Matemática
Contemporânea
Vol. 53, 213–232
http://doi.org/10.21711/231766362023/rmc5310

Categorical aspects of Gaffney's double structure of a module

Thiago da Silva [1]

[1] Universidade Federal do Espírito Santo, Av. Fernando Ferrari, 514, Vitória, Brazil

Abstract. In this work, we develop some categorical properties of the double structure of a module from the double morphism and we use it to get an equivalence between two Lipschitz saturation for a special class of modules.

Keywords: Double of Modules and morphisms, Bi-Lipschitz Equisingularity, Homological Algebra, Lipschitz saturation.

2020 Mathematics Subject Classification: 18B99, 14J17, 13C60.

Introduction

The study of bi-Lipschitz equisingularity was started at the end of the 1960s with the works of Zariski [13], Pham [11] and Teissier [10]. At the end of the 1980s, Mostowski [9] introduced a new technique for the study of Bi-Lipschitz equisingularity from the existence of Lipschitz vector fields.

In [5] Gaffney defined the concept of the double of an ideal and developed the infinitesimal Lipschitz conditions for a family of hypersurfaces using the integral closure of modules, namely, the double of some jacobian ideals. In [6] Gaffney used the double and the integral closure of modules to get algebraic conditions for bi-Lipschitz equisingularity of a family of irreducible curves. In [1] the authors also used the double and the integral closure of ideals to get an algebraic condition to get a canonical vector field defined along an Essentially Isolated Determinantal Singularities (EIDS) family, which is Lipschitz provided the matrix of deformation of the 1-unfolding which defines the EIDS is constant. In [2] they used these techniques to deal with normal forms of square matrices in this landscape.

In [4] it was extended the notion of the double for modules, and we generalize the infinitesimal theorem of [5]. In [3] we prove that the infinitesimal condition is necessary for the strongly bi-Lipschitz triviality, as developed by Fernandes and Ruas in [7].

In this work, our main goal is to look at the categorical properties of the double.

In the first section, we define the double morphism and we rephrase several results from Commutative Algebra that relate the standard properties of a morphism and its double.

In the second section we apply the double morphism to get an equivalence between the second and third Lipschitz saturation of modules, defined in [4], for a special class of modules.

In the third section, we develop some relations between the homological behavior of chain complexes and their doubles.

Finally, in the fourth section, we extend the notion of a double morphism between two submodules embedded on finite powers of local rings of possibly different analytic varieties which are linked by an analytic map-germ between them.

1 Background for the double morphism

Let $X \subseteq \mathbb{C}^n$ be an analytic space and let $\mathcal{O}_X$ be the analytic sheaf of local rings over X, and let $x \in X$. It is defined in [4] the concept of a double of a $\mathcal{O}_{X,x}$-submodule M of $\mathcal{O}_{X,x}^p$. We recall the definition now.

Consider the projection maps $\pi_1, \pi_2 : X \times X \to X$.

Definition 1.1. 1. Let $h \in \mathcal{O}^p_{X,x}$. The **double of** h is defined as

$$h_D := (h \circ \pi_1, h \circ \pi_2) \in \mathcal{O}^{2p}_{X\times X,(x,x)}.$$

2. The **double of** M is denoted by M_D and is defined as the $\mathcal{O}_{X\times X,(x,x)}$-submodule of $\mathcal{O}^{2p}_{X\times X,(x,x)}$ generated by $\{h_D \mid h \in M\}$.

It is well known we have the analytic tensor product in the analytic category in a way that $\mathcal{O}_{X\times X,(x,x)}$ can be viewed as $\mathcal{O}_{X,x}\hat{\underset{\mathbb{C}}{\otimes}}\mathcal{O}_{X,x}$. Once Gaffney's double structure was conceived to deal with bi-Lipschitz equisingularity, it is convenient to work on $\mathcal{O}_{X\times X,(x,x)}$ instead of $\mathcal{O}_{X,x}\hat{\underset{\mathbb{C}}{\otimes}}\mathcal{O}_{X,x}$. Because of it, in this section, we rephrase some classical results from Commutative Algebra from this point of view.

The first result is a quite useful tool many times when we work with the double.

Proposition 1.2. *Let $M, N \subseteq \mathcal{O}^p_{X,x}$ submodules and $h, g \in \mathcal{O}^p_{X,x}$. Then:*

a) h=g if, and only if, $h_D = g_D$;

b) $h \in M$ if, and only if, $h_D \in M_D$;

c) $M \subseteq N$ if, and only if, $M_D \subseteq N_D$;

d) $M = N$ if, and only if, $M_D = N_D$.

Corollary 1.3. *For each $\mathcal{O}_{X,x}$-submodule M of $\mathcal{O}^p_{X,x}$, the natural map*

$$\begin{array}{rccc} D_M : & M & \longrightarrow & M_D \\ & h & \longmapsto & h_D \end{array}$$

is an injective group morphism. In particular, we can see M as an additive subgroup of M_D.

Our main goal is to give a categorical sense of the double structure. The next theorem is the key to it.

Theorem 1.4. *Let $M \subseteq \mathcal{O}^p_{X,x}$ and $N \subseteq \mathcal{O}^q_{X,x}$ be $\mathcal{O}_{X,x}$-submodules. If $\phi : M \to N$ is an $\mathcal{O}_{X,x}$-module morphism then there exists a unique $\mathcal{O}_{X\times X,(x,x)}$-module morphism $\phi_D : M_D \to N_D$ such that*

$$\phi_D(h_D) = (\phi(h))_D, \forall h \in M,$$

i.e, the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[d, "D_M"] & N \arrow[d, "D_N"] \\ M_D \arrow[r, dashed, "\phi_D"] & N_D \end{tikzcd}$$

The map ϕ_D is called the double of ϕ.

From now on, all the modules are objects in $\mathcal{T}(\mathcal{O}_{X,x})$ and their doubles are objects in $\mathcal{T}(\mathcal{O}_{X\times X,(x,x)})$.

Notice that if $id_M : M \to M$ and $id_{M_D} : M_D \to M_D$ are the identity morphisms of M and M_D, then

$$(id_M)_D = id_{M_D}.$$

The next proposition gives us a relation between images and kernels of a module morphism.

