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Quasi-ordinary hypersurfaces obtained from others of smaller dimension

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Abstract. In this paper we consider quasi-ordinary hypersurface in $(\mathbb{C}^r,\underline{0})$ defined by quasi-ordinary hypersurface in $(\mathbb{C}^s,\underline{0})$ with $r\geq s$, we explore the relation between their numerical data and the behavior of their parameterizations under changes of coordinates.

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1 Introduction

An analytic germ of a hypersurface $(\mathcal{X},\underline{0})\subset(\mathbb{C}^{r+1},\underline{0})$ is a *quasi-ordinary hypersurface* if there exists a finite morphism $\varrho:(\mathcal{X},\underline{0})\rightarrow(\mathbb{C}^r,\underline{0})$ such that its discriminant locus is contained in a normal crossing divisor. If $(\mathcal{X},\underline{0})$ is irreducible, then there exist suitable coordinates (depending on ϱ) such that the hypersurface is defined by an equation $f=0$

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with $f \in \mathbb{C}\{\underline{X}\}[X_{r+1}] := \mathbb{C}\{X_1, \ldots, X_r\}[X_{r+1}]$ an irreducible Weierstrass polynomial with discriminant $\Delta_{X_{r+1}}(f) = \underline{X}^{\delta}.u(\underline{X}) := X_1^{\delta_1}.\cdots.X_r^{\delta_r}.u(\underline{X})$ where $u(\underline{X}) \in \mathbb{C}\{\underline{X}\}$, $u(\underline{0}) \neq 0$ and $\delta = (\delta_1, \ldots, \delta_r) \in \mathbb{N}^r$ where $\mathbb{N} := \mathbb{Z}_{\geq 0}$. In this case, we say that f is an irreducible *q.o. Weierstrass polynomial.*

In [13], Zariski presents an alternative method of resolution of surfaces singularities using quasi-ordinary surfaces and, in the Jung method (see [9]) of analysing a germ of surface singularity by embedded resolution of the discriminant of a finite morphism from the germ to a smooth surface, the quasi-ordinary hypersurface singularities arise naturally.

The only quasi-ordinary hypersurface isolated singularities are plane curves and normal surfaces in $\mathbb{C}^3$. Lipman (see [10], Remark 7.3.2) showed that a quasi-ordinary hypersurface $(\mathcal{X}, \underline{0}) \subset (\mathbb{C}^{r+1}, \underline{0})$ is normal if and only if it is analytically equivalent to a germ given by $X_{r+1}^n - \prod_{i=1}^{e} X_i = 0$ where $e \leq r$ is the *equisingular dimension* of $(\mathcal{X}, \underline{0})$, *i.e.*, the finite morphism $\varrho : (\mathcal{X}, \underline{0}) \to (\mathbb{C}^r, \underline{0})$ is an equisingular deformation of an e-dimensional quasi-ordinary hypersurface, but not of a smaller-dimensional germ. If $e = r$, than we say that $(\mathcal{X}, \underline{0})$ has maximal equisingular dimension.

Given a hypersurface $(\mathcal{X}, \underline{0}) \subset (\mathbb{C}^{r+1}, \underline{0})$ it is a non trivial task to verify if $(\mathcal{X}, \underline{0})$ is a quasi-ordinary hypersurface because we must to guarantee the existence of a finite morphism $\varrho : (\mathcal{X}, \underline{0}) \to (\mathbb{C}^r, \underline{0})$ such that its discriminant locus is contained in a normal crossing divisor.

In this work[1] we will consider irreducible quasi-ordinary hypersurface with maximal equisingular dimension (shortly, q.o.h.). We present a particular way to obtain a q.o.h. in $(\mathbb{C}^{r+1}, \underline{0})$ by a q.o.h. in $(\mathbb{C}^{s+1}, \underline{0})$ with $r \geq s$ by avoiding the presentation of the finite morphism whose discriminant locus is contained in a normal crossing divisor. We explore the relation between numerical data of $(\mathbb{C}^{r+1}, \underline{0})$ and $(\mathbb{C}^{s+1}, \underline{0})$, as the generalized characteristic exponents and the set of dominant exponent of Kähler forms (see Section 3). In addition, we analyze the behavior of their parameterizations under changes of coordinates (see Section 4).

[1] This work is based on part of the Ph.D. thesis of the first author under supervision of the second one (see [3]).

2 Notation and preliminaries

In what follows we consider an irreducible quasi-ordinary hypersurface $(\mathcal{X},\underline{0}) \subset (\mathbb{C}^{r+1},\underline{0})$ given by the germ of the set $\left\{P \in \mathbb{C}^{r+1}; f(P)=0\right\}$ where $f \in \mathbb{C}\{\underline{X}\}[X_{r+1}]$ is an irreducible q.o. Weierstrass polynomial of degree $n>1$.

By the Abhyankar-Jung theorem (see [9] and [1]), we can assume that any root q of f (a *quasi-ordinary branch*) belongs to $\mathbb{C}\left\{\underline{X}^{\frac{1}{n}}\right\} := \mathbb{C}\left\{X_1^{\frac{1}{n}},\cdots,X_r^{\frac{1}{n}}\right\}$. Consequently,

$$\Delta_{X_{r+1}}(f) = (-1)^{\frac{n(n-1)}{2}}\prod_{i\neq j}(q_i-q_j) = \underline{X}^{\delta}\cdot u(\underline{X}) \in \mathbb{C}\{\underline{X}\}$$

where $u(\underline{X}) \in \mathbb{C}\{\underline{X}\}$ a unit and q_k, $k=1,\dots,n$ are roots of f. In particular, $q_i-q_j = X_1^{\frac{\lambda_1(i,j)}{n}}\cdot X_2^{\frac{\lambda_2(i,j)}{n}}\cdot\ldots\cdot X_r^{\frac{\lambda_r(i,j)}{n}}\cdot u_{ij}(\underline{X}) \in \mathbb{C}\left\{\underline{X}^{\frac{1}{n}}\right\}$ with $u_{ij}(\underline{X})$ a unit and $\lambda_l(i,j)\in\mathbb{N}$ for $l=1,\dots,r$.

Let $\lambda_1,\dots,\lambda_g$ be the distinct r-tuples $\lambda_l(i,j)$. Considering $\preceq$ the partial order in $\mathbb{N}^r$ given by the usual order[2] $\leq$ coordinate wise, we can reindex the elements in $\{\lambda_1,\dots,\lambda_g\}$ in such a way that $\lambda_1\prec\ldots\prec\lambda_g$ (Lemma 5.6 of [10]).

The r-tuples $\lambda_1,\dots,\lambda_g$ are called *(generalized) characteristic exponents* of f (or q).

The following result characterizes elements in $\mathbb{C}\left\{\underline{X}^{\frac{1}{n}}\right\}$ that are quasi-ordinary branches.

Lemma 2.1. *A non unit* $q=\sum_\delta c_\delta\underline{X}^\delta \in \mathbb{C}\left\{\underline{X}^{\frac{1}{n}}\right\}$ *is a quasi-ordinary branch if and only if there are* $\{\lambda_1,\dots,\lambda_g\}\subset\mathbb{N}^r$*, such that:*

1. $\lambda_1\prec\lambda_2\prec\cdots\prec\lambda_g$ *with* $c_{\frac{\lambda_i}{n}}\neq 0$.
2. *If* $c_\delta\neq 0$ *then* $n\delta\in n\mathbb{Z}^r+\sum_{\lambda_i\preceq n\delta}\mathbb{Z}\lambda_i$.
3. $\lambda_j\notin Q_{j-1} := n\mathbb{Z}^r+\sum_{\lambda_i\prec\lambda_j}\mathbb{Z}\lambda_i$.

[2] As usual, $\prec$ means $\preceq$ and $\neq$.

Proof. See Proposition 1.3 in [6]. □

To illustrate the above concepts and to motivate the next definition we present the following example.

Example 2.2. Let us consider the irreducible Weierstrass polynomial $h(X,Y,Z) = Z^2 - X(Y-X)^2 \in \mathbb{C}\{X,Y\}[Z]$. Notice that $\Delta_Z(h) = 4X(Y-X)^2$, so h is not a q.o. Weierstrass polynomial. Considering the linear change of coordinates $L(X,Y,Z) = (X, X+Y, Z)$ we get $g = h \circ L = Z^2 - XY^2$ with $\Delta_Z(g) = 4XY^2$, that is, g defines a q.o. hypersurface, $X^{\frac{1}{2}}Y$ is a root of g and $(1,2)$ is the unique generalized characteristic exponent of g.

In addition, taking the linear change of coordinates $T(X,Y,Z) = (Y,X,Z)$ we have that $f = g \circ T = Z^2 - YX^2$ is an irreducible q.o. Weierstrass polynomial adminting root $XY^{\frac{1}{2}}$ and generalized characteristic exponent $(2,1)$.

The previous example shows that q.o. Weierstrass polynomials and generalized characteristic exponents are sensitive by change of coordinates. However, we can always to elect coordinates that keep a quasi-ordinary branch in a special form:

Definition 2.3. A quasi-ordinary branch $q = \sum c_\delta \underline{X}^\delta \in \mathbb{C}\left\{\underline{X}^{\frac{1}{n}}\right\}$ with generalized characteristic exponents $\lambda_1 \prec \ldots \prec \lambda_g$ is *normalized* if:

1. If $c_\delta \neq 0$ then $n\underline{\delta} \succeq \lambda_1$, that is, $q = c_{\frac{\lambda_1}{n}} \underline{X}^{\frac{\lambda_1}{n}} \cdot u(\underline{X})$ with $u(\underline{0}) = 1$.

2. The i-th coordinates λ^i of $\lambda_1, \ldots, \lambda_g$ satisfy $\lambda^i := (\lambda_{1i}, \ldots, \lambda_{gi}) \geq_{lex} \lambda^j := (\lambda_{1j}, \ldots, \lambda_{gj})$ for $1 \leq i < j \leq r$.

3. If $\lambda_1 = (\lambda_{11}, 0, \ldots, 0)$, then $\lambda_{11} > n$.

In [10], Lipman showed that for any irreducible quasi-ordinary hypersurface $(\mathcal{X}, \underline{0})$ there exists a system of coordinates in a such way that $(\mathcal{X}, \underline{0})$ can be defined by a irreducible q.o. Weierstrass polynomial whose roots are normalized quasi-ordinary branches. Moreover, all normalized quasi-ordinary branches associated to $(\mathcal{X},\underline{0})$ have the same generalized characteristic exponents.

