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Functions on a swallowtail

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Abstract.

We classify submersions from $(\mathbb{R}^3, 0)$ to $(\mathbb{R}, 0)$ up to diffeomorphisms which preserve the swallowtail and use this classification to study its flat geometry. The flat geometry is derived from the contact of the swallowtail with planes, which is measured by the singularities of the height function.

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1 Introduction

A *swallowtail* is the image of a germ $g : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ that is $\mathcal{A}$-equivalent to $f(x, y) = (x, -4y^3 - 2xy, 3y^4 + xy^2)$, that is, there exist germs of diffeomorphisms ϕ and ψ such that $g = \psi \circ f \circ \phi^{-1}$. We refer to the swallowtail parametrised by f as the *standard swallowtail* (see Figure 1.1) and to the swallowtail parametrised by any g as the *geometric swallowtail*. In [38] a normal form of a geometric swallowtail obtained using changes of coordinates in the source and isometries in the target is given.

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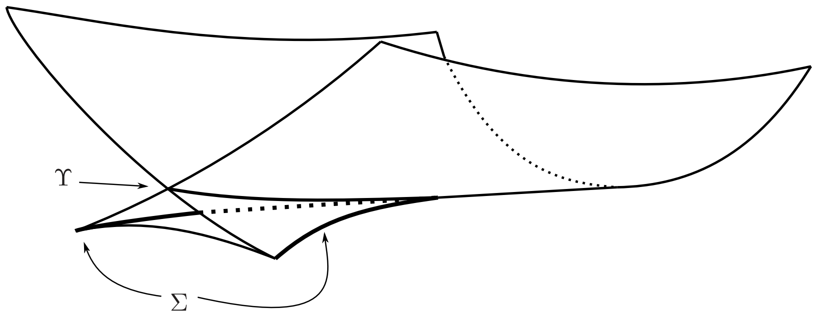

Figure 1.1: The swallowtail, its singular curve Σ and its double point curve Υ.

Swallowtail singularities arise in a natural way. For instance, the focal sets, duals and discriminants of curves and surfaces in the Euclidean space $\mathbb{R}^3$ can have swallowtail singularities (see for example [3], [6], [11], [39]). Hence it is important to study their differential geometry.

In this paper, we classify germs of submersions $f : (\mathbb{R}^3, 0) \to (\mathbb{R}, 0)$ up to diffeomorphisms in the source which preserve the swallowtail. We note that this classification in a more general context was stated in [8], but without the proof in details, this is done here. Part of this classification can be seen as a consequence of the results in [1] for the swallowtail in $\mathbb{C}^3$. We also study the flat geometry of a swallowtail which is derived from its contact with planes (flat objects). This contact is measured by the singularities of the height functions on the swallowtail.

This work is part of an ongoing study of the geometry of singular surfaces from Singularity Theory viewpoint (see for example [10], [14], [18], [20], [19], [21], [31], [32], [35], [41], [44] for cross-cap, [11], [23], [26], [27], [30], [36], [39], [42], [45] for cuspidal edge, [38] for swallowtail, [34] for cuspidal cross-cap and [25], [37] for corank 1 singularity).

We follow the approach in [10] : we fix the standard swallowtail $X = f(\mathbb{R}^2, 0)$ and consider its contact with fibres of submersions. (See §3.3 for details)

The paper is organized as follows. In §2 we give some concepts and results on classification of germs of functions on an analytic variety. In

§3 we give some properties of the standard swallowtail and classify submersions from $(\mathbb{R}^3, 0)$ to $(\mathbb{R}, 0)$ up to changes of coordinates in the source that preserves the standard swallowtail, in §4 we obtain the discriminants of versal unfoldings of each normal form obtained in the classification and analyze the contact between the zero fiber and the standard swallowtail in each case. We use in §5 the classification in §3 to study the flat geometry of a geometric swallowtail.

For background material on singularity theory we refer the reader to [24], [29], [43] and on its application in the differential geometry to [7], [22].

This paper is part of the PhD Thesis work of the author under supervision of Farid Tari. For more details see [16], [15].

2 Functions on analytic varieties

In this section we review some concepts and results from [8], [9], [10], [12] and [36] which are useful tools for classifying functions on analytic varieties.

Let $\mathcal{E}_n$ be the local ring of germs of smooth functions $(\mathbb{R}^n, 0) \to \mathbb{R}$ and $\mathcal{M}_n$ its unique maximal ideal.