Proposition 1.5. *Let $\phi : M \to N$ be an $\mathcal{O}_{X,x}$-module morphism and $\phi_D : M_D \to N_D$ its double. Then:*

a) $Im(\phi_D) = (Im(\phi))_D$;

b) $(ker(\phi))_D \subseteq ker(\phi_D)$.

The next proposition shows that double morphism has good behavior concerning sum and composition.

Proposition 1.6. *Let $\phi, \phi' : M \to N$ and $\gamma : N \to P$ be $\mathcal{O}_{X,x}$-module morphisms.*

a) $\phi = \phi' \iff \phi_D = \phi'_D$;

b) $(\gamma \circ \phi)_D = \gamma_D \circ \phi_D$;

c) $(\phi + \phi')_D = \phi_D + \phi'_D$.

Corollary 1.7. *Let $\phi : M \to N$ be an $\mathcal{O}_{X,x}$-module morphism. Then:*

a) $\phi : M \to N$ *is surjective if, and only if,* $\phi_D : M_D \to N_D$ *is a surjective;*

b) If $\phi_D : M_D \to N_D$ *is injective then* $\phi : M \to N$ *is injective;*

c) $\phi : M \to N$ *is an* $\mathcal{O}_{X,x}$*-isomorphism if, and only if,* $\phi_D : M_D \to N_D$ *is an* $\mathcal{O}_{X\times X,(x,x)}$*-isomorphism;*

d) $\phi : M \to N$ *is the zero morphism if, and only if,* $\phi_D : M_D \to N_D$ *is the zero morphism.*

As an application of the double morphism, we prove in the next theorem that the double structure is compatible with the finite direct sum of modules.

Theorem 1.8. *Let* $M \subseteq \mathcal{O}^p_{X,x}$ *and* $N \subseteq \mathcal{O}^q_{X,x}$ *be* $\mathcal{O}_{X,x}$*-submodules. Then*

$$(M \oplus N)_D \cong M_D \oplus N_D$$

as $\mathcal{O}_{X\times X,(x,x)}$*-submodules of* $\mathcal{O}^{2(p+q)}_{X\times X,(x,x)}$.

Furthermore, there exists an isomorphism

$$\eta : (M \oplus N)_D \longrightarrow M_D \oplus N_D$$

such that $\eta((h, g)_D) = (h_D, g_D)$, *for all* $h \in M$ *and* $g \in N$.

Corollary 1.9. *Let* $M_i \subseteq \mathcal{O}^{p_i}_{X,x}$ *be* $\mathcal{O}_{X,x}$*-submodules, for each* $i \in \{1, ..., r\}$. *Then*

$$(M_1 \oplus ... \oplus M_r)_D \cong (M_1)_D \oplus ... \oplus (M_r)_D$$

as $\mathcal{O}_{X\times X,(x,x)}$*-submodules of* $\mathcal{O}^{2(p_1+...+p_r)}_{X\times X,(x,x)}$ *through an isomorphism such that*

$$(h_1, ..., h_r)_D \longmapsto ((h_1)_D, ..., (h_r)_D)$$

for all $h_i \in M_i$.

Proposition 1.10. *Let $M \subseteq N$ be $\mathcal{O}_{X,x}$-submodules of $\mathcal{O}^p_{X,x}$.*

a) If M_D has finite length then M has finite length and $\ell(M) \leq \ell(M_D)$;

b) If M_D has finite colength in N_D then M has finite colength in N.

2 Applications on the equivalence of the Lipschitz saturation of modules

In this section, we want to apply the double morphism to compare two different types of Lipschitz saturation (which were defined in [4]) for a special class of modules. However, we need some tools first.

Definition 2.1. We say that an $\mathcal{O}_{X,x}$-morphism $\phi : M \subseteq \mathcal{O}^p_{X,x} \to N \subseteq \mathcal{O}^q_{X,x}$ is induced by a $q \times p$ matrix if there exists $A \in \mathrm{Mat}_{q\times p}(\mathcal{O}_{X,x})$ such that $\phi(h) = A \cdot h$, $\forall h \in M$.

Lemma 2.2. *An $\mathcal{O}_{X,x}$-morphism $\phi : M \subseteq \mathcal{O}^p_{X,x} \to N \subseteq \mathcal{O}^q_{X,x}$ is induced by a $q\times p$ matrix if, and only if, there exists an $\mathcal{O}_{X,x}$-morphism $\tilde{\phi} : \mathcal{O}^p_{X,x} \to \mathcal{O}^q_{X,x}$ such that $\tilde{\phi}(M) \subseteq N$ and $\tilde{\phi} \mid_M = \phi$.*

Proof. ($\Longrightarrow$) By hypothesis there exists a $q \times p$ matrix A with entries in $\mathcal{O}_{X,x}$ such that $\phi(h) = A \cdot h$, $\forall h \in M$. From this matrix A, we can define $\tilde{\phi} : \mathcal{O}^p_{X,x} \to \mathcal{O}^q_{X,x}$ given by $\tilde{\phi}(g) := A \cdot g$, which is an $\mathcal{O}_{X,x}$-morphism. Clearly, $\tilde{\phi} \mid_M = \phi$, and for all $h \in M$ we have $\tilde{\phi}(h) = \phi(h) \in N$, so $\tilde{\phi}(M) \subseteq N$.

($\Longleftarrow$) Let $e_1, ..., e_p$ be the canonical elements in $\mathcal{O}^p_{X,x}$. Let A be the $q\times p$ matrix whose columns are $\phi(e_1), ..., \phi(e_p)$. Then $\tilde{\phi}(g) = A \cdot g, \forall g \in \mathcal{O}^p_{X,x}$. Since $\tilde{\phi} \mid_M = \phi$ then $\phi(h) = \tilde{\phi}(h) = A \cdot h, \forall h \in M$. Therefore, ϕ is induced by a $q \times p$ matrix. $\square$

In the next proposition, we prove the double morphism inherits to be induced by a matrix from the original one.

