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Affine and projective Lê cycles

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Abstract. Lê cycles of germs of complex analytic functions are analytic cycles that describe, among other things, the topology of the local Milnor fibres: We know from [7, 8] that there is a Lê cycle in each dimension, from 0 to that of the singular set, and the multiplicity of the Lê cycles at each point says how many handles of the corresponding dimension we must attach to a ball in order to construct the local Milnor fibre (up to homeomorphism).

In [1], José Seade, Roberto Callejas-Bedregal and I defined the global Lê cycles (affine and projective), which are a global extension of the Lê cycles defined by Massey in [7]. Here, the relationship between affine and projective Lê cycles will be detailed, this is also mentioned in [2].

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1 Introdution

Lê cycles are analytic cycles encoding deep information about singularity germs $f : (\mathbb{C}^N, 0) \to (\mathbb{C}, 0)$ and allow describing the topology and diffeomorphism type of the local Milnor fibres. These were introduced by D. Massey in [8] and detailed in section 2, on local Lê cycles.

In the affine context, we have the description by Schürmann and Tibăr in [12] about the Schwartz-MacPherson classes of a complex algebraic proper subset $X \subset \mathbb{C}^N$ using algebraic cycles. Motivated by this description the definition of affine Lê cycles appears and they are a global extension of Massey's local Lê cycles, which can be found in section 3. In section 4, these are generalized to the compact projective setting via projective Lê cycles.

Large portions of this paper appear in section 7.5 of an earlier article ″Milnor numbers and Chern classes for singular varieties: an introduction by Callejas-Bedregal, Morgado, and Seade in 2022″ ([2]), but more detail is given here in several cases. The main result of this paper is Theorem 4.4, which relates the affine and projective Lê cycles, originally appearing without any proof in [2, Proposition 7.5.5].

Explicitly, let $X \subseteq \mathbb{C}P^N$ be a d-dimensional projective variety endowed with a Whitney stratification with connected strata, let β be a constructible function on X, with respect to this stratification, and let L_{k+2} be a linear subvariety of $\mathbb{C}P^N$ of codimension $k+2$. Then the projective Lê cycle $\Lambda_k^{\mathbb{P}}(\beta, L_{k+2})$ is the projection of the affine Lê cycle $\Lambda_{k+1}^{\mathbb{A}}\left(\tilde{\beta}, \mathrm{Cone}\,(L_{k+2})\right)$, where $\tilde{\beta}$ is a constructible function induced by β on Cone(X), i.e,

$$\Lambda_k^{\mathbb{P}}(\beta, L_{k+2}) = \mathbb{P}\left(\Lambda_{k+1}^{\mathbb{A}}\left(\tilde{\beta}, \mathrm{Cone}\,(L_{k+2})\right)\right).$$

2 Local Lê cycles

Let us recall first the definition of Lê cycles and Lê numbers of germs of complex analytic functions introduced by D. Massey in [7] (see also [8]). We assume that the reader is familiar with the notion of gap sheaves (see [13] and [8, Definition 1.1]). For a coherent sheaf of ideals α and an analytic subset W in an affine space U, we denote by α/W the corresponding gap sheaf, which is a coherent sheaf of ideals in $\mathcal{O}_U$, and by $V(\alpha)/W$ the analytic space defined by the vanishing of α/W. It is important to note that the analytic space $V(\alpha)/W$ does not depend on the structure of W as a scheme, but only as an analytic set (see [8, p. 10]).

Let U be an open subset of $\mathbb{C}^{n+1}$ containing the origin, $h:(U,0)\to(\mathbb{C},0)$ the germ of an analytic function, $z=(z_0,\cdots,z_n)$ a linear choice of coordinates in $\mathbb{C}^{n+1}$ and $\Sigma(h)=V\left(\frac{\partial h}{\partial z_0},\ldots,\frac{\partial h}{\partial z_n}\right)$ the critical set of h. To define the Lê cycles we need to define the relative polar cycles first, which are associated to the relative polar varieties:

Definition 2.1. For each k with k with $0\leq k\leq n$, the **k-th local polar variety** $\Gamma^k_{h,z}$ is the analytic space $V\left(\frac{\partial h}{\partial z_k},\ldots,\frac{\partial h}{\partial z_n}\right)/\Sigma(h)$.

Hence the analytic structure of $\Gamma^k_{h,z}$ does not depend on the structure $\Sigma(h)$ as a scheme, but only as an analytic set. At the level of ideals, $\Gamma^k_{h,z}$ consists of those components of $V\left(\frac{\partial h}{\partial z_k},\ldots,\frac{\partial h}{\partial z_n}\right)$ which are not contained in the set $\Sigma(h)$. Massey denotes by $\left[\Gamma^k_{h,z}\right]$ the cycle associated with the space $\Gamma^k_{h,z}$ (see [8, p. 9]).