The relevance of the generalized characteristic exponents in the theory of q.o.h. is highlighted when we consider the topological equivalence.

Definition 2.4. We say that two quasi-ordinary hypersurface $(\mathcal{X},\underline{0})$ and $(\mathcal{Y},\underline{0})$ in $(\mathbb{C}^{r+1},\underline{0})$ are *topologically equivalent*, if there are a homeomorphism Φ of $(\mathbb{C}^{r+1},\underline{0})$, neighborhoods U and V of origin $\underline{0}\in\mathbb{C}^{r+1}$ such that $\Phi(\mathcal{X}\cap U)=\mathcal{Y}\cap V$. When Φ is an analytic isomorphism, we say that $(\mathcal{X},\underline{0})$ and $(\mathcal{Y},\underline{0})$ are *analytically equivalent.*

Using local ring saturation results, Lipman (see [10]) showed that the sequence $\left(\frac{\lambda_i}{n}\right)_{i=1}^{g}$ obtained from a normalized quasi-ordinary branch associated to $(\mathcal{X},\underline{0})$ determines the topological class of it and Gau (in [6]) proved that the converse is true, that is, the topological class of $(\mathcal{X},\underline{0})$ allows to recover the sequence $\left(\frac{\lambda_i}{n}\right)_{i=1}^{g}$.

Theorem 2.5 (Lipman-Gau, [10] and [6])**.** *The topological class of a q.o.h. is completely characterized by the integers n and λ_i, for $i=1,\ldots,g$.*

Let $q=\sum_\delta b_\delta \underline{X}^\delta\in\mathbb{C}\left\{\underline{X}^{\frac{1}{n}}\right\}$ be a quasi-ordinary branch of an irreducible q.o. Weierstrass polynomial $f\in\mathbb{C}\{\underline{X}\}[X_{r+1}]$ defining $(\mathcal{X},\underline{0})$. Denoting

$$t_i=X_i^{\frac{1}{n}}\ \text{ for }\ 1\le i\le r\quad\text{and}\quad S(\underline{t})=\sum_\gamma c_\gamma \underline{t}^\gamma\in\mathbb{C}\{\underline{t}\}:=\mathbb{C}\{t_1,\ldots,t_r\}$$

where $\gamma:=n\delta\in\mathbb{N}^r$ and $c_\gamma:=b_\delta$ we say that

$$H:=H_f:=(t_1^n,\ldots,t_r^n,S(\underline{t}))$$

is a *quasi-ordinary parameterization* (q.o. parameterization) of f or q.

As f is the minimal polynomial of q, the epimorphism of $\mathbb{C}$-algebras

$$\begin{array}{rccl} H^*: & \mathbb{C}\{\underline{X},X_{r+1}\} & \longrightarrow & \mathbb{C}\{t_1^n,\ldots,t_r^n,S(\underline{t})\}\subset\mathbb{C}\{\underline{t}\}\\ & h(\underline{X},X_{r+1}) & \mapsto & h(t_1^n,\ldots,t_r^n,S(\underline{t}))\end{array}$$

give us $\mathcal{O} := \frac{\mathbb{C}\{\underline{X}, X_{r+1}\}}{\langle f\rangle} \cong \mathbb{C}\{t_1^n, \ldots, t_r^n, S(\underline{t})\}$ where $\mathcal{O}$ is the *analytic algebra* associated to $(\mathcal{X}, 0)$. Recall (see [4], section 8.2) that two q.o.h. are analytically equivalent if and only if their analytic algebras are isomorphic.

Remark 2.6. If two q.o.h. are defined by q.o. Weierstrass polynomials $f_1, f_2 \in \mathbb{C}\{\underline{X}\}[X_{r+1}]$ and we identify such polynomials with map germs from $(\mathbb{C}^{r+1}, \underline{0})$ to $(\mathbb{C}, \underline{0})$, then analytic equivalence is translated to the $\mathcal{K}$-equivalence of f_1 and f_2, that is, $f_2 = u \cdot \Psi(f_1)$ for some automorphism Ψ and some unit u of $\mathbb{C}\{\underline{X}, X_{r+1}\}$.

Notice that an isomorphism of analytic algebras $\mathbb{C}\{t_1^n, \ldots, t_r^n, S(\underline{t})\}$ corresponds to change of parameters and coordinates. In this way, identifying a parameterization with the map germ from $(\mathbb{C}^r, \underline{0})$ to $(\mathbb{C}^{r+1}, \underline{0})$ defined by $H(\underline{t}) = (t_1^n, \ldots, t_r^n, S(\underline{t}))$, we have that two q.o.h. with parameterizations H_1 and H_2 are analytically equivalent if and only if there exist germs of analytic isomorphisms $\sigma \in Iso(\mathbb{C}^{r+1}, \underline{0})$ and $\rho \in Iso(\mathbb{C}^r, \underline{0})$ of $(\mathbb{C}^{r+1}, \underline{0})$ and $(\mathbb{C}^r, \underline{0})$ respectively, such that $H_2 = \sigma \circ H_1 \circ \rho^{-1}$. Considering the group $\mathcal{A} = \{(\sigma, \rho) \in Iso(\mathbb{C}^{r+1}, \underline{0}) \times Iso(\mathbb{C}^r, \underline{0})\}$ and denoting its action on H_1 by $\sigma \circ H_1 \circ \rho^{-1}$, the analytic equivalence of q.o.h. can be translated by the $\mathcal{A}$-equivalence of H_1 and H_2.

Similarly to the case of plane curves, we can associate a discrete semigroup to the analytic algebra that encodes the topological aspects of a quasi-ordinary hypersurface. For this purpose, we introduce the following concept.

We say that $p \in \mathbb{C}\{\underline{t}\}$ has *dominant exponent* $\mathcal{V}(p) := \delta \in \mathbb{N}^r$ if $p = \underline{t}^\delta \cdot v(\underline{t})$ with $v(\underline{0}) \neq 0$. Given $h \in \mathbb{C}\{\underline{X}, X_{r+1}\} \setminus \langle f\rangle$ if $H^*(h)$ has dominant exponent we put $\mathcal{V}_H(h) := \mathcal{V}(H^*(h))$.

Remark 2.7. Let $p = \sum_\gamma c_\gamma \underline{t}^\gamma$ be a non zero element of $\mathbb{C}\{\underline{t}\}$ and consider $Supp(p) := \{\gamma \in \mathbb{N}^r; c_\gamma \neq 0\}$. Denoting $\mathcal{N}(p)$ the Newton polyhedron of p, that is, the convex closure in $\mathbb{R}^r$ of $Supp(p) + \mathbb{R}^r_+$, we have that p has dominant exponent $\mathcal{V}(p)$ if and only if $\mathcal{V}(p)$ is the unique vertex of $\mathcal{N}(p)$.

Given a q.o. parametrization $H = (t_1^n, \ldots, t_r^n, S(\underline{t}))$ we set

$$\Gamma_H = \{\mathcal{V}_H(h) : \ h \in \mathbb{C}\{\underline{X}, X_{r+1}\} \setminus \langle f\rangle \text{ such that } H^*(h) \text{ has dominant exponent}\}.$$

It is immediate that $\Gamma_H \subset \mathbb{N}^r$ is an additive semigroup.

For an irreducible plane curve defined by $f = 0$ with associated parameterizaton $H = (t^n, S(t))$ we have that $\mathcal{V}(H^*(h))$ equals the intersection multiplicity of f and h given by $I(f,h) := \dim_{\mathbb{C}} \frac{\mathbb{C}\{X_1,X_2\}}{\langle f,h\rangle}$ and, as $I(u\cdot\Psi(f), u\cdot\Psi(h)) = I(f,h)$ for any automorphism Ψ and any unit u of $\mathbb{C}\{X_1,X_2\}$, we conclude, by Remark 2.6, that Γ_H is an analytic invariant for plane curves. The same is true for an arbitrary q.o.h., that is, Γ_H is an analytic invariant (see [7] and [12]). Moreover, denoting $n_k = \sharp\frac{Q_k}{Q_{k-1}}$ (see definition of Q_k in Lemma 2.1), for $k = 1,\dots,g$ and

$$\begin{aligned} &\nu_j = n\theta_j, \text{ for } j = 1,\dots,r,\\ &\nu_{r+1} = \lambda_1;\\ &\nu_{r+i} = n_{i-1}\nu_{r+i-1} + \lambda_i - \lambda_{i-1} \text{ for all } i = 2,\dots,g \end{aligned} \tag{2.1}$$

where $\{\theta_j = (0,\dots,0,1,0,\dots,0),\ 1 \le j \le r\}$ is the set of canonical generators of the semigroup $\mathbb{N}^r$ then

$$\Gamma_H = \langle \nu_1,\dots,\nu_{r+g}\rangle := \mathbb{N}\cdot\nu_1 + \cdots + \mathbb{N}\cdot\nu_{r+g}.$$

In particular, if $\mathcal{M}_{r+1}$ denotes the maximal ideal of $\mathbb{C}\{\underline{X}, X_{r+1}\}$ then

$$\begin{aligned} \{\mathcal{V}_H(h) :\ h \in \mathcal{M}_{r+1}\setminus\mathcal{M}_{r+1}^2 \text{ such that } H^*(h) \text{ has dominant exponent}\} =\\ = \{\nu_1,\dots,\nu_{r+1}\}. \end{aligned} \tag{2.2}$$

Given $\gamma \in Q_k$ for some $1 \le k \le g$ there are unique $a_1,\dots,a_{r+k} \in \mathbb{Z}$ with $0 \le a_{r+j} < n_j$ and $j = 1,\dots,k$, such that $\gamma = \sum_{i=1}^{r+k} a_i\nu_i$ that we call the *standard representation* of γ. Moreover, if $\sum_{i=1}^{r+k} a_i\nu_i$ is the standard representation of $\gamma \in Q_k$ then $\gamma \in \Gamma_k := \langle \nu_1,\dots,\nu_{r+k}\rangle$ if and only if $a_i \ge 0$ for every $1 \le i \le r$. In particular, denoting

$$\mathcal{F}_H = \sum_{i=1}^{g}(n_i - 1)\nu_{r+i} - (\underline{n}) \tag{2.3}$$

with $(\underline{n}) := (n,\dots,n) \in \mathbb{N}^r$. The element $\mathcal{F}_H \in \mathbb{N}^r$ satisfies the following property: if $\gamma \succ \mathcal{F}_H$ then $\gamma \in \Gamma_H$. Therefore the element $\mathcal{F}_H$ is called the *Frobenius vector* of Γ_H (see [2]).