Let $(X, 0) \subset (\mathbb{R}^n, 0)$ be a germ of a reduced analytic subvariety of $\mathbb{R}^n$ at 0. We say that a germ of diffeomorphism $\varphi : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ *preserves* X if $\varphi(X)$ and X are equal as germs at 0, that is, $(\varphi(X), 0) = (X, 0)$. The set of such diffeomorphisms forms a subgroup of the group of all diffeomorphisms in $(\mathbb{R}^n, 0)$ (the group $\mathcal{R}$) and is denoted by $\mathcal{R}(X)$.

Given two germs $f, g \in \mathcal{E}_n$, we say that they are $\mathcal{R}(X)$-*equivalents* if there exists a germ of diffeomorphism $\varphi \in \mathcal{R}(X)$ such that $g \circ \varphi^{-1} = f$.

We denote by $\Theta(X)$ the $\mathcal{E}_n$-module of germs of vector fields tangent to X at 0. We define $\Theta(X) \cdot f = \{\xi \cdot f \in \mathcal{E}_n \,|\, \xi \in \Theta(X)\,, \xi(0) = 0\}$, which is an $\mathcal{E}_n$-module.

Let $\Theta_1(X) = \{\xi \in \Theta(X) \,|\, j^1\xi = 0\}$ which is an $\mathcal{E}_n$-module. If we integrate the vector fields in $\Theta_1(X)$ we obtain a group denoted by $\mathcal{R}_1(X)$, which is the set of germs of diffeomorphisms in $\mathcal{R}(X)$ with 1-jets is the identity. We also can define the subgroup $\mathcal{R}_k(X)$ of germ of diffeomorphism at $\mathcal{R}(X)$ with k-jets is the identity. It is a normal subgroup of $\mathcal{R}(X)$ and, consequentially, we can define the group $\mathcal{R}^{(k)}(X) = \dfrac{\mathcal{R}(X)}{\mathcal{R}_k(X)}$. The elements of $\mathcal{R}^{(k)}(X)$ are k-jets of elements of $\mathcal{R}(X)$. The action of $\mathcal{R}(X)$ on $\mathcal{M}_n$ induces an smooth action of the group $\mathcal{R}^{(k)}(X)$ on the k-jet space of function germs $J^k(n,1)$.

For $f \in \mathcal{E}_n$ the tangent spaces to the $\mathcal{R}(X)$ and $\mathcal{R}_1(X)$-orbits of f are, respectively

$$L\mathcal{R}(X) \cdot f = \Theta(X) \cdot f \quad \text{and} \qquad L\mathcal{R}_1(X) \cdot f = \Theta_1(X) \cdot f.$$

The tools for classifying germs of functions $(\mathbb{R}^n, 0) \to \mathbb{R}$, up to the $\mathcal{R}(X)$-equivalence, are generalizations of the classical results about the action of $\mathcal{R}$ over $\mathcal{E}_n$. The group $\mathcal{R}(X)$ is a Damon's geometric subgroup (for more details, see [12] and [13]), so the theorems on versal deformations and finite determinacy apply to this setting.

Definition 2.1. A germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ is k-$\mathcal{R}(X)$-*determined* if every germ of a function with the same k-jet as f is $\mathcal{R}(X)$-equivalent to f. We say that f is $\mathcal{R}(X)$-*finitely determined* if f is k-$\mathcal{R}(X)$-determined for same $k \in \mathbb{N}^*$.

Theorem 2.2. ([12]) *Consider a germ* $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$*. If there exists* $k \in \mathbb{N}^*$*, such that*

$$\mathcal{M}_n^k \subset L\mathcal{R}(X) \cdot f,$$

then f *is* $(k+1)$-$\mathcal{R}(X)$*-determined.*

We define the *extended pseudo-group* of diffeomorphisms preserving X, denoted by $\mathcal{R}_e(X)$, as being the pseudo-group obtained by integrating the vector fields $\xi \in \Theta(X)$, but excluding the condition $\xi(0) = 0$. Hence, for $f \in \mathcal{E}_n$ the extended tangent space to the $\mathcal{R}_e(X)$-orbit of f is $L\mathcal{R}_e(X) \cdot f = \{\xi \cdot f \in \mathcal{E}_n \,|\, \xi \in \Theta(X)\}$.

Note that when X is a swallowtail, every vector field vanishes at the origin, that is, in this case $\mathcal{R}_e(X) = \mathcal{R}(X)$.

The $\mathcal{R}(X)$-classification of finitely determined germs is carried out inductively on the jet level. The method used here is that of the complete transversal [8] adapted for the $\mathcal{R}(X)$-action in [10].