Definition 2.2. For each $0\leq k\leq n$, the **k-th local Lê cycle** $\Lambda^k_{h,z}$ of h with respect to the coordinate system z as the cycle is:

$$\Lambda^k_{h,z}:=\left[\Gamma^{k+1}_{h,z}\cap V\left(\frac{\partial h}{\partial z_k}\right)\right]-\left[\Gamma^k_{h,z}\right].$$

If a point $p=(p_0,\cdots,p_n)\in U$ is an isolated point of the intersection of $\Lambda^k_{h,z}$ with the cycle of $V(z_0-p_0,\cdots,z_{k-1}-p_{k-1})$, then the **$k$-th Lê number** $\lambda^k_{h,z}(p)$ is the intersection number at p :

$$\lambda^k_{h,z}(p):=\left(\Lambda^k_{h,z}\cdot V(z_0-p_0,\ldots,z_{k-1}-p_{k-1})\right)_p.$$

It is proved in [9, Theorem 7.5] (see also [8, Theorem 10.18]) that for a generic choice of linear coordinates, all the Lê numbers of h at p are defined and they are independent of the coordinates choice. Hence, these are called the generic Lê numbers of h at p and they are denoted simply by $\lambda^k_h(p)$.

An important feature of the generic Lê numbers is that they allow to describe a handle decomposition of the Milnor fiber $F_{h,p}$ of h at p. In fact, Massey proved in [8, Theorem 3.3; Theorem 10.3] the following:

Theorem 2.3. *Let U be an open subset of $\mathbb{C}^{n+1}$, let $h : (U, 0) \to (\mathbb{C}, 0)$ be a germ of an analytic function, let s denote $\dim_0 \Sigma(h)$, and let $z = (z_0, \cdots, z_n)$ be a generic choice of linear coordinates in $\mathbb{C}^{n+1}$. Then the local Lê cycles are a collection of analytic cycle germs $\Lambda^i_{h,z}$ in $\Sigma(h)$ at the origin such that each $\Lambda^i_{h,z}$ is purely i-dimensional and properly intersects $V(z_0, \ldots, z_{i-1})$ at the origin, and for all $p \in \Sigma(h)$ near 0 we have that*

1. *If $s \leq n-2$, then $F_{h,p}$ p is obtained up to diffeomorphism from a real $2n$-ball by successively attaching $\lambda^{n-k}_{h,z}(p)k$-handles, where $n - s \leq k \leq n$;*

2. *If $s = n - 1$, then $F_{h,p}$ is obtained up to diffeomorphism from a real $2n$-manifold with a homotopy-type of a bouquet $\lambda^{n-1}_{h,z}(p)$ circles by successively attaching $\lambda^{n-k}_{h,z}(p)k$-handles, where $2 \leq k \leq n$.*

3. *The reduced Euler characteristic of the Milnor fiber of h at p is given by*
$$\tilde{\chi}(F_{h,p}) = \sum_{i=0}^{n} (-1)^{n-i} \lambda^i_{h,z}(p).$$

Massey gives an alternative characterization of the local Lê cycles of a hypersurface singularity, which leads to a generalization of the Lê numbers that can be applied to any constructible complex of sheaves. From this more general viewpoint, the case of the Lê numbers of a function h is just the case where the underlying constructible complex of sheaves is the sheaf of vanishing cycles along h. Let us explain this. We assume some basic knowledge on derived categories, hypercohomology and sheaves of vanishing cycles as described in [3].

If X is a complex analytic space then $\mathcal{D}^b_c(X)$ denotes the derived category of bounded, constructible complexes of sheaves of $\mathbb{C}$-vector spaces on X. We denote the objects of $\mathcal{D}^b_c(X)$ by something of the form $F^\bullet$. The shifted complex $F^\bullet[l]$ is defined by $(F^\bullet[l])^k = F^{l+k}$ and its differential is $d^k_{[l]} = (-1)^l d^{k+l}$. The constant sheaf $\mathbb{C}_X$ on X induces an object $\mathbb{C}^\bullet_X \in \mathcal{D}^b_c(X)$ by letting $\mathbb{C}^0_X = \mathbb{C}_X$ and $\mathbb{C}^k_X = 0$ for $k \neq 0$.

If $h : X \to \mathbb{C}$ is an analytic map and $F^\bullet \in \mathcal{D}_c^b(X)$ then we denote the sheaf of vanishing cycles of $F^\bullet$ with respect to h by $\phi_h F^\bullet$.