By (2.1), the semigroup Γ_H determines and it is determined by the generalized characteristic exponents, consequently Γ_H is also a topological invariant of the q.o.h. with parameterization H.

Remark 2.8. If $H_1 = (t_1^n, \ldots, t_r^n, S_1(\underline{t}))$ is a q.o. parameterization for $(\mathcal{X}_1, \underline{0})$ with $S_1(\underline{t}) = \sum_{\delta \succeq \lambda_1} a_\delta \underline{t}^\delta$ then, by (2.2), for any $\lambda_1 \prec \delta'$ with $\delta' \in Supp(S_1(\underline{t})) \cap \Gamma_{H_1}$ there exists $h \in \mathcal{M}_{r+1}^2$ such that $H_1^*(h) = -a_{\delta'}\underline{t}^{\delta'}.v(\underline{t})$ with $v(\underline{0}) = 1$. Considering the analytic isomorphisms $\rho(\underline{t}) = \underline{t}$ and $\sigma(\underline{X}, X_{r+1}) = (\underline{X}, X_{r+1} + h)$ we get $\sigma \circ H_1 \circ \rho^{-1} = H_2 = (t_1^n, \ldots, t_r^n, S_2(\underline{t}))$ with $S_2(\underline{t}) := \sum_{\delta \succeq \lambda_1} b_\delta \underline{t}^\delta$, that is, we have a q.o.h. $(\mathcal{X}_2, \underline{0})$ analytically equivalent to $(\mathcal{X}_1, \underline{0})$ admiting a parametrization H_2 satisfying $b_\delta = a_\delta$ for all $\delta \notin Supp(H_1^*(h))$ and $b_{\delta'} = 0$. In particular, $Supp(S_2(\underline{t}) - S_1(\underline{t})) \subset \delta' + \mathbb{N}^r$ and $\delta' \notin Supp(S_2)$. We will refer to this property by saying that the term $\underline{t}^{\delta'}$ *can be eliminated* from H_1.

A more general subset of $\mathbb{N}^r$ that contains eliminable terms in a q.o. parameterization H is given in [8]. Such subset is related with elements in a subgroup $\widetilde{\mathcal{A}}$ of $\mathcal{A}$ as defined in the sequel.

Definition 2.9 ([8], Definition 2.3)**.** Fixing a topological class of a q.o.h. in $(\mathbb{C}^{r+1}, \underline{0})$ determined by $\{n, \lambda_1, \ldots, \lambda_g\}$ we denote by $\widetilde{\mathcal{A}}$ the subgroup of $\mathcal{A}$ consisting of all elements $(\sigma, \rho) \in \mathcal{A}$ given by $\rho = (t_1 \cdot u_1, \ldots, t_r \cdot u_r)$ and $\sigma = (\sigma_1, \ldots, \sigma_{r+1})$ such that

$$\sigma_i = a_i \cdot X_i + P_i, \qquad \sigma_{r+1} = X_{r+1} \cdot (a_{r+1} + \epsilon_{r+1}) + \underline{X}^\gamma \cdot \eta_{r+1},$$

where $u_i \in \mathbb{C}\{\underline{t}\}$ are units, $\gamma = \left(\left\lceil \frac{\lambda_{11}}{n} \right\rceil, \ldots, \left\lceil \frac{\lambda_{1r}}{n} \right\rceil\right)$, $a_i, a_{r+1} \in \mathbb{C} \setminus \{0\}$, $P_i = X_i \cdot \epsilon_i + X_{r+1} \cdot \eta_i$, $\epsilon_i, \epsilon_{r+1} \in \mathcal{M}_{r+1}$, $\eta_i, \eta_{r+1} \in \mathbb{C}\{\underline{X}, X_{r+1}\}$ for $i = 1, \ldots, r$ and $\eta_i = 0$ if $\lambda_{1i} < n$.

Notice that the action of $\tilde{\mathcal{A}}$ on a normalized q.o. parameterization provides us a parameterization with the same characteristic exponents but not necessarily a q.o. parameterization.

Proposition 2.10. *Given a q.o. parameterization $H = (t_1^n, \ldots, t_r^n, S(\underline{t}))$ and $(\sigma, \rho) \in \widetilde{\mathcal{A}}$, as Definition 2.9, we have that $\sigma \circ H \circ \rho^{-1}$ is a q.o. parameterization if and only if $u_i = \left(a_i + \frac{P_i(H)}{t_i^n}\right)^{\frac{1}{n}}$.*

Proof. First of all, notice that $\sigma \circ H \circ \rho^{-1}$ is a q.o. parameterization if and only if

$$t_i^n = \sigma_i \circ H \circ \rho^{-1} = a_i \cdot (\rho^{-1})_i^n + P_i(H \circ \rho^{-1}) = \left((\rho^{-1})_i \cdot \left(a_i + \frac{P_i(H \circ \rho^{-1})}{(\rho^{-1})_i^n} \right)^{\frac{1}{n}} \right)^n$$

for $i = 1, \ldots, r$.

As $\rho_i \circ \rho^{-1} = t_i$, we conclude that $\rho_i(\underline{t}) = t_i \cdot \kappa_i \cdot \left(a_i + \frac{P_i(H)}{t_i^n} \right)^{\frac{1}{n}}$ where $\kappa_i^n = 1$ for $i = 1, \ldots, r$.

Without loss of generality, we can consider $\kappa_i = 1$. □

Notice that the subset of elements in $\tilde{\mathcal{A}}$ that preserve q.o. parameterization H is not a subgroup since such elements depend on H. The previous result was presented by Panek in her thesis (see [11]).

The description of a set of eliminable terms in a q.o. parameterization by the $\tilde{\mathcal{A}}$-action can be related with the set of dominant exponents of Kähler r-forms.

Let $f \in \mathbb{C}\{\underline{X}\}[X_{r+1}]$ be an irreducible q.o. Weierstrass polynomial with a q.o. parameterization $H = (t_1^n, \ldots, t_r^n, S(\underline{t}))$. We denote by $\Omega^1_{\mathcal{O}}$ the *Kähler differentials module* of the analytic algebra $\mathcal{O} = \frac{\mathbb{C}\{\underline{X}, X_{r+1}\}}{\langle f \rangle}$, that is, the $\mathcal{O}$-module generated by $\{dX_i;\ i = 1, \ldots, r+1\}$ under the relation $df = \sum_{i=1}^{r+1} f_{X_i} dX_i = 0$. With this notations we indicate the $\mathcal{O}$-module of Kähler k-forms by $\Omega^k_{\mathcal{O}} = \bigwedge_{i=1}^k \Omega^1_{\mathcal{O}}$ where $k \in \{1, \ldots, r+1\}$ and by dx_i the image of dX_i in $\Omega^1_{\mathcal{O}}$. Notice that an element $\omega \in \Omega^k_{\mathcal{O}}$ can be expressed by $\omega = \sum_{|I|=k} \overline{h_I} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, where $I = \{i_1, \ldots, i_k\}$ ranges all increasing subsets of k elements from $\{1, \ldots, r+1\}$ and $\overline{h_I}$ denotes the class of $h_I \in \mathbb{C}\{\underline{X}, X_{r+1}\}$ in $\mathcal{O}$.

Considering the isomorphism $\mathcal{O} \cong \mathbb{C}\{t_1^n, \ldots, t_r^n, S(\underline{t})\} \subset \mathbb{C}\{\underline{t}\}$ and the $\mathbb{C}\{\underline{t}\}$-module $\Omega^k_{\mathbb{C}\{\underline{t}\}}$ of differentials k-forms of $\mathbb{C}\{\underline{t}\}$, we get the $\mathcal{O}$-homomorphism

$$\begin{array}{rccl} \Psi^k_H : & \Omega^k_{\mathcal{O}} & \rightarrow & \Omega^k_{\mathbb{C}\{\underline{t}\}} \\ & \omega & \mapsto & \sum_{|I|=k} H^*(h_I) dH^*(X_{i_1}) \wedge \cdots \wedge dH^*(X_{i_k}), \end{array}$$

where $\omega = \sum_{|I|=k} \overline{h_I} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$.

Similarly to the semigroup Γ_H we define

$$\Lambda_H = \left\{ \mathcal{V}_H\left(\frac{\Psi^r_H(\omega)}{dt_1\wedge\cdots\wedge dt_r}\right) + (\underline{1});\ \frac{\Psi^r_H(\omega)}{dt_1\wedge\cdots\wedge dt_r} \text{ has dominant exponent} \right\},$$

where $(\underline{1}) = (1,\ldots,1) \in \mathbb{N}^r$.

By Proposition 2.15 and Theorem 3.1 in [8], we can relate elements in Λ_H with tangent vectors to the $\tilde{\mathcal{A}}$-orbit of H and, applying the Complete Transversal Theorem (see [5]), we can identify eliminable terms in H. More precisely, if $\delta = \mathcal{V}_H\left(\frac{\Psi^r_H(\omega)}{dt_1\wedge\cdots\wedge dt_r}\right) + (\underline{1}) \in \Lambda_H$ where $\omega = \sum_{i=1}^{r+1}(-1)^{r+1-i}\overline{h_i}dx_1\wedge\ldots\wedge\widehat{dx_i}\wedge\ldots\wedge dx_{r+1}$ is such that $h_i = \sigma_i - a_i\cdot X_i \in \mathbb{C}\{\underline{X}, X_{r+1}\}$ with σ_i and a_i described in Definition 2.9, then $\underline{t}^{\delta-(\underline{n})}$ is eliminable from H.