Theorem 2.3. ***Complete Transversal*** *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a smooth germ and $\{h_1, ..., h_r\}$ a collection of homogeneous polynomials of degree $k+1$ such that*

$$\mathcal{M}_n^{k+1} \subset L\mathcal{R}_1(X) \cdot f + \mathbb{R} \cdot \{h_1, ..., h_r\} + \mathcal{M}_n^{k+2}.$$

Then any germ $g : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ with $j^k g(0) = j^k f(0)$ is $\mathcal{R}_1(X)$-equivalent to a germ of the form

$$f(x) + \sum_{i=1}^{r} \lambda_i h_i(x) + \varphi(x),$$

where $\varphi(x) \in \mathcal{M}_n^{k+2}$ and $\lambda_i \in \mathbb{R}$. The real vector space $T = \mathbb{R} \cdot \{h_1, ..., h_r\}$ is called by a complete $(k+1)$-transversal of f.

Proposition 2.4. (i) *A germ $f \in \mathcal{M}_n$ is k-$\mathcal{R}_1(X)$-determined if and only if*

$$\mathcal{M}_n^{k+1} \subset L\mathcal{R}_1(X) \cdot f + \mathcal{M}_n^{k+2}.$$

(ii) *In particular, if every vector field in $\Theta(X)$ vanishes at the origin and*

$$\mathcal{M}_n^{k+1} \subset L\mathcal{R}(X) \cdot f + \mathcal{M}_n^{k+2},$$

then f is $(k+1)$-$\mathcal{R}(X)$-determined.

Proof. This is a consequence of Theorem 2.3 and Theorem 2.5 in [5] applied to our setting. □

An *s-parameter deformation of $f \in \mathcal{E}_n$* is a family of germs of functions $F : (\mathbb{R}^n \times \mathbb{R}^s, (0, 0)) \to (\mathbb{R}, 0)$ such that $F_0(x) = F(x, 0) = f(x)$. An s-parameter deformation F is said to be *P-$\mathcal{R}^+(X)$-induced* from an r-parameter deformation G if there exist a germ $\phi : (\mathbb{R}^n \times \mathbb{R}^s, (0,0)) \to (\mathbb{R}^n \times \mathbb{R}^r, (0,0))$ of the form $\phi(x,u) = (\varphi(x,u), \psi(u))$ and a germ of a function $c : (\mathbb{R}^s, 0) \to \mathbb{R}$ such that $F(x,u) = G(\phi(x,u)) + c(u)$. When ϕ is a germ of a diffeomorphism we say that F and G are P-$\mathcal{R}^+(X)$-*equivalent* (see for example [7] for the notion of (p)-unfoldings).

We say that a deformation F of f is an $\mathcal{R}^+(X)$-*versal deformation of* f if any other deformation of f is P-$\mathcal{R}^+(X)$-induced from F.

Proposition 2.5. ([10]) *An s-parameter deformation F of a germ of a function f on X is an $\mathcal{R}^+(X)$-versal deformation if and only if*

$$L\mathcal{R}_e(X) \cdot f + \mathbb{R} \cdot \{1, \dot{F}_1, ..., \dot{F}_s\} = \mathcal{E}_n,$$

where $\dot{F}_i = \frac{\partial F}{\partial u_i}(x, 0)$, *for* $i = 1, ..., s$.

We define the $\mathcal{R}_e^+(X)$-codimension of f as $\mathrm{cod}(\mathrm{f}, \mathcal{R}_\mathrm{e}^+(\mathrm{X})) = \dim_{\mathbb{R}}(\dfrac{\mathcal{M}_\mathrm{n}}{\mathrm{L}\mathcal{R}_\mathrm{e}(\mathrm{X}) \cdot \mathrm{f}})$. It is the least number of parameters needed to have an $\mathcal{R}^+(X)$-versal deformation of f.

Another important tool in the classification is Mather's Lemma.

Lemma 2.6. ([28]) ***Mather's Lemma*** *Let $\alpha : G \times M \to M$ be a smooth action of a Lie group G over a smooth manifold M, and let V be a connected submanifold of M. Then the necessary and sufficient conditions for V been in a single orbit are the following:*

(i) $T_v V \subset T_v(G.v)$, *for every* $v \in V$.

(ii) $\mathrm{dim}(\mathrm{T_v}(\mathrm{G.v}))$ *is independent of* $v \in V$.

3 Classification of functions on a swallowtail

In this section, we shall use the results in §2 to classify smooth germs of functions from $(\mathbb{R}^3, 0) \to (\mathbb{R}, 0)$ up to changes of coordinates in the source which preserve the standard swallowtail. Note that when X is a swallowtail, every vector field vanishes at the origin, that is, in this case $\mathcal{R}_e(X) = \mathcal{R}(X)$.