Proposition 2.3. *If $\phi : M \subseteq \mathcal{O}^p_{X,x} \to N \subseteq \mathcal{O}^q_{X,x}$ is an $\mathcal{O}_{X,x}$-morphism induced by a $q \times p$ matrix then*

$$\phi_D : M_D \subseteq \mathcal{O}^{2p}_{X\times X,(x,x)} \to N_D \subseteq \mathcal{O}^{2q}_{X\times X,(x,x)}$$

is an $\mathcal{O}_{X\times X,(x,x)}$-morphism induced by a $2q \times 2p$ matrix.

Proof. By hypothesis there exists a $q \times p$ matrix A with entries in $\mathcal{O}_{X,x}$ such that $\phi(h) = A \cdot h$, $\forall h \in M$. Then, for all $h \in M$ we have

$$\phi_D(h_D) = \begin{bmatrix} \phi(h) \circ \pi_1 \\ \phi(h) \circ \pi_2 \end{bmatrix} = \begin{bmatrix} (A \cdot h) \circ \pi_1 \\ (A \cdot h) \circ \pi_2 \end{bmatrix} = \begin{bmatrix} (A \circ \pi_1) \cdot (h \circ \pi_1) \\ (A \circ \pi_2) \cdot (h \circ \pi_2) \end{bmatrix}.$$

So, taking the $2q \times 2p$ matrix

$$B := \begin{bmatrix} A \circ \pi_1 & 0_{q\times p} \\ 0_{q\times p} & A \circ \pi_2 \end{bmatrix}$$

we conclude that $\phi_D(h_D) = B \cdot h_D$, and the proposition is proved, once M_D is generated by h_D, $h \in M$. □

The next proposition gives the *persistence* of the integral closure of modules.

Proposition 2.4. *Let $\varphi : M \subseteq \mathcal{O}^p_{X,x} \to N \subseteq \mathcal{O}^q_{X,x}$ be a morphism of $\mathcal{O}_{X,x}$-modules which can be extended to a morphism $\tilde{\varphi} : \mathcal{O}^p_{X,x} \to \mathcal{O}^q_{X,x}$, given by $\tilde{\varphi}(h) = A \cdot h$, where A is a $q \times p$ matrix with entries in $\mathcal{O}_{X,x}$. Let $h \in \mathcal{O}^p_{X,x}$.*

a) If $h \in \overline{M}$ then $\tilde{\varphi}(h) \in \overline{\varphi(M)}$;

b) Suppose $q = p$. If A is an invertible matrix and φ is an isomorphism of $\mathcal{O}_{X,x}$-modules then:

$$h \in \overline{M} \text{ if and only if } \tilde{\varphi}(h) \in \overline{\varphi(M)};$$

c) Suppose $q = p$ and φ is injective. If $\tilde{\varphi}(h) \in \overline{\varphi(M)}$ then $h \in \overline{M}$.

Proof. (a) Let $\phi : (\mathbb{C}, 0) \to (X, x)$ be an arbitrary analytic curve. By hypothesis $\phi^*(h) \in \phi^*(M)$ and we can write $\phi^*(h) = \sum \alpha_i \phi^*(g_i)$, for some $g_i \in M$ and $\alpha_i \in \mathcal{O}_{\mathbb{C},0}$. Thus: $\tilde{\varphi}(h) \circ \phi = (A \cdot h) \circ \phi = [A \circ \phi] \cdot [h \circ \phi] = [A \circ \phi] \cdot \sum \alpha_i \phi^*(g_i) = \sum \alpha_i([A \cdot g_i] \circ \phi) \in \phi^*(\varphi(M))$.

Hence, $\tilde{\varphi}(h) \in \overline{\varphi(M)}$.

(b) It suffices apply (a) in $\tilde{\varphi}^{-1}$.

(c) It suffices consider the isomorphism $\hat{\varphi} : M \to \varphi(M)$ given by $\hat{\varphi}(h) = \varphi(h)$ and apply the item (b). $\square$

Following the notation in [4], let us recall the definitions of the second and third Lipschitz saturations of modules.

Definition 2.5. The **2-Lipschitz saturation** of the submodule $M \subseteq \mathcal{O}^p_{X,x}$ is denoted by M_{S_2}, and is defined by

$$M_{S_2} := \{h \in \mathcal{O}^p_{X,x} \mid h_D \in \overline{M_D} \text{ at } (x, x)\}.$$

Before defining the third Lipschitz saturation, let us fix some notations.

For each $\psi : X \to Hom(\mathbb{C}^p, \mathbb{C})$ analytic map, $\psi = (\psi_1, ..., \psi_p)$ and $h = (h_1, ..., h_p) \in \mathcal{O}^p_X$, we define $\psi \cdot h \in \mathcal{O}_X$ given by $(\psi \cdot h)(z) := \sum_{i=1}^{p} \psi_i(z) h_i(z)$.

We define $\psi \cdot M$ as the ideal of $\mathcal{O}_X$ generated $\{\psi \cdot h \mid h \in M\}$.

Definition 2.6. The **3-Lipschitz saturation** of the submodule $M \subseteq \mathcal{O}^p_{X,x}$ is denoted by M_{S_3}, and is defined by

$$M_{S_3} := \{h \in \mathcal{O}^p_{X,x} \mid (\psi \cdot h)_D \in \overline{(\psi \cdot M)_D}, \forall \psi : X \to Hom(\mathbb{C}^p, \mathbb{C}) \text{ analytic map}\}.$$

By Proposition 3.1.22 of [4], we already know that $M_{S_2} \subseteq M_{S_3}$. We want to use double morphism to get an equivalence between the second and third Lipschitz saturations.

For each $i \in \{1, ..., p\}$ consider the i-th canonical global section of the vector bundle $Hom(\mathbb{C}^p, \mathbb{C})$,

$$\xi_i : X \to Hom(\mathbb{C}^p, \mathbb{C})$$

given by $\xi_i(x) = (0, .., 1, ...0)$, where 1 is on the i-th place. Let us denote

$$\hat{M} := \xi_1 \cdot M \oplus ... \oplus \xi_p \cdot M.$$

Notice that if M is an $\mathcal{O}_{X,x}$-submodule of $\mathcal{O}^p_{X,x}$ then $M \subseteq \hat{M}$ and it is easy to see that $\hat{\hat{M}} = \hat{M}$.

Now we are ready for the main theorem.