For $F^\bullet \in \mathcal{D}_c^b(X)$ and $p \in X$, we denote by $\mathcal{H}^* (F^\bullet)_p$ the stalk cohomology of $F^\bullet$ at p, and by $\chi (F^\bullet)_p$ its Euler characteristic. That is

$$\chi (F^\bullet)_p = \sum_k (-1)^k \dim_{\mathbb{C}} \mathcal{H}^k (F^\bullet)_p .$$

We also denote by $\chi (X, F^\bullet)$ the Euler characteristic of X with coefficients in $F^\bullet$, i.e.,

$$\chi (X, F^\bullet) = \sum_k (-1)^k \dim_{\mathbb{C}} \mathbb{H}^k (X, F^\bullet),$$

where $\mathbb{H}^* (X, F^\bullet)$ denotes the hypercohomology groups of X with coefficients in $F^\bullet$.

When $F^\bullet \in \mathcal{D}_c^b(X)$ is $\mathcal{S}$-constructible, where $\mathcal{S}$ is a Whitney stratification of X, we denote it by $F^\bullet \in \mathcal{D}_{\mathcal{S}}^b(X)$. We would like to point out the following result which appears in [3, Theorem 4.1.22]:

$$\chi (X, F^\bullet) = \sum_{S \in \mathcal{S}} \chi (F_S^\bullet) \chi(S),$$

where $\chi (F_S^\bullet) = \chi (F^\bullet)_p$ for an arbitrary point $p \in S$.

Let M be a complex manifold. For a complex analytic subspace V of M, we denote its conormal space by $T_V^* M$. That is

$$T_V^* M := \text{closure} \left\{ (x, \theta) \in T^* M \mid x \in V_{\text{reg}} \text{ and } \theta_{|_{T_x V_{\text{reg}}}} \equiv 0 \right\},$$

where $T^* M$ is the cotangent bundle of M and V_{reg} is the regular part of V. The following definition is standard in the literature:

Definition 2.4. Let X be an analytic subspace of a complex manifold M, $\{S_\alpha\}$ a Whitney stratification of M adapted to X and $x \in S_\alpha$ a point in X. Consider $g : (M, x) \to (\mathbb{C}, 0)$ a germ of holomorphic function such that $d_x g$ is a non-degenerate covector at x with respect to the fixed stratification, that is, $d_x g \in T_{S_\alpha}^* M$ and $d_x g \notin T_{\overline{S'}}^* M$, for all stratum $S' \neq S_\alpha$. And let N

be a germ of a closed complex submanifold of M which is transversal to S_α, with $N \cap S_\alpha = \{x\}$. Define the **complex link** l_{S_α} of S_α by:

$$l_{S_\alpha} := X \cap N \cap B_\delta(x) \cap \{g = w\} \quad \text{for } 0 < |w| \ll \delta \ll 1.$$

The **normal Morse datum** of S_α is defined by:

$$NMD\,(S_\alpha) := (X \cap N \cap B_\delta(x), l_{S_\alpha})\,,$$

and the **normal Morse index** $\eta\,(S_\alpha, F^\bullet)$ of the stratum is:

$$\eta\,(S_\alpha, F^\bullet) := \chi\,(NMD\,(S_\alpha)\,, F^\bullet)\,,$$

where the right-hand-side means the Euler characteristic of the relative hypercohomology.

By the result of M. Goresky and R. MacPherson in [4, Theorem 2.3] we get that the number $\eta\,(S_\alpha, F^\bullet)$ does not depend on the choices of $x \in S_\alpha, g$ and N. Notice that by [3, Remark 2.4.5(ii)], it follows that

$$\eta\,(S_\alpha, F^\bullet) = \chi\,(X \cap N \cap B_\delta(x), F^\bullet) - \chi\,(l_{S_\alpha}, F^\bullet)\,.$$

Lemma 2.5. *Let $F^\bullet \in \mathcal{D}^b_{\mathcal{S}}(X)$ with $\mathcal{S} = \{S_\alpha\}$ a Whitney stratification of X. Let $p \in S_\alpha$ and $g : (M,p) \to (\mathbb{C},0)$ be a holomorphic function germ such that $d_p g$ is a non-degenerate covector at $p \in S_\alpha$ with respect to the fixed stratification. Set $d = \dim X, d_\alpha = \dim S_\alpha$ and $m_\alpha := (-1)^{d-d_\alpha-1}\chi\left(\phi_{g_{|_N}} F^\bullet_{|_N}\right)_p$, where $\phi_{g_{|_N}} F^\bullet_{|_N}$ is the sheaf of vanishing cycles of $F^\bullet_{|_N}$ with respect to $g_{|_N}, p \in S_\alpha$ and N is a germ of a closed complex submanifold which is transversal to S_α with $N \cap S_\alpha = \{p\}$. Then*