We consider here X been the standard swallowtail parametrised by $f(x,y) = (x, -4y^3 - 2xy, 3y^4 + xy^2)$ or with equation

$$16u^4w - 4u^3v^2 - 128u^2w^2 + 144uv^2w - 27v^4 + 256w^3 = 0.$$

We called this germs *functions on a swallowtail.* Note that the function f is a parametrisation of the discriminant set of the $\mathcal{R}$-versal deformation $F(t, u_0, u_1, u_2) = t^4 + u_2t^2 + u_1t + u_0$ of the A_3-singularity t^4.

Proposition 3.1. ([4]) *The $\mathcal{E}_3$-module of germs at the origin of vector fields in $\mathbb{R}^3$ tangents to the standard swallowtail is generated by the vector fields θ_1, θ_2 and θ_3 with*

$$\theta_1 = 2u\frac{\partial}{\partial u} + 3v\frac{\partial}{\partial v} + 4w\frac{\partial}{\partial w},$$

$$\theta_2 = 6v\frac{\partial}{\partial u} + (8w - 2u^2)\frac{\partial}{\partial v} - uv\frac{\partial}{\partial w},$$

$$\theta_3 = (16w - 4u^2)\frac{\partial}{\partial u} - 8uv\frac{\partial}{\partial v} - 3v^2\frac{\partial}{\partial w}.$$

Integrating the linear parts of $\theta_1, \theta_2, \theta_3$ in Proposition 3.1, gives the followings 1-jets of changes of coordinate in $\mathcal{R}(X)$

$$h_1(u,v,w) = (e^{2\lambda}u, e^{3\lambda}v, e^{4\lambda}w),$$

$$h_2(u,v,w) = (u + 3\beta v, v + 4\gamma w, w),$$

$$h_3(u,v,w) = (u + \alpha w, v, w),$$

with $\alpha, \beta, \gamma, \lambda \in \mathbb{R}$.

Consider the 1-jet $j^1f = au + bv + cw$ of a submersion f, with a, b or c non-zero.

Proposition 3.2. *The $\mathcal{R}^{(1)}(X)$-orbits of submersions $f : (\mathbb{R}^3, 0) \to (\mathbb{R}, 0)$ are $\pm u$, v, $\pm w$.*

Proof. The proof immediately follows considering the 1-jets of diffeomorphisms in $\mathcal{R}(X)$. In order to get the 1-jet v we use the diffeomorphism $(u,v,w) \to (u,-v,w)$ which also preserves the swallowtail. $\square$

Now we investigate each case in Proposition 3.2.

Lemma 3.3. *The germ* $g(u,v,w) = \pm u$ *is* 1-$\mathcal{R}(X)$*-determined and has* $\mathcal{R}_e^+(X)$*-codimension* 0.

Proof. We have

$$L\mathcal{R}(X) \cdot g = \mathcal{E}_3 \cdot \{u, v, 4w - u^2\} = \mathcal{M}_3$$

and the result follows. $\square$

Lemma 3.4. *Any* $\mathcal{R}(X)$*-finitely determined germ in* $\mathcal{E}_3$ *with* 1*-jet* $\mathcal{R}^{(1)}(X)$*-equivalent to* v *is* $\mathcal{R}(X)$*-equivalent to* $v + au^{k+1}$ *for some* $k \geq 1$ *and* $a \neq 0$*. The germ* $v + au^{k+1}$*,* $a \neq 0$*, is* $(k+1)$-$\mathcal{R}(X)$*-determined and has* $\mathcal{R}_e^+(X)$*-codimension* k*.*

Proof. Observe that the germ v is not $\mathcal{R}(X)$-finitely determined. We proceed by induction on the k-jets ($k \geq 1$) of germs g with 1-jet v.

Firstly, we find a complete $(k+1)$-transversal of $g(u,v,w) = v$.

Note that

$$L\mathcal{R}_1(X) \cdot g = \mathcal{M}_3 \cdot \{v, 4w - u^2\} = \mathcal{E}_3 \cdot \{uv, v^2, vw, 4uw - u^3, 4w^2 - u^2 w\}.$$

Hence,

$$\mathcal{M}_3^{(k+1)} \subset L\mathcal{R}_1(X) \cdot g + \mathbb{R} \cdot \{u^{k+1}\} + \mathcal{M}_3^{(k+2)},$$

so $T = \mathbb{R} \cdot \{u^{k+1}\}$ is a complete $(k+1)$-transversal of g. Then, by Theorem 2.3, any $(k+1)$-jet with k-jet equal to v is $\mathcal{R}_1(X)$-equivalent to $v + au^{k+1}$, $a \in \mathbb{R}$.

For $a \neq 0$, using Proposition 2.4, we can conclude that the germ $\overline{g}(u,v,w) = v + au^{k+1}$ is $(k+2) - \mathcal{R}(X)$-determined. However, we can use Theorem 2.3 and Lemma 2.6 to conclude that $\overline{g}$ is, in fact, $(k+1)$-$\mathcal{R}(X)$-determined.