Theorem 2.7. *Let $M \subseteq \mathcal{O}^p_{X,x}$ be a submodule. Suppose that M_D is a reduction of $\hat{M}_D$. Then, $M_{S_2} = M_{S_3}$.*

Proof. Consider the inclusion $i : M \hookrightarrow \xi_1 \cdot M \oplus ... \oplus \xi_p \cdot M$. Then we can consider the inclusion $i_D : M_D \to (\xi_1 \cdot M \oplus ... \oplus \xi_p \cdot M)_D$ which is induced by an invertible $2p \times 2p$ matrix. By Corollary 1.9 there is an isomorphism

$$\gamma : (\xi_1 \cdot M \oplus ... \oplus \xi_p \cdot M)_D \to (\xi_1 \cdot M)_D \oplus ... \oplus (\xi_p \cdot M)_D$$

and by the proof of this corollary, this isomorphism is induced by an invertible $2p \times 2p$ matrix. Taking the composition of i_D with γ, we get an injective morphism

$$\eta : M_D \to (\xi_1 \cdot M)_D \oplus ... \oplus (\xi_p \cdot M)_D$$

induced by an invertible $2p \times 2p$ matrix B which extends to the isomorphism

$$\tilde{\eta} : \mathcal{O}^{2p}_{X \times X,(x,x)} \to \mathcal{O}^{2p}_{X \times X,(x,x)}$$

given by the multiplication by B, which satisfies the property

$$(g_1, ..., g_p)_D \mapsto ((g_1)_D, ..., (g_p)_D).$$

We already have the inclusion $M_{S_2} \subseteq M_{S_3}$. So, it suffices to check another inclusion.

Let $h \in M_{S_3}$. In particular, $(\xi_i \cdot h)_D \in \overline{(\xi_i \cdot M)_D}$, $\forall i \in \{1, ..., p\}$. Let $\phi : (\mathbb{C}, 0) \to (X \times X, (x, x))$ be an arbitrary analytic curve. Then $\phi^*((\xi_i \cdot h)_D) \in \phi^*((\xi_i \cdot M)_D)$, $\forall i \in \{1, ..., p\}$ and

$$\phi^*(\tilde{\eta}(h_D)) = \phi^*((\xi_1 \cdot h)_D, ..., (\xi_p \cdot h)_D) = (\phi^*((\xi_1 \cdot h)_D), ..., \phi^*((\xi_p \cdot h)_D))$$

which belongs to $\phi^*((\xi_1 \cdot M)_D \oplus ... \oplus (\xi_p \cdot M)_D)$. Hence,

$$\tilde{\eta}(h_D) \in \overline{(\xi_1 \cdot M)_D \oplus ... \oplus (\xi_p \cdot M)_D}.$$

Since M_D is a reduction of $(\xi_1 \cdot M \oplus ... \oplus \xi_p \cdot M)_D$ then by the previous proposition we have that $\eta(M_D)$ is a reduction of $(\xi_1 \cdot M)_D \oplus ... \oplus (\xi_p \cdot M)_D$. Thus, $\tilde{\eta}(h_D) \in \overline{\eta(M_D)}$, and by the previous proposition we conclude that $h_D \in \overline{M_D}$, therefore $h \in M_{S_2}$. □

Corollary 2.8. *If M is an $\mathcal{O}_{X,x}$-submodule of $\mathcal{O}^p_{X,x}$ and $\hat{M} = M$ then $M_{S_2} = M_{S_3}$.*

Corollary 2.9. *If M is an $\mathcal{O}_{X,x}$-submodule of $\mathcal{O}^p_{X,x}$ then $\hat{M}_{S_2} = \hat{M}_{S_3}$.*

Example 2.10. Take $f, g \in \mathcal{O}_{X,x}$ and let M be the $\mathcal{O}_{X,x}$-submodule of $\mathcal{O}^2_{X,x}$ generated by $\{(f,0),(0,g)\}$. So, $\xi_1 \cdot M$ and $\xi_2 \cdot M$ are the principal ideals of $\mathcal{O}_{X,x}$ generated by f and g, respectively. Thus, if $\phi \in \xi_1 \cdot M \oplus \xi_2 \cdot M$, we can write $\phi = (\beta_1 f, \beta_2 g)$, for some $\beta_1, \beta_2 \in \mathcal{O}_{X,x}$. Then,

$$\phi = \beta_1(f,0) + \beta_2(0,g) \in M.$$

This proves that $M = \xi_1 \cdot M \oplus \xi_2 \cdot M = \hat{M}$ hence, $M_{S_2} = M_{S_3}$.

3 Homological aspects of the double structure

In this section, we explore the double morphism to deal with homological features in this context.

Proposition 3.1. *Let*

$$M \xrightarrow{\phi} N \xrightarrow{\gamma} P$$

be a sequence of $\mathcal{O}_{X,x}$-module morphism and consider the double sequence

$$M_D \xrightarrow{\phi_D} N_D \xrightarrow{\gamma_D} P_D$$

If $Im(\phi) \subseteq ker(\gamma)$ then $Im(\phi_D) \subseteq ker(\gamma_D)$.

Proof. Since $\mathrm{Im}(\phi) \subseteq \ker(\gamma)$ then $(\mathrm{Im}(\phi))_D \subseteq (\ker(\gamma))_D$. Hence, $\mathrm{Im}(\phi_D) = (\mathrm{Im}(\phi))_D \subseteq (\ker(\gamma))_D \subseteq \ker(\gamma_D)$. $\square$

We will see that double morphism gives a natural way to study the homology of the double structure.

Definition 3.2 (The double chain complex)**.** Let $\mathcal{C} = (M_\bullet, \phi_\bullet)$ be a chain complex in $\mathcal{T}(\mathcal{O}_{X,x})$. We define

$$\mathcal{C}_D := ((M_\bullet)_D, (\phi_\bullet)_D)$$

and by Proposition 3.1 we have that $\mathcal{C}_D$ is a chain complex in $\mathcal{T}(\mathcal{O}_{X\times X,(x,x)})$. The chain complex $\mathcal{C}_D$ is called **the double of** $\mathcal{C}$.

