thus, the following calculation proves the statement. Let $\omega \in \Omega^\bullet_{\partial\mathcal{MS}}(X\setminus\Sigma)$ be a closed form and let $x \in S_\bullet(\Sigma\times L_{<k})$ be a cycle. Then,

$$\begin{aligned}
&\Phi_{\mathrm{reg}}\left([\omega]\right)\left((s_M)_* g_*[x]\right) = \Phi_{\mathrm{reg}}\left([\omega]\right)\left(j_*(s_{\Sigma\times L})_*(\mathrm{id}\times f)_*[x]\right)\\
&= \Phi_{\Sigma\times L}\left(J^*[\omega]\right)\left((s_{\Sigma\times L})_*(\mathrm{id}\times f)_*[x]\right)\\
&= \Phi_{\Sigma\times L}\left(\mathrm{proj}^*J^*[\omega]\right)\left((s_{\Sigma\times L})_*(\mathrm{id}\times f)_*[x]\right)\\
&= \Phi_{<k}\left(\mathrm{proj}^*J^*[\omega]\right)\left([x]\right).
\end{aligned}$$

□

The compatibility of $\Phi_{<k}$ and $\Phi_{\bar{p}}$ is covered in the following lemma.

Lemma 5.8 (Compatibility of $\Phi_{\bar{p}}$ and $\Phi_{<k}$)**.** *Let* $\delta : H^\bullet\left(ft_{<k}\Omega^\bullet_{\mathcal{MS}}(\Sigma\times L)\right) \to H^{\bullet+1}\left(\Omega I^\bullet_{\bar{p}}(X)\right)$ *be the connecting homomorphism of the cohomology sequence associated with the short exact sequence (5.3) and let* $\xi : S^\infty_\bullet(g) \to$

$H_{\bullet-1}(\Sigma \times L_{<k})$ denote the projection. These two morphisms fit into the following commutative square, connecting $\Phi_{\bar{p}}$ and $\Phi_{<k}$.

$$\begin{tikzcd} H^{\bullet}\left(ft_{<k}\Omega^{\bullet}_{\mathcal{MS}}(\Sigma \times L)\right) \arrow[r, "\delta"] \arrow[d, "\Phi_{<k}", "\cong"'] & H^{\bullet+1}\left(\Omega I^{\bullet}_{\bar{p}}(X)\right) \arrow[d, "\Phi_{\bar{p}}"] \\ H_{\bullet}(\Sigma \times L_{<k})^{\dagger} \arrow[r, "{\xi_*}^{\dagger}"] & H_{\bullet+1}\left(S^{\propto}_{\bullet}(g)\right)^{\dagger} \end{tikzcd}$$

Proof. Let $\eta \in \mathrm{ft}_{<k}\Omega^{\bullet}_{\mathcal{MS}}(\Sigma \times L)$ be closed. By a standard technique - namely pulling back to a collar, cutting off and extending by zero - η is the pullback of a form $\omega \in \Omega^{\bullet}_{\partial\mathcal{MS}}(X \setminus \Sigma)$, that is $J^*\omega = \eta$. Then, $d\omega \in \Omega I^{\bullet}_{\bar{p}}(X)$ and $\delta[\eta] = [d\omega]$. Further, let $([x], v) \in S^{\propto}_{\bullet}(g)$ be closed, i.e. $\partial v = s_M\, g_* x$. There, the cycle $x \in S_{\bullet-1}(\Sigma \times L_{<k})$ is chosen in such a way, that $q\left([x]\right) = x$. Then,

$$\begin{aligned} \Phi_{\bar{p}}\left(\delta[\eta]\right)\left[([x], v)\right] &= \int_{\alpha_* v} d\omega = \int_{\alpha_* s_M g_* x} \omega = \int_{s_{\Sigma\times L}(\mathrm{id}\times f)_* x} J^*\omega \\ &= \int_{s_{\Sigma\times L}(\mathrm{id}\times f)_* x} \eta = \Phi_{<k}[\eta]\left([x]\right) = \Phi_{<k}[\eta]\left(\xi_*\left[([x], v)\right]\right). \end{aligned}$$

$\square$

6 Proof of the main thoerem

The long exact (co)homology sequences associated with the short exact sequences (5.2) and (5.3) fit into the following diagram.