We have $\dfrac{\mathcal{M}_3}{L\mathcal{R}(X)\cdot\overline{g}} = \mathbb{R}\cdot\{u, u^2, ..., u^k, u^{k+1}\}$ which implies that the $\mathcal{R}_e^+(X)$-codimension of $\overline{g}$ is $k+1$ and the codimension of the stratum of this singularity is k. □

Lemma 3.5. *Any $\mathcal{R}(X)$-finitely determined germ in $\mathcal{E}_3$ with 1-jet $\mathcal{R}(X)$-equivalent to $\pm w$ and $\mathcal{R}_e^+(X)$-codimension ≤ 2 is $\mathcal{R}(X)$-equivalent to $\pm w + au^2 + bu^3$, with $a \neq 0, \pm\frac{1}{12}, \pm\frac{1}{4}$ and $b \neq 0$. Furthermore, the germ $\pm w + au^2 + bu^3$, with a and b in the previous conditions, is 3-$\mathcal{R}(X)$-determined and has $\mathcal{R}_e^+(X)$-codimension 2 (on the stratum).*

Proof. For $g(u,v,w) = \pm w$ we have

$$L\mathcal{R}(X)\cdot g = \mathcal{E}_3\cdot\{w, uv, v^2\},$$

so g is not $\mathcal{R}(X)$-finitely determined. We proceed by induction on the k-jets of germs with 1-jet $\pm w$.

Note that

$$\mathcal{M}_3^2 \subset L\mathcal{R}_1(X)\cdot g + \mathbb{R}\cdot\{u^2, uv, v^2\} + \mathcal{M}_3^3,$$

so $T = \mathbb{R}\cdot\{u^2, uv, v^2\}$ is a complete 2-transversal of g. Hence, any 2-jet with 1-jet equal to $\pm w$ is $\mathcal{R}_1(X)$-equivalent to $g(u,v,w) = \pm w + au^2 + buv + cv^2$, with $a, b, c \in \mathbb{R}$.

When $a \neq 0$, using the linear change of coordinates h_2 with $\gamma = 0$ and $\beta = \frac{-b}{6a}$, we obtain $\overline{g}(u,v,w) = g(h_2(u,v,w)) = \pm w + au^2 + c'v^2$. For each a fixed, denote by V the set $\{\pm w + au^2 + c'v^2; c' \in \mathbb{R}\}$. Thus the tangent space of V at $\overline{g}_{c'}$ is $T_{\overline{g}_{c'}}V = \mathbb{R}\cdot\{v^2\}$.

Note that

$$\begin{aligned}
\theta_1\overline{g}_{c'} &= 4au^2 + 6c'vw - 4c'u^2v;\\
\theta_2\overline{g}_{c'} &= (12a \mp 1)uv + 16c'vw - 4c'u2v;\\
\theta_3\overline{g}_{c'} &= 32auw - 8au^3 - 16c'uv^2 \mp 3v^2.
\end{aligned}$$

Therefore, the tangent space of the $\mathcal{R}^{(2)}(X)$-orbit of $\overline{g}_{c'}$ is given by

$$\begin{aligned} L\mathcal{R}^{(2)}(X)\cdot\overline{g}_{c'} &= J^2(\langle 4au^2+6c'vw-4c'u^2v,(12a\mp 1)uv+16c'vw,\\ &\quad 32auw\mp 3v^2\rangle)\\ &= J^2(\langle au^2\pm w, uw, vw, w^2, v^2, (12a\mp 1)uv\rangle) \end{aligned}$$

which consist of the 2-jets of elements in the space generated by $au^2\pm w, uw, vw, w^2, v^2, (12a\mp 1)uv$ and it is independent of c'.

Hence, $T_{\overline{g}_{c'}}V\subset L\mathcal{R}^{(2)}(X)\cdot\overline{g}_{c'}$. Using Mather's Lemma, we conclude that $\pm w+au^2+c'v^2$ is $\mathcal{R}(X)$-equivalent to $\pm w+au^2$.

Consider $f(u,v,w)=\pm w+au^2$, with $a\neq 0$. Then

$$L\mathcal{R}(X)\cdot f=\mathcal{E}_3\cdot\{au^2\pm w,(12a\mp 1)uv, 32auw-8au^3\mp 3v^2\}.$$

A complete 3-transversal is given by

$$T=\begin{cases}\mathbb{R}\cdot\{u^3\} & \text{if}\quad a\neq\pm\frac{1}{12}\\ \mathbb{R}\cdot\{u^3,u^2v\} & \text{if}\quad a=\pm\frac{1}{12}\end{cases}.$$

Therefore, when $a\neq 0,\pm\frac{1}{12}$, any 3-jet with 2-jet equal to $\pm w+au^2$ is $\mathcal{R}_1(X)$-equivalent to $\pm w+au^2+bu^3$, $b\in\mathbb{R}$.