Proposition 3.3. *Let $C = (M_\bullet, \phi_\bullet)$ be a chain complex. If C_D is an exact sequence then $\mathcal{C}$ is an exact sequence. In other words, if $\mathcal{C}_D$ has trivial homology then $\mathcal{C}$ has trivial homology.*

Proof. Let $i \in \mathbb{Z}$ be arbitrary. We have the sequences

$$M_{i+1} \xrightarrow{\phi_{i+1}} M_i \xrightarrow{\phi_i} M_{i-1} \qquad\qquad (M_{i+1})_D \xrightarrow{(\phi_{i+1})_D} (M_i)_D \xrightarrow{(\phi_i)_D} (M_{i-1})_D$$

We already know that $\mathrm{Im}(\phi_{i+1}) \subseteq \ker(\phi_i)$. Since $\mathcal{C}_D$ is an exact sequence then $\mathrm{Im}((\phi_{i+1})_D) = \ker((\phi_i)_D)$. By Proposition 1.5 we have $(\ker(\phi_i))_D \subseteq \ker((\phi_i)_D) = \mathrm{Im}((\phi_{i+1})_D) = (\mathrm{Im}(\phi_{i+1}))_D$. By Proposition 1.2 (c), we conclude that $\ker(\phi_i) \subseteq \mathrm{Im}(\phi_{i+1})$. Therefore, $\mathrm{Im}(\phi_{i+1}) = \ker(\phi_i)$. $\square$

Proposition 3.4. *Let $\mathcal{C} = (M_\bullet, \phi_\bullet)$ and $\mathcal{C}' = (M'_\bullet, \phi'_\bullet)$ be chain complexes. If $\alpha : \mathcal{C} \longrightarrow \mathcal{C}'$ is a chain complex morphism then $\alpha_D : \mathcal{C}_D \longrightarrow \mathcal{C}'_D$ given by*

$$\{(\alpha_i)_D : (M_i)_D \to (M'_i)_D \mid i \in \mathbb{Z}\}$$

is a chain complex morphism, called the **the double morphism of** α.

Proof. Let $i \in \mathbb{Z}$. So we have the diagram

$$\begin{tikzcd} M_i \arrow[r, "\phi_i"] \arrow[d, "\alpha_i"] & M_{i-1} \arrow[d, "\alpha_{i-1}"] \\ M'_i \arrow[r, "\phi'_i"] & M'_{i-1} \end{tikzcd}$$

is commutative. By Proposition 1.6 (b) follows that $(\phi'_i)_D \circ (\alpha_i)_D = (\phi'_i \circ \alpha_i)_D = (\alpha_{i-1} \circ \phi_i)_D = (\alpha_{i-1})_D \circ (\phi_i)_D$, and the following diagram is also commutative:

$$\begin{tikzcd} (M_i)_D \arrow[r, "(\phi_i)_D"] \arrow[d, "(\alpha_i)_D"] & (M_{i-1})_D \arrow[d, "(\alpha_{i-1})_D"] \\ (M'_i)_D \arrow[r, "(\phi'_i)_D"] & (M'_{i-1})_D \end{tikzcd}$$

□

Corollary 3.5. *If $\alpha : \mathcal{C} \longrightarrow \mathcal{C}'$ and $\beta : \mathcal{C}' \longrightarrow \mathcal{C}''$ are chain morphisms then*

$$(\beta \circ \alpha)_D = \beta_D \circ \alpha_D.$$

Proof. It is a straightforward consequence of the Proposition 1.6 (b). □

Now, we will get some results related to chain homotopy.

Let $\mathcal{C} = (M_\bullet, \phi_\bullet)$ and $\mathcal{C}' = (M'_\bullet, \phi'_\bullet)$ be chain complexes. Let $\mu : \mathcal{C} \to \mathcal{C}'$ be a morphism of degree 1, i.e, μ is a collection of $\mathcal{O}_{X,x}$-module morphisms $\{\mu_i : M_i \to M'_{i+1} \mid i \in \mathbb{Z}\}$. We know this morphism induces a chain morphism $\tilde{\mu} : \mathcal{C} \to \mathcal{C}'$ given by $\{\tilde{\mu}_i : M_i \to M'_i \mid i \in \mathbb{Z}\}$, where $\tilde{\mu_i} := \phi'_{i+1} \circ \mu_i + \mu_{i-1} \circ \phi_i, \forall i \in \mathbb{Z}$.

If $\alpha, \beta : \mathcal{C} \to \mathcal{C}'$ are chain morphisms, remember that $\mu : \mathcal{C} \to \mathcal{C}'$ is defined as a **homotopy between** α **and** β when $\tilde{\mu} = \alpha - \beta$, and we denote $\alpha \simeq \beta$ by μ.

Lemma 3.6. *Consider $\mu_D : \mathcal{C}_D \to \mathcal{C}'_D$ the morphism of degree 1 given by the double morphisms of $\mu : \mathcal{C} \to \mathcal{C}'$. Then $\widetilde{\mu_D} = (\tilde{\mu})_D$.*

Proof. For all $i \in \mathbb{Z}$ we have $(\widetilde{\mu_D})_i = (\phi'_{i+1})_D \circ (\mu_i)_D + (\mu_{i-1})_D \circ (\phi_i)_D$ $= (\phi'_{i+1} \circ \mu_i + \mu_{i-1} \circ \phi_i)_D = (\tilde{\mu}_i)_D$, and the lemma is proved. $\square$

Proposition 3.7. *Let $\alpha, \beta : \mathcal{C} \to \mathcal{C}'$ be chain morphisms and $\mu : \mathcal{C} \to \mathcal{C}'$ a morphism of degree 1. Then: μ is a homotopy between α and β if, and only if, μ_D is a homotopy between α_D and β_D.*

Proof. We have that μ is a homotopy between α and β $\iff$ $\tilde{\mu} = \alpha - \beta$. By Proposition 1.6 (a) and (c) and the previous lemma we have: $\tilde{\mu} = \alpha - \beta \iff (\tilde{\mu})_D = (\alpha - \beta)_D \iff \widetilde{\mu_D} = \alpha_D - \beta_D \iff \mu_D$ is a homotopy between α_D and β_D. $\square$

Corollary 3.8. *If $\alpha : \mathcal{C} \to \mathcal{C}'$ is a chain homotopy equivalence then $\alpha_D : \mathcal{C}_D \to \mathcal{C}'_D$ is a chain homotopy equivalence.*