$$m_\alpha = (-1)^{d-d_\alpha}\eta\,(S_\alpha, F^\bullet)\,.$$

Proof. By [3, Equation (4.1), p. 106] we have that

$\mathcal{H}^i\,(\phi_g F^\bullet)_p \simeq \mathbb{H}^{i+1}\left(B_\epsilon(p) \cap X, B_\epsilon(p) \cap X \cap g^{-1}(\varsigma), F^\bullet\right)$, for $0 < |\varsigma| \ll \epsilon \ll 1$. Hence

$$\chi\left(\phi_{g_{|_N}} F^\bullet_{|_N}\right)_p = -\chi\left(B_\epsilon(p) \cap X \cap N, B_\epsilon(p) \cap X \cap N \cap g^{-1}(\varsigma), F^\bullet\right)\,,$$

and therefore $m_\alpha = (-1)^{d-d_\alpha}\eta\,(S_\alpha, F^\bullet)$. □

Remark 2.6. Everything we have defined so far for a constructible complex of sheaves is defined by J. Schürmann and M. Tibăr in [12] for constructible functions, and the two equivalent constructions. In fact, given $F^\bullet \in \mathcal{D}_c^b(X)$, we have naturally associated the constructible function on X given by

$$\beta(p) = \chi\left(F^\bullet\right)_p.$$

Moreover, the converse also holds (see [11]), i.e., given any constructible function β on X there is $F^\bullet \in \mathcal{D}_c^b(X)$ such that

$$\beta(p) = \chi\left(F^\bullet\right)_p.$$

In particular, for any constructible function β on X we have that

$$\eta\left(S_\alpha, \beta\right) = \chi\left(X \cap N \cap B_\delta(x), \beta\right) - \chi\left(l_{S_\alpha}, \beta\right). \tag{2.1}$$

Let X be an analytic germ of an s-dimensional space which is embedded in some affine space, $M := \mathbb{C}^{n+1}$, so that the origin is a point of X. Consider a bounded, constructible sheaf $F^\bullet$ on X or M.

For a generic choice of linear coordinates $z = (z_0, \ldots, z_n)$ for $\mathbb{C}^{n+1}$, Massey in [9, Proposition 0.1] proves that there exists analytic cycles $\Lambda^i_{F^\bullet,z}$ in X which are purely i-dimensional, such that $\Lambda^i_{F^\bullet,z}$ and $V\left(z_0 - p_0, \ldots, z_{i-1} - p_{i-1}\right)$ intersect properly at each point $p = (p_0, \cdots, p_n)$ $\in X$ near the origin, and such that

$$\chi\left(F^\bullet\right)_p = \sum_{i=0}^{s} (-1)^{s-i} \left(\Lambda^i_{F^\bullet,z} \cdot V\left(z_0 - p_0, \ldots, z_{i-1} - p_{i-1}\right)\right)_p.$$

Moreover, whenever such analytic cycles $\Lambda^i_{F^\bullet,z}$ exist, they are unique. He also sets $\lambda^i_{F^\bullet,z}(p) = \left(\Lambda^i_{F^\bullet,z} \cdot V\left(z_0 - p_0, \ldots, z_{i-1} - p_{i-1}\right)\right)_p$ and calls it the i-th characteristic polar multiplicity $F^\bullet$. When $\beta(p) = \chi\left(F^\bullet\right)_p$ we also deno $\Lambda^i_{F^\bullet,z}$ by $\Lambda^i_{\beta,z}$.

In [8, Corollary 10.15] was proved that, for a generic choice of linear coordinates $z = (z_0, \ldots, z_n)$, if we let L^i be the i-dimensional linear subspace $V\left(z_0, \ldots, z_{n-i}\right)$ then,

$$\Lambda^k_{F^\bullet,z} = \sum_\alpha m_\alpha P_k\left(\overline{S_\alpha}\right) = \sum_\alpha (-1)^{s-d_\alpha} \eta\left(S_\alpha, F^\bullet\right) P_k\left(\overline{S_\alpha}\right). \tag{2.2}$$

where $P_k\left(\overline{S_\alpha}\right)$ is the absolute affine k-dimensional polar variety, with respect to the flag given by the L^i above, as defined by Lê and Teissier in [5]. We are going to define these affine polar varieties later on.