For $\overline{f}(u,v,w)=\pm w+au^2+bu^3$, with $a\neq 0,\pm\frac{1}{12}$, we have

$$\mathcal{M}_3^4\subset L\mathcal{R}_1(X)\cdot\overline{f}+\mathcal{M}_3^5,$$

if and only if $a\neq\pm\frac{1}{4}$, that is, by Proposition 2.4, $\overline{f}$ is 3-$\mathcal{R}(X)$-determined if and only if $a\neq\pm\frac{1}{4}$.

Furthermore,

$$\frac{\mathcal{M}_3}{L\mathcal{R}(X)\cdot\overline{f}}=\begin{cases}\mathbb{R}\cdot\{u,v,u^2,u^3\} & \text{if}\quad b\neq 0\\ \mathbb{R}\cdot\{u,v,u^2,u^3,v^2\} & \text{if}\quad b=0\end{cases}$$

which implies that the $\mathcal{R}_e^+(X)$-codimension of the stratum of the singularity of $\overline{f}$ is 2 if $b\neq 0$ and 4 if $b=0$.

When $a=0$, any $\mathcal{R}(X)$-finitely determined germ in $\mathcal{E}_3$ with 2-jet $\mathcal{R}(X)$-equivalent to $\pm w+buv+cv^2$ has $\mathcal{R}_e^+(X)$-codimension >2. □

Theorem 3.6. *Let X be the swallowtail parameterised by $f(x,y) = (x, -4y^3 - 2xy, 3y^4 + xy^2)$. Denote by (u,v,w) the coordinates in the target. Then any germ $g : (\mathbb{R}^3,0) \to (\mathbb{R},0)$ of an $\mathcal{R}(X)$-finitely determined submersion with $\mathcal{R}_e^+(X)$-codimension ≤ 2 of the stratum in the presence of moduli is $\mathcal{R}(X)$-equivalent to one of the germs in Table 3.1.*

Table 3.1: $\mathcal{R}_e^+(X)$-codimension ≤ 2 germs of submersions.

Normal form	$\mathrm{cod}(\mathrm{f}, \mathcal{R}_\mathrm{e}^+(\mathrm{X}))$	**$\mathcal{R}^+(X)$-versal deformation**
$\pm u$	0	$\pm u$
$v + au^2, a \neq 0$	1	$v + au^2 + a_1 u$
$v + au^3, a \neq 0$	2	$v + au^3 + a_1 u + a_2 u^2$
$\pm w + au^2 + bu^3, a \neq 0, \pm\frac{1}{12}, \pm\frac{1}{4}; b \neq 0$	2	$\pm w + au^2 + bu^3 + a_1 u + a_2 v$

Proof. The proof follows from Proposition 3.2 and Lemmas 3.3, 3.4, 3.5. □

Remark 3.7. The $\mathcal{K}(X)$-classification of germs of submersions $(\mathbb{R}^3,0) \to (\mathbb{R},0)$ of $\mathcal{K}_e(X)$-codimension ≤ 2 can be obtained from Theorem 3.6 by setting $a = \pm 1$. Furthermore, we observe that if we are interested in the fibers of these submersions, then both classifications can be used, since the fibers will be diffeomorphic.

In [33] they used this classification to obtain a classification of simple bigerms from $\mathbb{R}^3$ to $\mathbb{R}^3$ where one branch is a swallowtail and the other is a folding plane.

4 The geometry of functions on a swallowtail

The standard swallowtail has equation $16u^4w - 4u^3v^2 - 128u^2w^2 + 144uv^2w - 27v^4 + 256w^3 = 0$. By Shafarevich [40], if X is an irreducible affine variety in $\mathbb{R}^n$ defined by the ideal I then the equations of the tangent cone of X are the lowest degree terms of the polynomials in I. Therefore, the tangent cone to the standard swallowtail is the repeated plane $w^3 = 0$.

The tangential line of the standard swallowtail at the origin is the line with direction $(1,0,0)$ passing through the origin. The germ $f(x,y) = (x, -4y^3 - 2xy, 3y^4 + xy^2)$ is singular along a curve Σ parametrised by $\alpha(t) = f(-6t^2, t) = (-6t^2, 8t^3, -3t^4)$. Furthermore f has a double point curve Υ parametrised by $\beta(t) = f(-2t^2, t) = (-2t^2, 0, t^4)$ which ends at the swallowtail point. See Figure 1.1.