Notice that Λ_H is a Γ_H-monomodule, that is, $\Gamma_H + \Lambda_H \subset \Lambda_H$ and, as $\mathcal{V}_H\left(\frac{\Psi^r_H(dx_1\wedge\cdots\wedge\widehat{dx_i}\wedge\cdots\wedge dx_{r+1})}{dt_1\wedge\cdots\wedge dt_r}\right) = \sum_{\substack{j=1\\ j\neq i}}^{r+1}\nu_j$, we get

$$\bigcup_{i=1}^{r+1}\left(\Gamma_H + \sum_{\substack{j=1\\ j\neq i}}^{r+1}\nu_j\right) \subset \Lambda_H.$$

In particular, given a q.o. parameterization $H = (t_1^n,\ldots,t_r^n, S(\underline{t}))$ and $\lambda_1 \prec \delta \in Supp(S(\underline{t}))\cap\Gamma_H$, there exists $h = X_{r+1}\cdot\epsilon_{r+1} + \underline{X}^\gamma\cdot\eta_{r+1} \in \mathcal{M}^2_{r+1}$, with $\epsilon_{r+1}, \eta_{r+1}$ and γ as described in the Definition 2.9, such that $\delta + (\underline{n}) = \mathcal{V}_H(h) + (\underline{n}) = \mathcal{V}_H\left(\frac{\Psi^r_H(\overline{h}dx_1\wedge\cdots\wedge dx_r)}{dt_1\wedge\cdots\wedge dt_r}\right) \in \Lambda_H$, that is, δ can be eliminable as mentioned in the Remark 2.8.

3 Q.O.H. defined by others of smaller dimension

As mentioned in the introduction, it is not easy to verify if a hypersurface $(\mathcal{X},\underline{0}) \subset (\mathbb{C}^{r+1},\underline{0})$ is a quase ordinary hypersurface. In addition, as we illustrate in the Example 2.2, q.o. Weierstrass polynomials are sensitive by change of coordinates. In this section, given a q.o.h. in $(\mathbb{C}^{s+1},\underline{0})$ we define a q.o.h. in $(\mathbb{C}^{r+1},\underline{0})$ with $r > s$ and we explore relations among the semigroups and the set Λ associated to them.

Fixing $s, r \in \mathbb{N}$ with $0 < s < r$, we consider an *ordered partition* of $\{1, 2, \dots, r\}$ given by

$$P = \left\{ \begin{array}{c} \{\alpha_0 + 1, \alpha_0 + 2, \dots, \alpha_1\}, \{\alpha_1 + 1, \alpha_1 + 2, \dots, \alpha_2\}, \dots, \\ \dots, \{\alpha_{s-1} + 1, \alpha_{s-1} + 2, \dots, \alpha_s\} \end{array} \right\}, \quad (3.1)$$

where $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_s = r$ and we define the monomorphism of $\mathbb{C}$-algebras

$$\begin{array}{rccl} T_P: & \mathbb{C}\{u_1, \dots, u_s\} & \longrightarrow & \mathbb{C}\{t_1, \dots, t_r\} \\ & u_i & \mapsto & \prod_{\alpha_{i-1} < j \leq \alpha_i} t_j \quad 1 \leq i \leq s. \end{array} \quad (3.2)$$

Given $(\mathcal{X}_s, \underline{0}) \subset (\mathbb{C}^{s+1}, \underline{0})$ a q.o.h. with parameterization

$$H_s = (u_1^n, \dots, u_s^n, S(u_1, \dots, u_s)),$$

we define

$$H_r = (t_1^n, \dots, t_r^n, T_P(S(u_1, \dots, u_s)))$$

and we indicate $H_s \underset{P}{\rightsquigarrow} H_r$.

By construction, if $\delta = (\delta_1, \dots, \delta_s) \in Supp(S(u_1, \dots, u_s))$, then $\delta_P := (\delta_1, \dots, \delta_1, \delta_2, \dots, \delta_2, \dots, \delta_s, \dots, \delta_s) \in Supp(T_P(S(u_1, \dots, u_s)))$ where δ_i appears $\alpha_i - \alpha_{i-1}$ times in δ_P for each $i = 1, \dots, s$.

Notice that $\delta_P = \delta \cdot M_P$ where

$$M_P := (m_{ij})_{\substack{1 \leq i \leq s \\ 1 \leq j \leq r}} = \begin{pmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 1 & \cdots & 1 \end{pmatrix}$$

and $m_{ij} = \begin{cases} 1, \text{ if } \alpha_{i-1} < j \leq \alpha_i \\ 0, \text{ otherwise.} \end{cases}$

Remark 3.1. We have that $\delta \cdot M_P = \gamma \cdot M_P$ if and only if $\delta = \gamma \in \mathbb{N}^s$.

Proposition 3.2. *Given a q.o.h.* $(\mathcal{X}_s, \underline{0}) \subset (\mathbb{C}^{s+1}, \underline{0})$ *with parameterization* $H_s = (u_1^n, \dots, u_s^n, S(u_1, \dots, u_s))$ *and an ordered partition* P *of*

$\{1,\ldots,r\}$ as (3.1) then H_r such that $H_s \underset{P}{\rightsquigarrow} H_r$ is a q.o. parameterization, that is, there exists a q.o.h. $(\mathcal{X}_r,\underline{0}) \subset (\mathbb{C}^{r+1},\underline{0})$ that admits H_r as a parameterization.

Proof. Since H_s is a q.o. parameterization it corresponds to a quasi-ordinary branch q satisfying the Lemma 2.1. Let $(\lambda_i)_{i=1}^g$ be the generalized characteristic exponents of q.

(i) As $\lambda_i \in Supp(S(u_1,\ldots,u_s))$ and $\lambda_i \prec \lambda_j$ it follows immediately that $\lambda_i \cdot M_P \in Supp(T_P(S(u_1,\ldots,u_s)))$ and $\lambda_i \cdot M_P \prec \lambda_j \cdot M_P$ for $i<j$.

(ii) If $\delta \in Supp(S(u_1,\ldots,u_s))$, then $\delta \in n\mathbb{Z}^s + \sum_{\lambda_i \preceq \delta} \mathbb{Z}\lambda_i$. In this way, $\delta \cdot M_P \in Supp(T_P(S(u_1,\ldots,u_s)))$ and

$$\delta \cdot M_P \in \left(n\mathbb{Z}^s + \sum_{\lambda_i \preceq \delta} \mathbb{Z}\lambda_i \right) \cdot M_P \subset n\mathbb{Z}^r + \sum_{\lambda_i \cdot M_P \preceq \delta \cdot M_P} \mathbb{Z}\lambda_i \cdot M_P.$$

(iii) Since $\lambda_i \notin n\mathbb{Z}^s + \sum_{\lambda_j \prec \lambda_i} \mathbb{Z}\lambda_j$ we get

$$\lambda_i \cdot M_P \notin n\mathbb{Z}^r + \sum_{\lambda_j \cdot M_P \prec \lambda_i \cdot M_P} \mathbb{Z}\lambda_j \cdot M_P.$$

By Lemma 2.1, H_r is a q.o. parametrization of a q.o.h. $(\mathcal{X}_r,\underline{0})$ in $(\mathbb{C}^{r+1},\underline{0})$. □

If $(\lambda_i)_{i=1}^g$ are the generalized characteristic exponent of a q.o.h. $(\mathcal{X}_s,\underline{0})$ with parameterization H_s then, by the above proposition, $(\lambda_i \cdot M_P)_{i=1}^g$ are the generalized characteristic exponent of $(\mathcal{X}_r,\underline{0})$ with parameterization H_r where $H_s \underset{P}{\rightsquigarrow} H_r$. In this way, as an immediate consequence we obtain the following corollary.

Corollary 3.3. *Given q.o.h. $(\mathcal{X}_s,\underline{0}),(\mathcal{X}'_s,\underline{0})$ in $(\mathbb{C}^{s+1},\underline{0})$ and $(\mathcal{X}_r,\underline{0}),(\mathcal{X}'_r,\underline{0})$ in $(\mathbb{C}^{r+1},\underline{0})$ with parameterizations H_s, H'_s, H_r and H'_r respectively, where $H_s \underset{P}{\rightsquigarrow} H_r$ and $H'_s \underset{P}{\rightsquigarrow} H'_r$ for some ordered partition P of $\{1,\ldots,r\}$. The q.o.h. $(\mathcal{X}_s,\underline{0})$ and $(\mathcal{X}'_s,\underline{0})$ are topologically equivalent if and only if $(\mathcal{X}_r,\underline{0})$ and $(\mathcal{X}'_r,\underline{0})$ are topologically equivalent.*

Given a q.o.h. $(\mathcal{X}_s, \underline{0}) \subset (\mathbb{C}^{s+1}, \underline{0})$ admitting a parameterization $H_s = (u_1^n, \ldots, u_s^n, S(\underline{u}))$ and fixing an ordered partition P of $\{1, \ldots, r\}$ as (3.1) we define the $\mathbb{C}$-algebra monomorphism

$$\begin{array}{rccl} T^P: & \mathbb{C}\{Y_1, \ldots, Y_{s+1}\} & \to & \mathbb{C}\{X_1, \ldots, X_{r+1}\} \\ & Y_i & \mapsto & \prod_{\alpha_{i-1} < j \leq \alpha_i} X_j \text{ for } 1 \leq i \leq s \\ & Y_{s+1} & \mapsto & X_{r+1} \end{array} \tag{3.3}$$

and, considering the restriction of T_P on $\mathbb{C}\{u_1^n, \ldots, u_s^n, S(\underline{u})\}$, we get the following commutative diagram of $\mathbb{C}$-algebras

$$\begin{array}{ccc} \mathbb{C}\{Y_1, \ldots, Y_{s+1}\} & \overset{H_s^*}{\to} & \mathbb{C}\{u_1^n, \ldots, u_s^n, S(\underline{u})\} \\ \downarrow T^P & & \downarrow T_P \\ \mathbb{C}\{X_1, \ldots, X_{r+1}\} & \overset{H_r^*}{\to} & \mathbb{C}\{t_1^n, \ldots, t_r^n, T_P(S(\underline{u}))\}. \end{array} \tag{3.4}$$

Consequently, if $(\mathcal{X}_s, \underline{0})$ is defined by a q.o. Weierstrass polynomial $f \in \mathbb{C}\{Y_1, \ldots, Y_s\}[Y_{s+1}]$ then the q.o.h. $(\mathcal{X}_r, \underline{0}) \subset (\mathbb{C}^{r+1}, \underline{0})$ that admits the parameterization H_r with $H_s \underset{P}{\rightsquigarrow} H_r$ is defined by the q.o. Weierstrass polynomial $T^P(f) \in \mathbb{C}\{\underline{X}\}[X_{r+1}]$.

As $T_P(\mathbb{C}\{u_1^n, \ldots, u_s^n, S(\underline{u})\}) \subset \mathbb{C}\{t_1^n, \ldots, t_r^n, T_P(S(\underline{u}))\}$, if the element $h \in \mathbb{C}\{\underline{Y}, Y_{s+1}\}$ admits dominant exponent $\mathcal{V}_{H_s}(h)$ then $\mathcal{V}_{H_r}(T^P(h)) = \mathcal{V}_{H_s}(h) \cdot M_P$.