Proof. By hypothesis there exists a chain morphism $\beta : \mathcal{C}' \to \mathcal{C}$ such that $\beta \circ \alpha \simeq id_{\mathcal{C}}$ and $\alpha \circ \beta \simeq id_{\mathcal{C}'}$. By the previous proposition, we have $(\beta \circ \alpha)_D \simeq (id_{\mathcal{C}})_D$ and $(\alpha \circ \beta)_D \simeq (id_{\mathcal{C}'})_D$, and therefore $(\beta)_D \circ (\alpha)_D \simeq id_{\mathcal{C}_D}$ and $\alpha_D \circ \beta_D \simeq id_{\mathcal{C}'_D}$. $\square$

Corollary 3.9. *If $\mathcal{C}$ is a contractible chain complex then $\mathcal{C}_D$ is also contractible.*

Proof. Since $\mathcal{C}$ is contractible then $id_{\mathcal{C}} \simeq 0_{\mathcal{C}}$, and by Proposition 3.7 follows that $(id_{\mathcal{C}})_D \simeq (0_{\mathcal{C}})_D$, thus $id_{\mathcal{C}_D} \simeq 0_{\mathcal{C}_D}$. Hence, $\mathcal{C}_D$ is contractible. $\square$

Notice the nice relation between the Proposition 3.3 and Corollary 3.9. We already know every contractible chain complex is an exact sequence. The Proposition 3.3 says that the exactness on the double level implies the exactness on the single level. The Corollary 3.9, which treats contractible (stronger than exactness), says the opposite.

All the results obtained in this section can be naturally translated into the cohomology language.

4 The double morphism relative to an analytic map germ

Let (Y, y) and (X, x) be germs of analytic spaces, and let $\varphi : (Y, y) \to (X, x)$ be an analytic map germ. So, the pullback map $\varphi^* : \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is a ring morphism, which induces an $\mathcal{O}_{X,x}$-algebra structure in $\mathcal{O}_{Y,y}$. Thus, every $\mathcal{O}_{Y,y}$-module is also an $\mathcal{O}_{X,x}$-module through this ring morphism.

We will see that there is a natural $\mathcal{O}_{X\times X,(x,x)}$-algebra structure in $\mathcal{O}_{Y\times Y,(y,y)}$ induced by the pullback of φ. In fact, let

$$\mu_{X,x} : \mathcal{O}_{X,x}\hat{\underset{\mathbb{C}}{\otimes}}\mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X\times X,(x,x)}$$

be the $\mathbb{C}$-algebra morphism such that $\mu_{X,x}(f\hat{\underset{\mathbb{C}}{\otimes}}g)$ is the germ of the map

$$\begin{array}{ccc} U \times U & \to & \mathbb{C} \\ (u, v) & \mapsto & f(u).g(v) \end{array}$$

and let

$$\mu_{Y,y} : \mathcal{O}_{Y,y}\hat{\underset{\mathbb{C}}{\otimes}}\mathcal{O}_{Y,y} \longrightarrow \mathcal{O}_{Y\times Y,(y,y)}$$

the same for (Y, y).

Since $\varphi^* : \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is a ring morphism then we have a natural $\mathbb{C}$-algebra morphism

$$\varphi^{\hat{\otimes}} : \mathcal{O}_{X,x}\hat{\underset{\mathbb{C}}{\otimes}}\mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{Y,y}\hat{\underset{\mathbb{C}}{\otimes}}\mathcal{O}_{Y,y}$$

such that $\varphi^{\hat{\otimes}}(f\hat{\underset{\mathbb{C}}{\otimes}}g) = (\varphi^*(f))\hat{\underset{\mathbb{C}}{\otimes}}(\varphi^*(g))$, $\forall f, g \in \mathcal{O}_{X,x}$. Indeed, the map

$$\begin{array}{ccc} \mathcal{O}_{X,x} \times \mathcal{O}_{X,x} & \longrightarrow & \mathcal{O}_{Y,y}\hat{\underset{\mathbb{C}}{\otimes}}\mathcal{O}_{Y,y} \\ (f, g) & \longmapsto & (\varphi^*(f))\hat{\underset{\mathbb{C}}{\otimes}}(\varphi^*(g)) \end{array}$$

is $\mathbb{C}$-bilinear. So, the existence and uniqueness of $\varphi^{\hat{\otimes}}$ are provided by the universal property of the tensor product. It is known that $\mu_{X,x}$ and $\mu_{Y,y}$ are

$\mathbb{C}$-algebra isomorphisms, so we can consider the $\mathbb{C}$-algebra morphism $\epsilon_\varphi : \mathcal{O}_{X\times X,(x,x)} \to \mathcal{O}_{Y\times Y,(y,y)}$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{O}_{X,x}\hat{\underset{\mathbb{C}}{\otimes}}\mathcal{O}_{X,x} & \xrightarrow{\mu_{X,x}} & \mathcal{O}_{X\times X,(x,x)} \\ \big\downarrow{\scriptstyle \varphi^{\hat{\otimes}}} & & \big\downarrow{\scriptstyle \epsilon_\varphi} \\ \mathcal{O}_{Y,y}\hat{\underset{\mathbb{C}}{\otimes}}\mathcal{O}_{Y,y} & \xrightarrow{\mu_{Y,y}} & \mathcal{O}_{Y\times Y,(y,y)} \end{array}$$

Since $\mu_{X,x}$ and $\mu_{Y,y}$ are $\mathbb{C}$-algebra isomorphisms then we can identify $\epsilon_\varphi \cong \varphi^{\hat{\otimes}}$, and $\varphi^{\hat{\otimes}} : \mathcal{O}_{X\times X,(x,x)} \to \mathcal{O}_{Y\times Y,(y,y)}$ induces in $\mathcal{O}_{Y\times Y,(y,y)}$ an $\mathcal{O}_{X\times X,(x,x)}$-algebra structure.