Remark 2.7. By [8, Remark 10.5, Remark 10.7] it follows that if we have $h : (U, 0) \to (\mathbb{C}, 0)$ with U an open neighborhood of the origin in $\mathbb{C}^{n+1}$, $X = \Sigma(h)$ the critical set of h, $s = \dim_0 X$ and we let

$$P^\bullet = \left(\phi_h \mathbb{C}^\bullet_U\right)_{|_{\Sigma(h)}}[n-s],$$

then for generic linear coordinates z, for all i and for all $p \in X$ near the origin, we have $\Lambda^i_{P^\bullet,z} = \Lambda^i_{h,z}$ and $\lambda^i_{P^\bullet,z} = \lambda^i_{h,z}(p)$. Also

$$m_\alpha = (-1)^{s-d_\alpha}\eta\left(S_\alpha, P^\bullet\right) = (-1)^{s-d_\alpha}\eta\left(S_\alpha, w\right),$$

where w is the constructible function defined by $w(p) = \chi\left(P^\bullet\right)_p = \chi\left(F_{h,p}\right) - 1$ with $F_{h,p}$ being the Milnor fiber of h at p. Hence, by equation (2.2) we have that

$$\Lambda^i_{h,z} = \sum_\alpha (-1)^{s-d_\alpha}\eta\left(S_\alpha, w\right) P_i\left(\overline{S_\alpha}\right).$$

This is the description of the local Lê cycles in terms of local polar varieties we need in order to define the global Lê cycles for compact projective varieties.

3 Affine Lê cycles

In the affine context, Schürmann and Tibăr in [12] describe the Schwartz-MacPherson classes of a complex algebraic proper subset $X \subset \mathbb{C}^N$ using algebraic cycles, which were called MacPherson cycles. In this construction a key role is played by the affine polar varieties, which we now describe (see [5]).

Definition 3.1. For each $0 \leq i \leq N$, let L_i be a linear subvariety of $\mathbb{C}^N$ of codimension i. If X is of pure dimension $d < N$, the **k-th affine polar variety** of X, with $0 \leq k \leq d$, is the following algebraic set

$$P_k\left(X, L_{k+1}\right) := \overline{\left\{x \in X_{reg} \mid \dim\left(T_x X_{reg} \cap L_{k+1}\right) \geq d-k\right\}}.$$

For L_{k+1} sufficiently generic, the polar variety $P_k(X, L_{k+1})$ has pure dimension k. We have $P_d(X) := X$ and we set $P_k(X) := \emptyset$ for $k > d$.

We fix an algebraic Whitney stratification $\{S_\alpha\}$ of X with connected strata. In this context X does not need to be pure dimensional and we only assume $d = \dim X < N$. Let β be a constructible function on X with respect to this Whitney stratification.

Schürmann and Tibăr make the following definition.

Definition 3.2. The ***k*-th MacPherson cycle** of β $(0 \leq k \leq d)$ by:

$$MP_k(\beta, L_{k+1}) := \sum_\alpha (-1)^{d_\alpha} \eta(S_\alpha, \beta) P_k\left(\overline{S_\alpha}, L_{k+1}\right),$$

where $d_\alpha = \dim S_\alpha$ and $P_k\left(\overline{S_\alpha}, L_{k+1}\right)$ is the k-th global affine polar variety of the algebraic closure $\overline{S_\alpha} \subset \mathbb{C}^N$ of the stratum S_α.

The most important result of [12] is that, for generic L_{k+1}, the cycle $MP_k(\beta, L_{k+1})$ represents the k-th dual Schwartz-MacPherson class $\check{c}_k^{SM}(\beta)$ in the Chow group $A_k(X)$, where $\check{c}_k^{SM}(\beta) = (-1)^k c_k^{SM}(\beta)$. That is,

$$c_k^{SM}(\beta) = (-1)^k [MP_k(\beta)] = (-1)^k \sum_\alpha (-1)^{d_\alpha} \eta(S_\alpha, \beta) \left[P_k\left(\overline{S_\alpha}\right)\right]. \quad (3.1)$$

Hence, this way, Schürmann and Tibăr describe the Schwartz-MacPherson classes via affine polar varieties.

Definition 3.3. We define the ***k*-th affine Lê cycle** of β by

$$\Lambda_k^{\mathbb{A}}(\beta, L_{k+1}) := \sum_\alpha (-1)^{d-d_\alpha} \eta(S_\alpha, \beta) P_k\left(\overline{S_\alpha}, L_{k+1}\right).$$

Notice that $\Lambda_k^{\mathbb{A}}(\beta, L_{k+1}) = (-1)^d MP_k(\beta, L_{k+1})$. Hence, by equation (3.1) we have that

$$c_k^{SM}(\beta) = (-1)^{k+d} \left[\Lambda_k^{\mathbb{A}}(\beta)\right] = (-1)^{k+d} \sum_\alpha (-1)^{d-d_\alpha} \eta(S_\alpha, \beta) \left[P_k\left(\overline{S_\alpha}\right)\right]. \quad (3.2)$$