Let $Q_i^s := n\mathbb{Z}^s + \sum_{\lambda_j \prec \lambda_i} \mathbb{Z}\lambda_j$ and $Q_i^r := n\mathbb{Z}^s + \sum_{\lambda_j \prec \lambda_i} \mathbb{Z}\lambda_j \cdot M_P$, with $0 \leq i \leq g$ be the corresponding groups associated to the quasi-ordinary branches determined by H_s and H_r as described in Lemma 2.1. Notice that

$$Q_0^s \cdot M_P = n\mathbb{Z}^s \cdot M_P \subset n\mathbb{Z}^r = Q_0^r \quad \text{and}$$

$$Q_i^s \cdot M_P = Q_{i-1}^s \cdot M_P + \mathbb{Z}\lambda_i \cdot M_P \subset Q_{i-1}^r + \mathbb{Z}\lambda_i \cdot M_P = Q_i^r \ \text{ for } 1 \leq i \leq g.$$

As $\left(\sharp \frac{Q_i^s}{Q_{i-1}^s}\right) \lambda_i \in Q_{i-1}^s$, we get $\left(\sharp \frac{Q_i^s}{Q_{i-1}^s}\right) \lambda_i \cdot M_P \in Q_{i-1}^s \cdot M_P \subset Q_{i-1}^r$ and consequently, $\sharp \frac{Q_i^r}{Q_{i-1}^r} \leq \sharp \frac{Q_i^s}{Q_{i-1}^s}$. On the other hand, we have

$$\left(\sharp \frac{Q_i^r}{Q_{i-1}^r}\right) \lambda_i \cdot M_P = (d_{i-1}\lambda_{i-1} + \cdots + d_1\lambda_1) \cdot M_P + nd' \in Q_{i-1}^r$$

with $d_j \in \mathbb{Z}$ for $1 \leq j < i$ and $d' \in \mathbb{Z}^r$. In this way, $d' = d \cdot M_P$ for some $d \in \mathbb{Z}^s$ and, by the Remark 3.1, we have

$$\left(\sharp \frac{Q_i^r}{Q_{i-1}^r}\right) \lambda_i = d_{i-1}\lambda_{i-1} + \cdots + d_1\lambda_1 + nd \in Q_{i-1}^s,$$

that is, $\sharp \frac{Q_i^r}{Q_{i-1}^r} \geq \sharp \frac{Q_i^s}{Q_{i-1}^s}$. Hence, we must have

$$\sharp \frac{Q_i^r}{Q_{i-1}^r} = \sharp \frac{Q_i^s}{Q_{i-1}^s}. \tag{3.5}$$

We denote the former number by n_i for all $i = 1, \ldots, g$.

The above explanation allow us to relate the semigroup of H_s and H_r for $H_s \underset{P}{\rightsquigarrow} H_r$.

Theorem 3.4. *Let $(\mathcal{X}_s, \underline{0}) \subset (\mathbb{C}^{s+1}, \underline{0})$ be a q.o.h. with parameterization H_s. If $(\mathcal{X}_r, \underline{0}) \subset (\mathbb{C}^{r+1}, \underline{0})$ is the q.o.h. admitting parameterization H_r and $H_s \underset{P}{\rightsquigarrow} H_r$ for some ordered partition P of $\{1, \ldots, r\}$ as (3.1), then*

$$\Gamma_{H_r} = n\mathbb{N}^r + \Gamma_{H_s} \cdot M_P \quad \textit{and} \quad \mathcal{F}_{H_r} = \mathcal{F}_{H_s} \cdot M_P.$$

Proof. If $(\lambda_i)_{i=1}^g$ are the generalized characteristic exponent of $(\mathcal{X}_s, \underline{0})$ then, by the Proposition 3.2, $(\lambda_i \cdot M_P)_{i=1}^g$ are the generalized characteristic exponent of $(\mathcal{X}_r, \underline{0})$. Consequently, if $\Gamma_{H_s} = \langle \nu_1, \ldots, \nu_{r+g} \rangle$, by (2.1) and (3.5), we get

$$\Gamma_{H_r} = n\mathbb{N}^r + \sum_{j=1}^{r}(\nu_{r+j} \cdot M_P) = n\mathbb{N}^r + \left(n\mathbb{N}^s + \sum_{j=1}^{r} \nu_{r+j}\right) \cdot M_P = n\mathbb{N}^r + \Gamma_{H_s} \cdot M_P.$$

In addition, by (2.3), we obtain

$$\mathcal{F}_{H_s} \cdot M_P = \sum_{i=1}^{g}(n_i - 1)\nu_i \cdot M_P + (n, \ldots, n) \cdot M_P = \mathcal{F}_{H_r},$$

here $(n, \ldots, n) \in \mathbb{N}^s$. □

According to the above theorem, if $\delta \in \Gamma_{H_s}$ then $\delta \cdot M_P \in \Gamma_{H_r}$ for any ordered partition P of $\{1, \ldots, r\}$. On the other hand, it is clear that

$\delta \in \Gamma_{H_r}$ does not imply that $\delta \in \Gamma_{H_s} \cdot M_P$. In fact, considering an ordered partition P as (3.1) with $\alpha_1 > 1$, we have $(n, 0, \cdots, 0) \in \Gamma_{H_r}$ but $n\mathbb{N}^r \notin \Gamma_{H_s} \cdot M_P$ for any such partition. However, we have the following result.

Proposition 3.5. *If $\delta \in \mathbb{N}^s$ is such that $\delta \cdot M_P \in \Gamma_{H_r}$ then $\delta \in \Gamma_{H_s}$.*

Proof. Denoting $\Gamma_{H_s} = \langle \nu_1, \ldots, \nu_{r+g} \rangle$ if $\gamma = \delta \cdot M_P \in \Gamma_{H_r}$ we can consider its standard representation $\gamma = n \cdot (a_1, \ldots, a_r) + \sum_{j=1}^{g} b_j \nu_{r+j} \cdot M_P$ with $(a_1, \ldots, a_r) \in \mathbb{N}^r$ and $0 \leq b_j < n_j = \sharp \frac{Q_j^r}{Q_{j-1}^r}$ for all $j = 1, \ldots, g$.

As $\gamma_i = n a_i + \sum_{j=1}^{g} b_j (\nu_{r+j} \cdot M_P)_i$ and $\gamma_{\alpha_{k-1}+1} = \ldots = \gamma_{\alpha_k}$ for all $1 \leq k \leq s$ we get $a_{\alpha_{k-1}+1} = \ldots = a_{\alpha_k}$ for $1 \leq k \leq s$. Consequently, there exists $(c_1, \ldots, c_s) \in \mathbb{N}^s$ such that $(a_1, \ldots, a_r) = (c_1, \ldots, c_s) \cdot M_P$ and we may write

$$\gamma = \delta \cdot M_P = \left(n \cdot (c_1, \ldots, c_s) + \sum_{j=1}^{g} b_j \nu_{r+j} \right) \cdot M_P.$$

So, by Remark 3.1, we get $\delta = n \cdot (c_1, \ldots, c_s) + \sum_{j=1}^{g} b_j \nu_{r+j} \in \Gamma_{H_s}$. □

As we have mentioned in the previous section, given a q.o.h. with parameterization H, the set Λ_H can be used to identify some eliminable terms in H. In this way, it is relevant to relate the sets $\Lambda_s := \Lambda_{H_s}$ and $\Lambda_r := \Lambda_{H_r}$ where $H_\bullet$ denotes a parameterization of a q.o.h. in $(\mathbb{C}^{\bullet+1}, \underline{0})$ and $H_s \underset{P}{\rightsquigarrow} H_r$ for some ordered partition P of $\{1, \ldots, r\}$.

Theorem 3.6. *Given Λ_s and Λ_r as above we have*

$$\Lambda_s \cdot M_P + (\underline{n}) - n \sum_{i=1}^{s} \theta_{\beta_i} \subset \Lambda_r$$

where $\{\theta_j = (0, \ldots, 0, 1, 0, \ldots, 0),\ 1 \leq j \leq r\}$ is the canonical generators of the semigroup $\mathbb{N}^r$, P is an ordered partition of $\{1, \ldots, r\}$ and $\alpha_{i-1} < \beta_i \leq \alpha_i$ with $i = 1, \ldots, s$.

Proof. Let $(\mathcal{X}_\bullet,\underline{0})$ be q.o.h. in $(\mathbb{C}^{\bullet+1},\underline{0})$ with parameterizaion $H_\bullet$ and denote $\Omega^k_{\mathcal{O}_\bullet}$ the module of Kähler k-forms of the analytic algebra $\mathcal{O}_\bullet$ of $(\mathcal{X}_\bullet,\underline{0})$.

Given $\delta\in\Lambda_s$ there exists $\omega_s=\sum_{i=1}^{s+1}\overline{h_i}dy_1\wedge\cdots\wedge\widehat{dy_i}\wedge\cdots\wedge dy_{s+1}\in\Omega^s_{\mathcal{O}_s}$ where $\overline{h_i}$ denotes the class of $h_i\in\mathbb{C}\{Y_1,\ldots,Y_{s+1}\}$ in $\mathcal{O}_s$ such that

$$\Psi^s_{H_s}(\omega_s)=\prod_{i=1}^{s}u_i^{\delta_i-1}\cdot v(\underline{u})\cdot\bigwedge_{i=1}^{s}du_i,$$

and $v(\underline{u})\in\mathbb{C}\{\underline{u}\}$ a unit.