We study here the discriminants of the singularities given in Theorem 3.6. Let $g : (\mathbb{R}^3, 0) \to (\mathbb{R}, 0)$ be a germ on $X = f(\mathbb{R}^2, 0)$ and $F : (\mathbb{R}^3 \times \mathbb{R}^2, (0,0)) \to (\mathbb{R}, 0)$ be a deformation of g. We consider the families $G(x, y, a_1, a_2) = F(f(x, y), a_1, a_2)$, $H_1(t, a_1, a_2) = F(\alpha(t), a_1, a_2)$ and $H_2(t, a_1, a_2) = F(\beta(t), a_1, a_2)$.

The discriminant of the family G is the set

$$\mathcal{D}_1(F) = \{(a_1, a_2, G(x, y, a_1, a_2)) \in \mathbb{R}^2 \times \mathbb{R};\ \frac{\partial G}{\partial x} = \frac{\partial G}{\partial y} = 0 \ at\ (x, y, a_1, a_2)\},$$

the discriminant of the family G restricted to the singular curve Σ is given by

$$\mathcal{D}_2(F) = \{(a_1, a_2, H_1(t, a_1, a_2)) \in \mathbb{R}^2 \times \mathbb{R};\ \frac{\partial H_1}{\partial t} = 0 \ at\ (t, a_1, a_2)\}$$

and the discriminant of the family G restricted to the double point curve Υ is the set

$$\mathcal{D}_3(F) = \{(a_1, a_2, H_2(t, a_1, a_2)) \in \mathbb{R}^2 \times \mathbb{R};\ \frac{\partial H_2}{\partial t} = 0 \ at\ (t, a_1, a_2)\}.$$

If F_1 and F_2 are two P-$\mathcal{R}^+(X)$-equivalent deformations of a germ g, then it is not difficult to show that the sets $\mathcal{D}_i(F_1)$ and $\mathcal{D}_i(F_2)$ are diffeomorphic for $i = 1, 2, 3$. Therefore, it is enough to compute the sets $\mathcal{D}_i(F)$ for the deformations given in Theorem 3.6.

- The case $g(u, v, w) = \pm u$.

In this case, an $\mathcal{R}^+(X)$-versal deformation of g is $F(u, v, w, a_1, a_2) = \pm u$. Then the other families are

$$G(x, y, a_1, a_2) = \pm x \qquad H_1(t, a_1, a_2) = \mp 6t^2 \qquad H_2(t, a_1, a_2) = \mp 2t^2.$$

Hence $\mathcal{D}_1(F)$ is the empty set and $\mathcal{D}_2(F) = \mathcal{D}_3(F)$ is a plane.

Here, the fiber $g = 0$ is a plane transverse to both the tangential line and the tangent cone of X.

- The case $g(u, v, w) = v + au^2, a \neq 0$.

In this case, an $\mathcal{R}^+(X)$-versal deformation is $F(u, v, w, a_1, a_2) = v + au^2 + a_1u$. Then the other families are

$$G(x, y, a_1, a_2) = -4y^3 - 2xy + ax^2 + a_1x,$$

$$H_1(t, a_1, a_2) = 8t^3 + 36at^4 - 6a_1t^2,$$

$$H_2(t, a_1, a_2) = 4at^4 - 2a_1t^2.$$

Note that H_2 is a versal deformation of the boundary B_2-singularity in the terminology of [2]. We have

$$\mathcal{D}_1(F) = \{(2y + 12ay^2, a_2, -4y^3 - 36ay^4)\},$$

$$\mathcal{D}_2(F) = \{(a_1, a_2, 0)\} \cup \{(2t + 12at^2, a_2, -4t^3 - 36at^4)\},$$

$$\mathcal{D}_3(F) = \{(a_1, a_2, 0)\} \cup \{(4at^2, a_2, -4at^4)\}.$$

These discriminants are illustrated in the Figure 4.1.

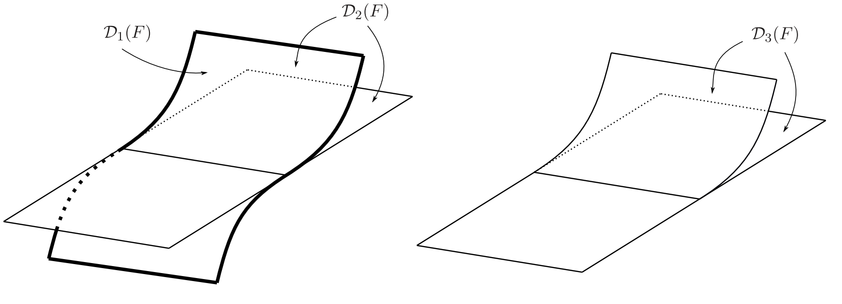

Figure 4.1: The discriminants $\mathcal{D}_2(F)$ and its subset $\mathcal{D}_1(F)$ in bold (left) and the discriminant $\mathcal{D}_3(F)$ (right) of $F = v + au^2 + a_1u$.