An interesting feature of these affine Lê cycles of X is that they are a global extension of the Lê cycles defined by Massey:

Proposition 3.4. *([2, Proposition 7.5.3]) Let X be a closed subvariety of $\mathbb{C}^N$ and let β be a constructible function on X with respect to a Whitney stratification $\{S_\alpha\}$ of X. Let $x \in X$ and let $U \subseteq \mathbb{C}^N$ be an open neighborhood of x. Let $\{x\} = L_N \subset L_{N-1} \subset \cdots \subset L_1 \subset L_0 = \mathbb{C}^N$ be a generic flag of linear subvarieties of $\mathbb{C}^N$ with L_i being of codimension i and such that $L_i \cap U = V(z_0, \ldots, z_{i-1})$ where $z = (z_0, \ldots, z_{N-1})$ is a generic linear coordinates around x. Let $\iota : U \cap X \longrightarrow \mathbb{C}^N$ be the inclusion. Then, the flat pull-back of the affine Lê cycles satisfies the following property*

$$\iota^* \Lambda_k^{\mathbb{A}}(\beta, L_{k+1}) = \Lambda^k_{\iota^*(\beta), z}.$$

Proof. In fact,

$$\begin{aligned}
\iota^* \Lambda_k^{\mathbb{A}}(\beta, L_{k+1}) &= \iota^* \left(\sum_\alpha (-1)^{d-d_\alpha} \eta(S_\alpha, \beta) P_k\left(\overline{S_\alpha}, L_{k+1}\right) \right) \\
&= \sum_\alpha (-1)^{d-d_\alpha} \eta(S_\alpha, \beta)\, \iota^* \left(P_k\left(\overline{S_\alpha}, L_{k+1}\right)\right) \\
&= \sum_\alpha (-1)^{d-d_\alpha} \eta(S_\alpha \cap U, \iota^*(\beta))\, P_k\left(\overline{S_\alpha \cap U}\right) = \Lambda^k_{\iota^*(\beta), z}.
\end{aligned}$$

□

4 Projective Lê cycles

Let X be a complex analytic space in $\mathbb{C}P^N$ of pure dimension d. For each $0 \le k \le N$, let L_k be a linear subspace of $\mathbb{C}P^N$ codimension k.

Definition 4.1. The k**-th projective polar variety** of X, with respect to L_{k+2}, is defined by

$$\mathbb{P}_k(X, L_{k+2}) := \overline{\{x \in X_{reg} \mid \dim(T_x X_{reg} \cap L_{k+2}) \ge d - k - 1\}},$$

where $T_x X_{\mathrm{reg}}$ is the projective tangent space of X at a regular point x.

We observe that for L_{k+2} sufficiently general, the dimension of $\mathbb{P}_k(X, L_{k+2})$ is equals to k. Thus, we are indexing the polar varieties by their dimension and not by their codimension, as it is usually done.

Also observe that the class $[\mathbb{P}_k(X, L_{k+2})]$ of $\mathbb{P}_k(X, L_{k+2})$ modulo rational equivalence in the Chow group $A_k(X)$ does not depend on L_{k+2} provided this is sufficiently general. This class is denoted by $[\mathbb{P}_k(X)]$ and it is called the ***k*-th projective polar class** of X.

Remark 4.2. For any subvariety Z of $\mathbb{C}P^N$ we denote by Cone(Z) the cone in $\mathbb{C}^{N+1}$ induced by Z. Analogously, for any conical subvariety through the origin V of $\mathbb{C}^{N+1}$ we denote by $\mathbb{P}(V)$ the induced projective variety in $\mathbb{C}P^N$. Let X be a subvariety of $\mathbb{C}P^N$ and let L_{k+2} be a linear subvariety of $\mathbb{C}P^N$ of codimension $k+2$. In this case, Cone(L_{k+2}) is a linear subspace of codimension $k+2$ in $\mathbb{C}^{N+1}$ and $P_{k+1}(\text{Cone}(X), \text{Cone}(L_{k+2}))$ is a conical subvariety of $\mathbb{C}^{N+1}$ of dimension $k+1$. The relationship between the projective and the affine polar varieties is given by

$$\mathbb{P}_k(X, L_{k+2}) = \mathbb{P}(P_{k+1}(\text{Cone}(X), \text{Cone}(L_{k+2}))).$$

Definition 4.3. For any given $F^\bullet \in \mathcal{D}^b_{\mathcal{S}}(X)$, where $\mathcal{S} = \{S_\alpha\}$ is a Whitney stratification of X, define the ***k*-th projective Lê cycle**, with respect to L_{k+2}, by

$$\Lambda^{\mathbb{P}}_k(F^\bullet, L_{k+2}) := \sum_\alpha (-1)^{d-d_\alpha} \eta(S_\alpha, F^\bullet)\, \mathbb{P}_k\left(\bar{S}_\alpha, L_{k+2}\right),$$

where $d_\alpha = \dim S_\alpha$.