Keeping the notation T^P and T_P for the $\mathbb{C}$-linear maps

$$\begin{array}{ccc}\Omega^s_{\mathcal{O}_s} & \overset{T^P}{\to} & \Omega^s_{\mathcal{O}_r}\\ \sum_{|I|=s}\overline{h_I}dy_{i_1}\wedge\cdots\wedge dy_{i_s} & \mapsto & \sum_{|I|=s}\overline{\overline{T^P(h_I)}}dT^P(Y_{i_1})\wedge\cdots\wedge dT^P(Y_{i_s})\end{array}$$

where $\overline{\overline{T^P(h_I)}}$ denotes the class of $T^P(h_I)\in\mathbb{C}\{\underline{X},X_{r+1}\}$ in $\mathcal{O}_r$ and

$$\begin{array}{ccc}\Omega^s_{\mathbb{C}\{\underline{u}\}} & \overset{T_P}{\to} & \Omega^s_{\mathbb{C}\{\underline{t}\}}\\ q(\underline{u})du_1\wedge\cdots\wedge du_s & \mapsto & T_P(q(\underline{u}))dT_P(u_1)\wedge\cdots\wedge dT_P(u_s)\end{array}$$

we have the following commutative diagram of $\mathbb{C}$-linear maps

$$\begin{array}{ccc}\Omega^s_{\mathcal{O}_s} & \overset{T^P}{\longrightarrow} & \Omega^s_{\mathcal{O}_r}\\ \downarrow\Psi^s_{H_s} & & \downarrow\Psi^s_{H_r}\\ \Omega^s_{\mathbb{C}\{\underline{u}\}} & \overset{T_P}{\to} & \Omega^s_{\mathbb{C}\{\underline{t}\}}.\end{array}$$

Notice that

$$\begin{aligned}T_P\circ\Psi^s_{H_s}(\omega_s) &= \prod_{i=1}^{s}\left(\prod_{\alpha_{i-1}<j\leq\alpha_j}t_j\right)^{\delta_i-1}\cdot v(\underline{t})\cdot\bigwedge_{i=1}^{s}d\left(\prod_{\alpha_{i-1}<j\leq\alpha_j}t_j\right)\\ &= \underline{t}^{(\delta-(\underline{1}))\cdot M_P}\cdot v(\underline{t})\cdot\bigwedge_{i=1}^{s}\left(\sum_{\alpha_{i-1}<j\leq\alpha_i}\prod_{\substack{\alpha_{i-1}<k\leq\alpha_j\\ k\neq j}}t_kdt_j\right)\\ &= \underline{t}^{(\delta-(\underline{1}))\cdot M_P}\cdot v(\underline{t})\cdot\sum_{\substack{(\beta_1,\ldots,\beta_s)\\ \alpha_{i-1}<\beta_j\leq\alpha_i}}\frac{t_1\cdot\ldots\cdot t_r}{t_{\beta_1}\cdot\ldots\cdot t_{\beta_s}}\bigwedge_{i=1}^{s}dt_{\beta_i},\end{aligned}$$

where $v(\underline{t}) := v\left(\prod_{\alpha_0<j\leq\alpha_1} t_j, \dots, \prod_{\alpha_{s-1}<j\leq\alpha_s} t_j\right) \in \mathbb{C}\{\underline{t}\}$ is a unit.

Now, for each $\beta = (\beta_1, \dots, \beta_s)$ with $\alpha_{i-1} < \beta_i \leq \alpha_i$ we take $\omega_\beta := \bigwedge_{i=1}^{s} dx_{\alpha_{i-1}+1} \wedge \cdots \wedge \widehat{dx_{\beta_i}} \wedge \cdots \wedge dx_{\alpha_i} \in \Omega^{r-s}_{\mathcal{O}_r}$. As

$$\Psi^{r-s}_{H_r}(\omega_\beta) = n^{r-s} \frac{t_1^{n-1} \cdot \ldots \cdot t_r^{n-1}}{t_{\beta_1}^{n-1} \cdot \ldots \cdot t_{\beta_s}^{n-1}} \cdot \bigwedge_{i=1}^{s} dt_{\alpha_{i-1}+1} \wedge \cdots \wedge \widehat{dt_{\beta_i}} \wedge \cdots \wedge dt_{\alpha_i}$$

we obtain the commutative diagram

$$\begin{array}{ccccc}
\Omega^s_{\mathcal{O}_s} & \xrightarrow{T^P} & \Omega^s_{\mathcal{O}_r} & \xrightarrow{\omega_\beta\wedge} & \Omega^r_{\mathcal{O}_r} \\
\downarrow \Psi^s_{H_s} & & \downarrow \Psi^s_{H_r} & & \downarrow \Psi^r_{H_r} \\
\Omega^s_{\mathbb{C}\{\underline{u}\}} & \xrightarrow{T_P} & \Omega^s_{\mathbb{C}\{\underline{t}\}} & \xrightarrow{\Psi^{r-s}_{H_r}(\omega_\beta)\wedge} & \Omega^r_{\mathbb{C}\{\underline{t}\}}.
\end{array}$$

In this way, we get $T^P(\omega_\beta) \wedge \omega_s \in \Omega^r_{\mathcal{O}_r}$ and

$$\Psi^r_{H_r}(T^P(\omega_\beta) \wedge \omega_s) = \Psi^{r-s}_{H_r}(\omega_\beta) \wedge T_P\left(\Psi^s_{H_r}(\omega_s)\right) \in \Omega^r_{\mathbb{C}\{\underline{t}\}}$$

adimitting dominant expoent $\delta \cdot M_P + (\underline{n}) - n\sum_{i=1}^{s} \theta_{\beta_i} \in \Lambda_r$, where $\{\theta_j = (0, \dots, 0, 1, 0, \dots, 0),\ 1 \leq j \leq r\}$ is the canonical generators of the semigroup $\mathbb{N}^r$. □

Notice that the inclusion presented in the above theorem is proper. In fact, considering $\omega = dx_1 \wedge \cdots \wedge dx_r \in \Omega^r_{H_r}$ we get $\mathcal{V}_{H_r}(\omega) = (\underline{n}) \in \Lambda_r$ but $(\underline{n}) \notin \Lambda_s \cdot M_P + \underline{n} - n\sum_{i=1}^{s} \theta_i$ for any ordered partition P of $\{1, \dots, r\}$ and any choice of β_i with $\alpha_{i-1} < \beta_i \leq \alpha_i$ with $i = 1, \dots, s$.

Corollary 3.7. *With the same above notation we have* $\Lambda_s \cdot M_P + (\underline{n}) \subset \Lambda_r$.

Proof. By the previous theorem, we get $\delta \cdot M_P + (\underline{n}) - n\sum_{i=1}^{s} \theta_i \in \Lambda_r$ for $\delta \in \Lambda_s$ and $\alpha_{i-1} < \beta_i \leq \alpha_i$ with $i = 1, \dots, s$. As, $n\sum_{i=1}^{s} \theta_i \in \Gamma_{H_r}$ and $\Gamma_{H_r} + \Lambda_r \subset \Lambda_r$, it follows that

$$n\sum_{i=1}^{s} \theta_i + \delta \cdot M_P + (\underline{n}) - n\sum_{i=1}^{s} \theta_i = \delta \cdot M_P + (\underline{n}) \in \Lambda_r.$$

□

4 $\widetilde{\mathcal{A}}$-action on H_r with $H_s \underset{P}{\rightsquigarrow} H_r$

In the previous section we show that q.o.h. with parameterizations H_s and H'_s are topologically equivalent if and only if the q.o.h. admitting parameterizations H_r and H'_r with $H_s \underset{P}{\rightsquigarrow} H_r$ and $H'_s \underset{P}{\rightsquigarrow} H'_r$ are topologically equivalent. So, it is natural to ask about the behavior of such q.o.h. with respect to the $\tilde{\mathcal{A}}$-equivalence.

Firstly we remark that the property $H_s \underset{P}{\rightsquigarrow} H_r$ is sensitive with respect to the $\tilde{\mathcal{A}}$-equivalence. In fact, considering the parameterization $H_1 = (u^2, u^3)$ of a plane branch, that is a q.o.h. in $\mathbb{C}^2$ and taking $r = 2$, then the unique ordered partition of $\{1, 2\}$ as (3.1) is $P = \{\{1, 2\}\}$ and we obtain $H_1 \underset{P}{\rightsquigarrow} H_2$ with $H_2 = (t_1^2, t_2^2, t_1^3 t_2^3)$. Now, taking $(\sigma, \rho) \in \tilde{\mathcal{A}}$ given by $\sigma(X_1, X_2, X_3) = (X_1, X_2, X_3 + X_1^2 X_2^3)$ and $\rho(t_1, t_2) = (t_1, t_2)$ we get $H'_2 = \sigma \circ H_2 \circ \rho^{-1} = (t_1^2, t_2^2, t_1^3 t_2^3 + t_1^4 t_2^6)$ and obviously there is no a plane curve parameterization H'_1 in a such way that $H'_1 \underset{P}{\rightsquigarrow} H'_2$.

As we are considering q.o. parameterizations, our focus will be on elements in $\tilde{\mathcal{A}}$ satisfying the Proposition 2.10.

In that follows we consider a q.o.h. $(\mathcal{X}_s, \underline{0}) \in (\mathbb{C}^{s+1}, \underline{0})$ (resp. $(\mathcal{X}_r, \underline{0}) \in (\mathbb{C}^{r+1}, \underline{0})$) defined by a q.o. Weierstrass polynomial $f_s \in \mathbb{C}\{Y_1, \ldots, Y_{s+1}\}$ (resp. $f_r \in \mathbb{C}\{X_1, \ldots, X_{r+1}\}$) with q.o. parameterization H_s (resp. H_r) such that $H_s \underset{P}{\rightsquigarrow} H_r$ for some ordered partition

$$P = \{\{\alpha_0 + 1 = 1, \ldots, < \alpha_1\}, \ldots, \{\alpha_{s-1}, \ldots, \alpha_s = r\}\} \text{ of } \{1, \ldots, r\}.$$

Recall that a q.o. parameterization $H_r = (t_1^n, \ldots, t_r^n, S(\underline{t}))$ satisfies $H_s \underset{P}{\rightsquigarrow} H_r$ for some H_s if and only if $S(\underline{t}) \in Im(T_P)$ where T_P is the $\mathbb{C}$-algebras monomorphism given in (3.2).

With the above notations we have the following result.

Proposition 4.1. *Given $H_s \underset{P}{\rightsquigarrow} H_r$ and $(\sigma, \rho) \in \tilde{\mathcal{A}}$ with*

$$\sigma_i = a_i \cdot X_i + P_i, \quad \sigma_{r+1} = X_{r+1} \cdot (a_{r+1} + T^P(E_{r+1})) + \underline{X}^{\gamma \cdot M_P} \cdot T^P(N_{r+1})$$

and $\rho_i = t_i \cdot \left(a_i + \frac{P_i(H_r)}{t_i^n}\right)^{\frac{1}{n}}$, with $a_i, a_{r+1} \in \mathbb{C} \setminus \{0\}$, $\gamma \in \mathbb{N}^s$, $P_i = X_i \cdot T^P(E_i + Y_{s+1} \cdot N_i)$, where T^P is the $\mathbb{C}$-monomorphism given in (3.3),

$E_i, E_{r+1}, N_i \in \mathcal{M}_{s+1}$ for $i = 1, \ldots, r$ and $N_{r+1} \in \mathbb{C}\{Y_1, \ldots, Y_{s+1}\}$, then there exists a q.o. parameterization H'_s such that $H'_s \underset{P}{\rightsquigarrow} \sigma \circ H_r \circ \rho^{-1}$.