Lemma 4.1. *Let $\alpha \in \mathcal{O}_{X\times X,(x,x)}$. Suppose that U is an open subset of X containing x where a representative of α is defined on $U \times U$. For each $w \in U$ let $\alpha^w \in \mathcal{O}_{X,x}$ be the germ of the map*

$$\begin{array}{rccc} \alpha^w : & U & \to & \mathbb{C} \\ & z & \mapsto & \alpha(z,w) \end{array}$$

For each $y_2 \in \varphi^{-1}(U)$ let $(\varphi^{\hat{\otimes}}(\alpha))^{y_2} \in \mathcal{O}_{Y,y}$ be the germ of the map

$$\begin{array}{rccc} (\varphi^{\hat{\otimes}}(\alpha))^{y_2} : & \varphi^{-1}(U) & \to & \mathbb{C} \\ & y_1 & \mapsto & (\varphi^{\hat{\otimes}}(\alpha))(y_1,y_2) \end{array}$$

Then

$$\varphi^*(\alpha^{\varphi(y_2)}) = (\varphi^{\hat{\otimes}}(\alpha))^{y_2}, \forall y_2 \in \varphi^{-1}(U).$$

Proof. We can write $\alpha = \sum(f_i \hat{\underset{\mathbb{C}}{\otimes}} g_i)$, with $f_i, g_i \in \mathcal{O}_{X,x}$. For all $y_1 \in \varphi^{-1}(U)$ we have:

$\varphi^*(\alpha^{\varphi(y_2)})(y_1) = \alpha^{\varphi(y_2)}(\varphi(y_1)) = \alpha(\varphi(y_1),\varphi(y_2)) = \sum(f_i(\varphi(y_1))\hat{\underset{\mathbb{C}}{\otimes}} g_i(\varphi(y_2))) = \left(\sum(\varphi^*(f_i))\hat{\underset{\mathbb{C}}{\otimes}}(\varphi^*(g_i))\right)(y_1,y_2) = (\varphi^{\hat{\otimes}}(\alpha))(y_1,y_2) = (\varphi^{\hat{\otimes}}(\alpha))^{y_2}(y_1)$, and the lemma is proved. □

We get the analogous result if we fix the first coordinate instead the second one.

Consider the projections $\pi_1^X, \pi_2^X : X \times X \to X$ and $\pi_1^Y, \pi_2^Y : Y \times Y \to Y$.

Theorem 4.2. *Let $M \subseteq \mathcal{O}^p_{X,x}$ and $N \subseteq \mathcal{O}^q_{Y,y}$ be submodules. If $\phi : M \to N$ is an $\mathcal{O}_{X,x}$-module morphism then there exists a unique $\mathcal{O}_{X\times X,(x,x)}$-module morphism $\phi_{D,\varphi} = \phi_D : M_D \to N_D$ such that*

$$\phi_D(h_D) = (\phi(h))_D, \forall h \in M.$$

The map $\phi_{D,\varphi} = \phi_D$ is called the double of ϕ relative to $\varphi : (Y, y) \to (X, x)$.

Proof. Since M_D is generated by $\{h_D \mid h \in M\}$ then we can define $\phi_D : M_D \to N_D$ in a natural way: for each $u = \sum\limits_i \alpha_i(h_i)_D$ with $\alpha_i \in \mathcal{O}_{X\times X,(x,x)}$ and $h_i \in M$ we define

$$\phi_D(u) := \sum_i \alpha_i(\phi(h_i))_D = \sum_i \varphi^{\hat{\otimes}}(\alpha_i)(\phi(h_i))_D$$

which belongs to N_D.

Claim: ϕ_D is well defined. In fact, suppose that $\sum\limits_i \alpha_i(h_i)_D = \sum\limits_j \beta_j(g_j)_D$, with $\alpha_i, \beta_j \in \mathcal{O}_{X\times X,(x,x)}$ and $h_i, g_j \in M$. So, we get two equations:

$$\sum_i \alpha_i(h_i \circ \pi_1^X) = \sum_j \beta_j(g_j \circ \pi_1^X) \tag{1}$$

$$\sum_i \alpha_i(h_i \circ \pi_2^X) = \sum_j \beta_j(g_j \circ \pi_2^X). \tag{2}$$

Take U an open neighborhood of x in X where α_i, β_j are defined on $U \times U$, and h_i, g_j are defined on U. For each $w \in U$ define $\alpha_i^w, \beta_j^w \in \mathcal{O}_{X,x}$ given by the germs of the maps

$$\begin{array}{rccc} \alpha_i^w : & U & \longrightarrow & \mathbb{C} \\ & z & \longmapsto & \alpha_i(z,w) \end{array} \qquad \begin{array}{rccc} \beta_j^w : & U & \longrightarrow & \mathbb{C} \\ & z & \longrightarrow & \beta_j(z,w) \end{array}$$

The equation (1) implies that $\sum_i \alpha_i^w h_i = \sum_j \beta_j^w g_j$, $\forall w \in U$. Applying ϕ (which is a $\mathcal{O}_{X,x}$-morphism) in both sides of the last equation we get

$$\sum_i \alpha_i^w \phi(h_i) = \sum_j \beta_j^w \phi(g_j), \forall w \in U.$$

By the $\mathcal{O}_{X,x}$-module structure on N induced by φ^*, the last equation boils down to

$$\sum_i \varphi^*(\alpha_i^w)\phi(h_i) = \sum_j \varphi^*(\beta_j^w)\phi(g_j), \forall w \in U.$$

By Lemma 4.1 we conclude that

$$\sum_i (\varphi^{\hat{\otimes}}(\alpha_i))^{y_2}\phi(h_i) = \sum_j (\varphi^{\hat{\otimes}}(\beta_j))^{y_2}\phi(g_j), \forall y_2 \in \varphi^{-1}(U).$$

Hence,

$$\sum_i \varphi^{\hat{\otimes}}(\alpha_i)(\phi(h_i) \circ \pi_1^Y) = \sum_j \varphi^{\hat{\otimes}}(\beta_j)(\phi(g_j) \circ \pi_1^Y).$$

Working with the analogous result of the Lemma 4.1, the equation (2) implies that

$$\sum_i \varphi^{\hat{\otimes}}(\alpha_i)(\phi(h_i) \circ \pi_2^Y) = \sum_j \varphi^{\hat{\otimes}}(\beta_j)(\phi(g_j) \circ \pi_2^Y).$$

Therefore,

$$\sum_i \varphi^{\hat{\otimes}}(\alpha_i)(\phi(h_i))_D = \sum_j \varphi^{\hat{\otimes}}(\beta_j)(\phi(g_j))_D$$

and ϕ_D is well-defined.