Hence, the class of this cycle in the Chow group $A_k(X)$ does not depend on L_{k+2} provided this is sufficiently general. This class is denoted by $\left[\Lambda^{\mathbb{P}}_k(F^\bullet)\right]$.

If β is the constructible function associated to $F^\bullet$ as in Remark 2.6 we also denote this cycle $\Lambda^{\mathbb{P}}_k(F^\bullet, L_{k+2})$ by $\Lambda^{\mathbb{P}}_k(\beta, L_{k+2})$ and the class $\left[\Lambda^{\mathbb{P}}_k(F^\bullet)\right]$ by $\left[\Lambda^{\mathbb{P}}_k(\beta)\right]$. That is,

$$\Lambda^{\mathbb{P}}_k(\beta, L_{k+2}) := \sum_\alpha (-1)^{d-d_\alpha} \eta(S_\alpha, \beta)\, \mathbb{P}_k\left(\bar{S}_\alpha, L_{k+2}\right).$$

The next result is going to relate the affine and projective Lê cycles, the main result of this paper. This is only mentioned in [2, Proposition 7.5.5].

Theorem 4.4. *Let $X \subseteq \mathbb{C}P^N$ be a d-dimensional projective variety endowed with a Whitney stratification $\mathcal{S} = \{S_\alpha\}$ with connected strata. Let L_{k+2} be a linear subvariety of $\mathbb{C}P^N$ of codimension $k+2$. Let $\pi : \mathbb{C}^{N+1}\backslash\{0\} \longrightarrow \mathbb{C}P^N$ be the natural projection. Let β be a constructible function on X, with respect to this stratification. Then*

1. $\mathcal{S}' := \left\{\pi^{-1}(S_\alpha)\right\} \cup \{\{0\}\}$ *is a Whitney stratification of* $\operatorname{Cone}(X)$.
2. β *induces a constructible function* $\tilde{\beta}$ *on* $\operatorname{Cone}(X)$ *with respect to the Whitney stratification* $\mathcal{S}'$.
3. $\Lambda_k^{\mathbb{P}}(\beta, L_{k+2}) = \mathbb{P}\left(\Lambda_{k+1}^{\mathbb{A}}\left(\tilde{\beta}, \operatorname{Cone}(L_{k+2})\right)\right)$.

Proof. Item (1) is easily verified. For (2), define $\tilde{\beta}(x) = \beta(\pi(x))$ if $x \neq 0$ and $\tilde{\beta}(0) = 0$. Then clearly $\tilde{\beta}$ is a constructible function on $\operatorname{Cone}(X)$ with respect to the Whitney stratification $\mathcal{S}'$. We prove now (3). Since $P_{k+1}(\{0\}) = \emptyset$ we have that

$$\Lambda_{k+1}^{\mathbb{A}}\left(\tilde{\beta}, \operatorname{Cone}(L_{k+2})\right) =$$

$$\sum_\alpha (-1)^{d-d_\alpha} \eta\left(\pi^{-1}(S_\alpha), \tilde{\beta}\right) P_{k+1}\left(\overline{\pi^{-1}(S_\alpha)}, \operatorname{Cone}(L_{k+2})\right).$$

But, by Remark 4.2, we have that

$$\mathbb{P}\left(P_{k+1}\left(\overline{\pi^{-1}(S_\alpha)}, \operatorname{Cone}(L_{k+2})\right)\right) = \mathbb{P}_k\left(\bar{S}_\alpha, L_{k+2}\right).$$

Thus

$$\mathbb{P}\left(\Lambda_{k+1}^{\mathbb{A}}\left(\tilde{\beta}, \operatorname{Cone}(L_{k+2})\right)\right) = \sum_\alpha (-1)^{d-d_\alpha} \eta\left(\pi^{-1}(S_\alpha), \tilde{\beta}\right) \mathbb{P}_k\left(\bar{S}_\alpha, L_{k+2}\right).$$

Since

$$\Lambda_k^{\mathbb{P}}(\beta, L_{k+2}) = \sum_\alpha (-1)^{d-d_\alpha} \eta(S_\alpha, \beta)\, \mathbb{P}_k\left(\bar{S}_\alpha, L_{k+2}\right)$$

it remains to prove that $\eta\left(\pi^{-1}(S_\alpha), \tilde{\beta}\right) = \eta(S_\alpha, \beta)$. Let $x \in \pi^{-1}(S_\alpha)$. We can choose a normal slice N to $\pi^{-1}(S_\alpha)$ at x such that $\pi|_N : N \longrightarrow \pi(N)$ is an isomorphism and $\pi(N)$ is a normal slice to S_α at $\pi(x)$. Let