As $H_r = (t_1^n, \ldots, t_r^n, S(\underline{t}))$ is a q.o. parameterization and, by hypothesis, $(\sigma, \rho) \in \widetilde{\mathcal{A}}$ satifies the assumption of the Proposition 2.10 we have $\sigma \circ\ H_r \circ \rho^{-1} = (t_1^n, \ldots, t_r^n, \sigma_{r+1} \circ H_r \circ \rho^{-1})$ is a q.o. parameterization. So, the Proposition 4.1 is equivalent to show that

$$\sigma_{r+1} \circ H_r \circ \rho^{-1} \in Im(T_P). \tag{4.1}$$

We will proof (4.1) by using the following claims:

Claim 1: Considering ρ_k, $1 \leq k \leq r$ given in Proposition 4.1 we get $\prod_{i=1}^{s} \left(\prod_{\alpha_{i-1} < j \leq \alpha_i} (\rho^{-1})_j \right)^{\beta_i} \in Im(T_P)$ for all $\beta_i \in \mathbb{N}$.

Initially we will show that

$$G(\underline{t}) := \prod_{\alpha_{i-1} < j \leq \alpha_i} (\rho^{-1})_j = \sum_{\delta} a_\delta \underline{t}^\delta \in Im(T_P).$$

Denoting $\rho_k(\underline{t}) = t_k \cdot u_k(\underline{t})$ with $u_k(\underline{0}) \neq 0$ we get

$$\prod_{\alpha_{i-1} < j \leq \alpha_i} t_j = G(\rho(\underline{t})) = \sum_{\delta} a_\delta \prod_{k=1}^{r} t_k^{\delta_k} \cdot u_k^{\delta_k}(\underline{t}). \tag{4.2}$$

By the above equality, we must have $\delta \in \mathbb{N}^s \cdot M_P$ for all $\delta \in Supp(G(\underline{t}))$, otherwise taking $\gamma = \min_{\preceq_{Lex}} \{\delta \in Supp\,(G(\underline{t}))\,;\ \delta \notin \mathbb{N}^s \cdot M_P\}$, where the symbol $\preceq_{Lex}$ denotes the lexicographical order in $\mathbb{N}^r$, the monomial $\underline{t}^\gamma$ does not vanish in the right side of (4.2).

Now, $\delta \in \mathbb{N}^s \cdot M_P$ for all $\delta \in Supp(G(\underline{t}))$ implies that $G(\underline{t}) \in Im(T_P)$ and, as T_P is a $\mathbb{C}$-algebra homomorphism, we get the Claim 1.

Claim 2: With the hypothesis of Proposition 4.1, if $H_r = (t_1^n, \ldots, t_r^n, S(\underline{t}))$ then we get $S(\rho^{-1}) \in Im(T_P)$.

As $H_s \underset{P}{\rightsquigarrow} H_r$, we have $S(\underline{t}) = \sum_\beta a_\beta \prod_{i=1}^{s} \left(\prod_{\alpha_{j-1} < j \leq \alpha_i} t_j \right)^{\beta_i}$ with $\beta_i \in \mathbb{N}$. In this way, by the Claim 1, we get

$$S(\rho^{-1}) = \sum_{\beta} a_\beta \prod_{i=1}^{s} \left(\prod_{\alpha_{i-1} < j \leq \alpha_i} (\rho^{-1})_j \right)^{\beta_i} \in Im(T_P).$$

Claim 3: Given $q_1(\underline{t}),\dots,q_{r+1}(\underline{t})\in\mathbb{C}\{\underline{t}\}$ satisfying

$$\prod_{\alpha_{i-1}<j\leq\alpha_i} q_j(\underline{t}), q_{r+1}(\underline{t})\in Im(T_P);\quad 1\leq i\leq s$$

then $G(q_1(\underline{t}),\dots,q_{r+1}(\underline{t}))\in Im(T_P)$ for all $G\in Im(T^P)$. In particular, $G(H_r)\in Im(T_P)$ for any $G\in Im(T^P)$.

If $G=T^P(F)$ with $F=\sum_\delta c_\delta\prod_{i=1}^{s+1}Y_i^{\delta_i}$ then

$$G(\underline{X},X_{r+1})=\sum_\delta c_\delta\prod_{i=1}^{s}\left(\prod_{\alpha_{i-1}<j\leq\alpha_i}X_j\right)^{\delta_i}\cdot X_{r+1}^{\delta_{s+1}}.$$

Denoting $\prod_{\alpha_{i-1}<j\leq\alpha_i} q_j(\underline{t}) = T_P(p_i(\underline{u}))$ for $1\leq i\leq s$ and $q_{r+1}(\underline{t}) = T_P(p_{s+1}(\underline{u}))$ with $p_k(\underline{u})\in\mathbb{C}\{\underline{u}\}$ for $1\leq k\leq s+1$, we get

$$\begin{aligned} G(q_1(\underline{t}),\dots,q_{r+1}(\underline{t})) &= \textstyle\sum_\delta c_\delta\prod_{i=1}^{s}\left(\prod_{\alpha_{i-1}<j\leq\alpha_i} q_j(\underline{t})\right)^{\delta_i}\cdot q_{r+1}(\underline{t})^{\delta_{s+1}}\\ &= \textstyle\sum_\delta c_\delta\prod_{i=1}^{s+1}T_P(p_i(\underline{u}))^{\delta_i}\\ &= T_P\left(\textstyle\sum_\delta c_\delta\prod_{i=1}^{s+1}p_i(\underline{u})^{\delta_i}\right)\in Im(T_P).\end{aligned}$$

In particular, $G(H_r) = G(t_1^n,\dots,t_r^n,T_P(S(\underline{u})))\in Im(T_P)$ because $\prod_{\alpha_{i-1}<j\leq\alpha_i} t_j^n = T_P(u_i^n)$.

Proof. of Proposition 4.1:

As we have mentioned it is sufficient to show (4.1).

Notice that $H_r\circ\rho^{-1} = ((\rho^{-1})_1^n,\dots,(\rho^{-1})_r^n,S(\rho^{-1}))$, by the Claim 1 and the Claim 2, we get $\prod_{\alpha_{i-1}<j\leq\alpha_i}(\rho^{-1})_j, S(\rho^{-1})\in Im(T_P)$. By hypothesis,

$$\begin{aligned}\sigma_{r+1} &= X_{r+1}\cdot(a_{r+1}+T^P(E_{r+1}))+\underline{X}^{\gamma\cdot M_P}\cdot T^P(N_{r+1})\\ &= T^P(Y_{s+1}\cdot(a_{r+1}+E_{r+1})+\underline{Y}^{\gamma}\cdot N_{r+1}) \qquad \in Im(T^P)\end{aligned}$$

and, by the Claim 3, we have $\sigma_{r+1}\circ H_r\circ\rho^{-1}\in Im(T_P)$ proving the proposition. □

At the beginning of this section we ask about the behavior of q.o.h. $H_s \underset{P}{\rightsquigarrow} H_r$ under the $\tilde{\mathcal{A}}$-equivalence. The next example shows that if H_s is $\tilde{\mathcal{A}}$-equivalent to H'_s with $H_s \underset{P}{\rightsquigarrow} H_r$ and $H'_s \underset{P}{\rightsquigarrow} H'_r$, then H_r is not necessarily $\tilde{\mathcal{A}}$-equivalent to H'_r.

Example 4.2. Let $H_1 = (u^3, u^4 + u^5)$ and $H_1' = (u^3, u^4)$ be parameterizations of q.o.h. $(\mathcal{X}_1, \underline{0})$ and $(\mathcal{X}_1', \underline{0})$ in $(\mathbb{C}^2, \underline{0})$ respectively, that is, plane curves. Considering $\omega := y_2 dy_2 \in \Omega^1_{\mathcal{O}_1}$ where $\mathcal{O}_1$ is the analytic algebra of $(\mathcal{X}_1, \underline{0})$, we get $\mathcal{V}_{H_1}\left(\frac{\Psi^1_{H_1}(\omega)}{du}\right) + 1 = 8 = 5 + 3 \in \Lambda_{H_1}$ and, according to the explanation in the end of the Section 2, u^5 is an eliminable term in H_1. So, H_1 is $\tilde{\mathcal{A}}$-equivalent to $(u^3, u^4 + \sum_{i \geq 6} a_i u^i)$. As the semigroup associated to $(\mathcal{X}_1, \underline{0})$ is $\Gamma_{H_1} = \langle 3, 4 \rangle$, its Frobenius vector is 5, consequently $\gamma \in \Gamma_{H_1}$ for any $\gamma \geq 6$ and, by Remark 2.8, u^γ is eliminable. In this way, H_1 is $\tilde{\mathcal{A}}$-equivalent[3] to H_1'.

Now, let us consider the q.o. parameterizations $H_2 = (t_1^3, t_2^3, t_1^4 t_2^4 + t_1^5 t_2^5)$ and $H_2' = (t_1^3, t_2^3, t_1^4 t_2^4)$, that is, $H_1 \underset{P}{\rightsquigarrow} H_2$ and $H_1' \underset{P}{\rightsquigarrow} H_2'$. If H_2 is $\tilde{\mathcal{A}}$-equivalent to H_2' then there exists $(\sigma, \rho) \in \tilde{\mathcal{A}}$ such that $\sigma \circ H_2 \circ \rho^{-1} = H_2'$. In particular, $t_1^5 t_2^5$ should be eliminable in H_2 by change of coordinates and parameters described in the Proposition 2.10, that is,

$$\rho_i(t_1, t_2) = t_i \cdot (a_i + P_i(\underline{t}))^{\frac{1}{3}} \quad \text{where} \quad P_i(\underline{t}) = \epsilon_i + \frac{(t_1^4 t_2^4 + t_1^5 t_2^5) \cdot \eta_i}{t_i^3}$$

$$\sigma_i(X_1, X_2, X_3) = X_i \cdot (a_i + \epsilon_i) + X_3 \cdot \eta_i, \;\; \sigma_3(X_1, X_2, X_3) = X_3 \cdot (a_3 + \epsilon_3) + X_1^2 X_2^2 \cdot \eta_3$$

with $a_i, a_3 \in \mathbb{C} \setminus \{0\}, \epsilon_i, \epsilon_3 \in \mathcal{M}_3$ and $\eta_i, \eta_3 \in \mathbb{C}\{\underline{X}, X_3\}$ for $i = 1, 2$.