Now, by the definition of ϕ_D, it is clear that ϕ_D is an $\mathcal{O}_{X\times X,(x,x)}$-module morphism and is the unique satisfying the property $\phi_D(h_D) = (\phi(h))_D$, $\forall h \in M$, i.e,

$$\phi_D(h \circ \pi_1^X, h \circ \pi_2^X) = (\phi(h) \circ \pi_1^Y, \phi(h) \circ \pi_2^Y).$$

$\square$

Notice that this approach generalizes what we have defined in Section 1, taking $\varphi : (X,x) \to (X,x)$ as the identity map. The main motivation of this approach is the fact that when we work with integral closure of modules, the analytic curves $\varphi : (\mathbb{C},0) \to (X,x)$ have a key role.

The Propositions 1.5, 1.6 (a,c), and the Corollary 1.7 (a,b,d) still hold for the double morphism relative to an analytic map.

We can write the Proposition 1.6 (b) on this new language as follows:

Proposition 4.3. *Let $\varphi : (Y,y) \to (X,x)$ and $\varphi' : (Z,z) \to (Y,y)$ be analytic map germs, $M \subseteq \mathcal{O}^p_{X,x}$, $N \subseteq \mathcal{O}^q_{Y,y}$ and $P \subseteq \mathcal{O}^r_{Z,z}$ submodules. Let $\phi : M \to N$ be an $\mathcal{O}_{X,x}$-module morphism and $\phi' : N \to P$ be an $\mathcal{O}_{Y,y}$-module morphism. Then, $\phi' \circ \phi : M \to P$ is an $\mathcal{O}_{X,x}$-module morphism, considering P with the $\mathcal{O}_{X,x}$-module structure induced by the pullback of $\varphi \circ \varphi' : (Z,z) \to (X,x)$. Furthermore,*

$$(\phi' \circ \phi)_{D,\varphi\circ\varphi'} = \phi'_{D,\varphi'} \circ \phi_{D,\varphi}.$$

Proof. For all $\alpha \in \mathcal{O}_{X,x}$ and $h \in M$, working with the module structures induced by the pullbacks of the analytic map germs, we have:

$\phi' \circ \phi(\alpha h) = \phi'(\alpha\phi(h)) = \phi'(\varphi^*(\alpha)\phi(h)) = \varphi^*(\alpha)\phi'(\phi(h)) = \varphi'^*(\varphi^*(\alpha))(\phi' \circ \phi(h)) = (\varphi \circ \varphi')^*(\alpha)(\phi' \circ \phi(h)) = \alpha(\phi' \circ \phi(h))$. So $\phi' \circ \phi : M \to P$ is an $\mathcal{O}_{X,x}$-module morphism and $(\phi' \circ \phi)_{D,\varphi\circ\varphi'}$ is well defined and clearly is equal to $\phi'_{D,\varphi'} \circ \phi_{D,\varphi}$. □

Acknowledgements

The author is grateful to Terence Gaffney for the inspiration and support for this work and to Nivaldo Grulha for his careful reading and valuable suggestions. Furthermore, the author would like to thank the reviewer for taking the time and effort necessary to review the manuscript, which helped to improve the quality of the manuscript.

The author was supported by Fundação de Amparo à Pesquisa do Estado de São Paulo - FAPESP, Brazil, grant 2013/22411-2.

References

[1] T. da Silva, N. Grulha and M. Pereira, *The Bi-Lipschitz Equisingularity of Essentially Isolated Determinantal Singularities*, Bulletin of the Brazilian Mathematical Society, New Series, v. 49, p. 637-645, (2018).

[2] T. da Silva, N. Grulha and M. Pereira, *Real and complex integral closure, Lipschitz equisingularity and applications on square matrices*, Journal of Singularities, v. 22, p. 215-226 (2020).

[3] T. Gaffney and T. da Silva, *Infinitesimal Lipschitz conditions on family of analytic varieties* , arXiv:1902.03194 [math.AG], (2019).

[4] T. Gaffney and T. da Silva, *The Lipschitz Saturation of a Module*, arXiv:2012.12239 [math.AG], (2020).

[5] T. Gaffney, *The genericity of the infinitesimal Lipschitz condition for hypersurfaces*, J. Singul. 10, (2014) 108-123.

[6] T. Gaffney, *Bi-Lipschitz equivalence, integral closure and invariants,* Proceedings of the 10th International Workshop on Real and Complex Singularities. Edited by: M. Manoel, Universidade de São Paulo, M. C. Romero Fuster, Universitat de Valencia, Spain, C. T. C. Wall, University of Liverpool, London Mathematical Society Lecture Note Series (No.380) November (2010).

[7] A. Fernandes and M. A. S. Ruas, *Bilipschitz determinacy of quasihomogeneous germs*, Glasgow Mathematical Journal, 46 (1), 77-82 (2004).

[8] R. C. Gunning and H. Rossi, *Analytic Functions of Several Complex Variables*, AMS Chelsea Publishing, (1965) vol. 368.

[9] T. Mostowski, *A criterion for Lipschitz equisingularity.* Bull. Polish Acad. Sci. Math. 37 (1989), no. 1-6, (1990) 109-116.

[10] F. Pham and B. Teissier, *Fractions lipschitziennes d'une algèbre analytique complexe et saturation de Zariski*, Centre Math. l'École Polytech., Paris, (1969).

[11] F. Pham, *Fractions lipschitziennes et saturation de Zariski des algèbres analytiques complexes.* Exposé d'un travail fait avec Bernard Teissier. Fractions lipschitziennes d'une algèbre analytique complexe et saturation de Zariski, Centre Math. lÉcole Polytech., Paris, 1969. Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2, pp. 649-654. Gauthier-Villars, Paris, (1971).

[12] J. Rotman, *An Introduction to Homological Algebra*, Springer (2009).

[13] O. Zariski, *General theory of saturation and of saturated local rings. II. Saturated local rings of dimension 1.* Amer. J. Math. 93, (1971) 872-964.