$g : \left(\mathbb{C}P^N, \pi(x)\right) \longrightarrow (\mathbb{C}, 0)$ be a non-degenerate covector at $\pi(x)$ with respect to the stratification $\mathcal{S}$. Clearly $\pi \circ g : \left(\mathbb{C}^{N+1}, x\right) \longrightarrow (\mathbb{C}, 0)$ is a non-degenerate covector at x with respect to the stratification $\mathcal{S}'$.

Let $\{T_\gamma\}$ be a Whitney stratification of $X \cap B_\delta(\pi(x)) \cap \pi(N)$. Then, $\left\{\pi^{-1}\left(T_\gamma\right) \cap \ B_\epsilon(x)\right\}$, with $\epsilon \ll \delta$, is a Whitney stratification of $\mathrm{Cone}(X) \cap B_\epsilon(x) \cap N$. Hence

$$\begin{aligned}\chi\left(\mathrm{Cone}(X) \cap B_\epsilon(x) \cap N, \tilde{\beta}\right) &= \sum_\gamma \tilde{\beta}\left(\pi^{-1}\left(T_\gamma\right) \cap B_\epsilon(x)\right) \chi\left(\pi^{-1}\left(T_\gamma\right) \cap B_\epsilon(x)\right) \\ &= \sum_\gamma \beta\left(T_\gamma\right) \chi\left(T_\gamma\right) \\ &= \chi\left(X \cap B_\delta(\pi(x)) \cap \pi(N)\right).\end{aligned}$$

Analogously we can prove that $\chi\left(\mathrm{Cone}(X) \cap B_\epsilon(x) \cap N \cap \{\pi \circ g = w\}, \tilde{\beta}\right)$ $= \chi\left(X \cap B_\delta(\pi(x)) \cap \pi(N) \cap \{g = w\}, \beta\right)$, which ends the proof. □

The following result could be seen as a projective version of equation (3.2).

Proposition 4.5. *Let X be a projective variety endowed with a Whitney stratification with connected strata S_α. Consider $\varphi : X \to \mathbb{C}P^N$ a closed immersion and $\mathcal{L} = \mathcal{O}_{\mathbb{C}P^N}(1)$. If $\beta : X \to \mathbb{Z}$ is a constructible function with respect to this stratification, then*

$$c_k^{SM}(\beta) = \sum_{i \geq k} (-1)^{d-i} \begin{pmatrix} i+1 \\ k+1 \end{pmatrix} c_1\left(\varphi^*\mathcal{L}\right)^{i-k} \cap \left[\Lambda_i^{\mathbb{P}}(\beta)\right].$$

Proof. For any purely dimensional projective variety V of dimension d we have, by R. Piene's work [10], the following characterization of the Mather classes via polar varieties:

$$c_k^{Ma}(V) = \sum_{i=k}^{d} (-1)^{d-i} \begin{pmatrix} i+1 \\ k+1 \end{pmatrix} c_1\left(\varphi^*\mathcal{L}\right)^{i-k} \cap \left[\mathbb{P}_i(V)\right]. \tag{4.1}$$

Since $\beta = \sum_\alpha \eta\left(S_\alpha, \beta\right) Eu_{\bar{S}_\alpha}$, where $Eu_{\bar{S}_\alpha}$ is the local Euler obstruction function of $\bar{S}_\alpha$ as defined by MacPherson [6], we have that

$$c_k^{SM}(\beta) = \sum_\alpha \eta\left(S_\alpha, \beta\right) c_k^{Ma}\left(\bar{S}_\alpha\right). \tag{4.2}$$

Hence, by equations (4.1) and (4.2) we have

$$c_k^{SM}(\beta) = \sum_{\alpha} \eta\left(S_\alpha, \beta\right) \sum_{i=k}^{d_\alpha} (-1)^{d_\alpha - i} \begin{pmatrix} i+1 \\ k+1 \end{pmatrix} c_1\left(\varphi^* \mathcal{L}\right)^{i-k} \cap \left[\mathbb{P}_i\left(\bar{S}_\alpha\right)\right]$$
$$= \sum_{i \geq k} (-1)^{d-i} \begin{pmatrix} i+1 \\ k+1 \end{pmatrix} c_1\left(\varphi^* \mathcal{L}\right)^{i-k} \cap \left[\Lambda_i^{\mathbb{P}}(\beta)\right].$$

□

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