Notice that $(\rho^{-1})_i = t_i \cdot v_i(\underline{t})$ with $v_i(0,0) \neq 0$, then for any $h \in \mathcal{M}^2 \setminus \{0\}$ we have $(\delta_1, \delta_2) \in Supp(h(\rho^{-1}))$ with $\delta_1 \geq 6$ or $\delta_2 \geq 6$. So, in order to eliminate $t_1^5 t_2^5$ in H_2 it is suficient to consider $\epsilon_3 = \eta_3 = 0$ and in this way, we get

$$\sigma_3 \circ H_2 \circ \rho^{-1} = a_3 \cdot \left(t_1^4 t_2^4 \cdot v_1^4(\underline{t}) \cdot v_2^4(\underline{t}) + t_1^5 t_2^5 \cdot v_1^5(\underline{t}) \cdot v_2^5(\underline{t})\right).$$

The only possibility to cancel the monomial $t_1^5 t_2^5$ in the above expression is to obtain $t_i \in Supp(v_i(\underline{t}))$ for $i = 1, 2$, but this is equivalent to get $t_i \in Supp(P_i(\underline{t}))$ for $i = 1, 2$ that is impossible.

Hence, the term $t_1^5 t_2^5$ is not eliminable in H_2 and consequently H_2 is not $\tilde{\mathcal{A}}$-equivalent to H_2'.

[3] In [14], Zariski showed that plane curves with parameterization $(u^3, u^4 + u^5)$ and (u^3, u^4) are analytical equivalent.

On the other hand, if there exists $(\sigma,\rho)\in\tilde{\mathcal{A}}$ as in the Proposition 4.1 and $\sigma\circ H_r\circ\rho^{-1}=H'_r$ with $H_s \underset{P}{\rightsquigarrow} H_r$ and $H'_s \underset{P}{\rightsquigarrow} H'_r$ then we can conclude that H_s is $\tilde{\mathcal{A}}$-equivalent to H'_s. This is the conclusion of our last result, for which we will use the following lemmas.

Lemma 4.3. *Considering ρ_k, $1\le k\le r$ given in Proposition 4.1, we get $\prod_{\alpha_{i-1}<j\le\alpha_i}\rho_j\in Im(T_P)$.*

Proof. Recall that $\rho_j=t_j\cdot\left(a_j+\frac{P_j(H_r)}{t_j^n}\right)^{\frac{1}{n}}$ and

$$\frac{P_j(H_r)}{t_j^n}=T^P(E_j+Y_{s+1}\cdot N_j)(H_r).$$

As T^P is a $\mathbb{C}$-algebras homomorphism, we get

$$\prod_{\alpha_{i-1}<j\le\alpha_i} t_j\cdot\left(a_j+\frac{P_j(H_r)}{t_j^n}\right)^{\frac{1}{n}}=\left(\prod_{\alpha_{i-1}<j\le\alpha_i} t_j\right)\cdot(c_i+G_i(H_r))^{\frac{1}{n}}$$

with $c_i=\prod_{\alpha_{i-1}<j\le\alpha_i}a_j$ and $G_i\in Im(T^P)$ for all $1\le i\le s$. Notice that $G_i=T^P(J_i+Y_{s+1}\cdot K_i)$ with $J_i,K_i\in\langle Y_1,\dots,Y_{s+1}\rangle$ then, by (3.4) we obtain $G_i(H_r)=H_r^*(G_i)=T_P(H_s^*(J_i+Y_{s+1}\cdot K_i))$ and

$$\prod_{\alpha_{i-1}<j\le\alpha_i}\rho_j=T_P\left(u_i\cdot\left(c_i+\frac{Q_i(H_s)}{u_i^n}\right)^{\frac{1}{n}}\right)\in Im(T_P). \tag{4.3}$$

where $Q_i=Y_i\cdot(J_i+Y_{s+1}\cdot K_i)$. □

By Claim 1 and (4.3), we have that

$$\prod_{\alpha_{i-1}<j\le\alpha_i}(\rho^{-1})_j(\underline{t})=T_P(\mu_i(\underline{u}))\quad\text{and}\quad\prod_{\alpha_{i-1}<j\le\alpha_i}\rho_j(\underline{t})=T_P(\theta_i(\underline{u})) \tag{4.4}$$

with $\mu_i(\underline{u}),\theta_i(\underline{u})=u_i\cdot\left(c_i+\frac{Q_i(H_s)}{u_i^n}\right)^{\frac{1}{n}}\in\mathbb{C}\{\underline{u}\}$ where $c_i\in\mathbb{C}\setminus\{0\}$ and $Q_i=Y_i\cdot(J_i+Y_{s+1}\cdot K_i)$.

Lemma 4.4. *Considering the above notation, we have $\theta^{-1}=\mu$ where $\theta=(\theta_1,\dots,\theta_s)$ and $\mu=(\mu_1,\dots,\mu_s)$.*

Proof. By (4.4) we get

$$\prod_{\alpha_{i-1}<j\leq\alpha_i} t_j = T_P(\theta_i)\circ\rho^{-1}(\underline{t}) = \theta_i\left(\prod_{\alpha_0<j\leq\alpha_1}(\rho^{-1})_j,\ldots,\prod_{\alpha_{s-1}<j\leq\alpha_s}(\rho^{-1})_j\right).$$

As T_P is a $\mathbb{C}$-algebra monomorphism and

$$T_P(u_i) = \prod_{\alpha_{i-1}<j\leq\alpha_i} t_j = \theta_i(T_P(\mu_1),\ldots,T_P(\mu_s)) = T_P(\theta_i(\mu_1,\ldots,\mu_s))$$

we obtain $\theta_i(\mu_1(\underline{u}),\ldots,\mu_s(\underline{u})) = u_i$ for $1\leq i\leq s$, that is, $\theta = (\theta_1,\ldots,\theta_s)$ admits inverse $\mu = (\mu_1,\ldots,\mu_s)$. □

The previous results allow us to obtain the following theorem.

Theorem 4.5. *Given* $H_s = (u_1^n,\ldots,u_s^n,S(\underline{u}))$ *such that* $H_s \underset{P}{\rightsquigarrow} H_r$, $(\sigma,\rho)\in\tilde{\mathcal{A}}$ *as described in the Proposition 4.1 and* $H_s' \underset{P}{\rightsquigarrow} \sigma\circ H_r\circ\rho^{-1}$, *then* H_s *is* $\tilde{\mathcal{A}}$*-equivalent to* H_s'.

Proof. Given $(\sigma,\rho)\in\tilde{\mathcal{A}}$, as described in the Proposition 4.1, we consider $(\tau,\theta)\in Iso(\mathbb{C}^{s+1},\underline{0})\times Iso(\mathbb{C}^s,\underline{0})$, where $\theta=(\theta_1,\ldots,\theta_s)$ is determined in (4.4), $\tau = (\tau_1,\ldots,\tau_{s+1})$ with

$$\tau_i = c_i\cdot Y_i + Q_i,\ 1\leq i\leq s \ \text{ and } \ \tau_{s+1} = Y_{s+1}\cdot(a_{r+1}+E_{r+1}) + \underline{Y}^{\gamma}\cdot N_{r+1}.$$

In this way, we get $\tau\circ H_s\circ\theta^{-1} = (\tau_1\circ H_s\circ\theta^{-1},\ldots,\tau_{s+1}\circ H_s\circ\theta^{-1})$ with

$$\begin{aligned}\tau_i\circ H_s\circ\theta^{-1} &= c_i\cdot(\theta^{-1})_i^n + Q_i(H_s(\theta^{-1}))\\ &= \left((\theta^{-1})_i\cdot\left(c_i+\frac{Q_i(H_s(\theta^{-1}))}{(\theta^{-1})_i^n}\right)^{\frac{1}{n}}\right)^n\\ &= (\theta_i\circ\theta^{-1})^n = u_i^n, \qquad \text{for } 1\leq i\leq s\end{aligned}$$

and, using that $\theta^{-1}=\mu$,

$$\tau_{s+1}\circ H_s\circ\theta^{-1} = S(\mu)\cdot(a_{r+1}+E_{r+1}(H_s(\mu))) + \prod_{i=1}^{s}(\mu_i^n)^{\gamma_i}\cdot N_{r+1}(H_s(\mu)) := S'(\underline{u}),$$

that is, $\tau\circ H_s\circ\theta^{-1}(\underline{u}) = (u_1^n,\ldots,u_s^n,S'(\underline{u}))$.

Now, as T_P is a $\mathbb{C}$-algebra homomorphism, by (4.4), we get

$$T_P(S(\mu)) = S\left(\prod_{\alpha_0<j\leq\alpha_1}(\rho^{-1})_j, \ldots, \prod_{\alpha_{s-1}<j\leq\alpha_s}(\rho^{-1})_j\right) = T_P(S)\circ\rho^{-1},$$

$$\begin{aligned} T_P(E_{r+1}(H_s(\mu))) &= E_{r+1}(T_P(\mu_1^n), \ldots, T_P(\mu_s^n), T_P(S(\mu)))) \\ &= E_{r+1}\left(\prod_{\alpha_0<j\leq\alpha_1}(\rho^{-1})_j^n, \ldots, \prod_{\alpha_{s-1}<j\leq\alpha_s}(\rho^{-1})_j^n, T_P(S)\circ\rho^{-1}\right) \\ &= T^P(E_{r+1})(H_r\circ\rho^{-1}) \end{aligned}$$

and similarly,

$$T_P\left(\prod_{i=1}^{s})\mu_i^n)\gamma_i \cdot N_{r+1}(H_s(\mu))\right) = T^P(\underline{Y}^\gamma \cdot N_{r+1})(H_r\circ\rho^{-1}).$$

Consequently, $T_P(S'(\underline{u})) = \sigma_{r+1}\circ H_r\circ\rho^{-1}$ and $H'_s = \tau\circ H_s\circ\theta$ that proofs the theorem. □

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