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
H^{\bullet-1}\big(\mathrm{ft}_{<k}\Omega^\bullet_{\mathcal{MS}}(\Sigma\times L)\big) \arrow[r, "\Phi_{<k}", "\cong"'] \arrow[d, "\delta"] & H_{\bullet-1}(\Sigma\times L_{<k})^\dagger \arrow[d, "{\xi_*}^\dagger"] \\
H^\bullet\big(\Omega I^\bullet_{\bar p}(X)\big) \arrow[r, "\Phi_{\bar p}"] \arrow[d, "i^*"] & H_\bullet\left(S^\pitchfork_\bullet(g)\right)^\dagger \arrow[d, "(\mathrm{inc}_*)^\dagger"] \\
H^\bullet\left(\Omega^\bullet_{\partial\mathcal{MS}}(X\setminus\Sigma)\right) \arrow[r, "\Phi_{\mathrm{reg}}", "\cong"'] \arrow[d, "\mathrm{proj}^* J^*"] & H^\infty_\bullet(M)^\dagger \arrow[d, "{g_*}^\dagger {(s_M)_*}^\dagger"] \\
H^\bullet\left(\mathrm{ft}_{<k}\Omega^\bullet_{\mathcal{MS}}(\Sigma\times L)\right) \arrow[r, "\Phi_{<k}", "\cong"'] \arrow[d] & H_\bullet(\Sigma\times L_{<k})^\dagger \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

This diagram commutes by the results of Section 5 and thus, the Five-Lemma proves the statement of the main theorem.

The following two questions are not addressed in this note.

Remark 6.1 (Is $\Phi_{\bar p}$ a ring isomorphism?)**.** By the main theorem, the de Rham map $\Phi_{\bar p}$ is an isomorphism of vector spaces. The proof does not imply, that it is a ring isomorphism. A natural question is: Do the arguments of [14] for spaces with isolated singularities generalize to the setting of this paper?

Remark 6.2 (Twisted link bundles)**.** Another question is, whether the present result generalizes to depth one pseudomanifolds with twisted link bundles. Intersection spaces have been defined in this context by Banagl and Chriestenson in [6] if the link bundles have equivariant Moore approximations. The de Rham model for intersection space cohomology is avaidable in the more general context of fiber bundles with locally constant transition functions and structure group the isometries of the fiber (for some Riemannian metric). Such bundles are called geometrically flat. Unfortunately, not all geometrically flat fiber bundles admit equivariant Moore approximations. The author expects that the de Rham Theorem for intersection spaces can be extended to pseudomanifolds with geometrically flat fiber bundles, which can be endowed with equivariant Moore approximations. To show that, one needs to relate the cohomology of fiberwisely cotruncated forms with the cohomology of the homotopy cofiber of the equivariant Moore approximation of the fiber bundles, e.g. using local to global techniques and small singular simplices.

Equivariant Moore approximations for geometrically flat bundles are obstructed by certain cohomology classes, which were introduced in [6, Section 6]. These classes are rather subtle, which can be seen in Example 6.13 of [6]. There, the cohomology classes do not vanish for a bundle with completely trivial differentials in the Serre spectral sequence.

Remark 6.3 (Greater stratification depth)**.** In greater stratification depth, first examples of intersection spaces, given in [4], were generalized by Agustín and Bobadilla in [2], e.g. for pseudomanifolds with compatibly trivializable link bundles, such as projective toric varieties. In ongoing work, the de Rham model for greater stratification depth, which was introduced by the author in [9], is compared to the cohomology of Agustin and Bobadilla's intersection spaces. The results of [1] are important tools in this setting, especially the uniqueness results when passing to the derived category.

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