The tangent plane to the fiber $g = 0$ contains the tangential line and is transverse to the tangent cone of X. The contact of the tangential line with the fiber $g = 0$ is measured by the singularities of $g(f(x,0)) = ax^2$ and is of type A_1.

- The case $g(u,v,w) = v + au^3, a \neq 0$.

In this case, an $\mathcal{R}^+(X)$-versal deformation is $F(u,v,w,a_1,a_2) = v + au^3 + a_1u + a_2u^2$ and the other families are

$$G(x,y,a_1,a_2) = -4y^3 - 2xy + ax^3 + a_1x + a_2x^2,$$

$$H_1(t,a_1,a_2) = 8t^3 - 216at^6 - 6a_1t^2 + 36a_2t^4,$$

$$H_2(t,a_1,a_2) = -8at^6 - 2a_1t^2 + 4a_2t^4.$$

Note that H_2 is a versal deformation of the boundary B_3-singularity in the terminology of [2]. Hence

$$\mathcal{D}_1(F) = \{(2y - 108ay^4 + 12a_2y^2, a_2, -4y^3 + 432ay^6 - 36a_2y^4)\},$$

$$\mathcal{D}_2(F) = \{(a_1,a_2,0)\} \cup \{(2t - 108at^4 + 12a_2t^2, a_2, -4t^3 + 432at^6 - 36a_2t^4)\},$$

$$\mathcal{D}_3(F) = \{(a_1,a_2,0)\} \cup \{(-12at^4 + 4a_2t^2, a_2, 16at^6 - 4a_2t^4)\}.$$

See Figure 4.2.

The second component of the discriminant $\mathcal{D}_3(F)$ is a surface which is singular along the set $\{(0,a_2,0)\} \cup \{(12at^4, 6at^2, -8at^6)\}$. The singularity along $(12at^4, 6at^2, -8at^6)$ is a cuspidal edge when $t \neq 0$.

Here, as in the previous case, the tangent plane to the fiber $g = 0$ contains the tangential line and is transverse to the tangent cone of X. However the contact of the tangential line with the fiber $g = 0$ is measured by the singularities of $g(f(x,0)) = ax^3$ and is of type A_2.

- The case $g(u,v,w) = \pm w + au^2 + bu^3,\ a \neq 0, \pm\frac{1}{12}, \pm\frac{1}{4},\ b \neq 0$.

In this case, an $\mathcal{R}^+(X)$-versal deformation is $F(u,v,w,a_1,a_2) = \pm w + au^2 + bu^3 + a_1u + a_2v$, and the other families are

$$G(x,y,a_1,a_2) = \pm 3y^4 \pm xy^2 + ax^2 + bx^3 + a_1x - 4a_2y^3 - 2a_2xy,$$

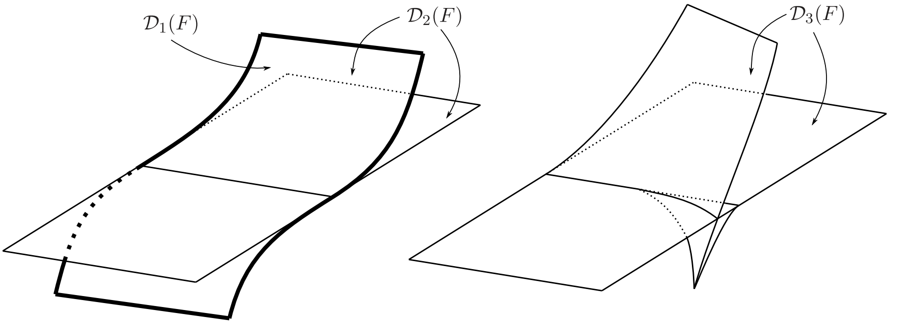

Figure 4.2: The discriminants $\mathcal{D}_2(F)$ and its subset $\mathcal{D}_1(F)$ in bold (left) and the discriminant $\mathcal{D}_3(F)$ (right) of $F = v + au^3 + a_1u + a_2u^2$.

$$H_1(t, a_1, a_2) = \mp 3t^4 + 36at^4 - 216bt^6 - 6a_1t^2 + 8a_2t^3,$$

$$H_2(t, a_1, a_2) = \pm t^4 + 4at^4 - 8bt^6 - 2a_1t^2.$$

Therefore, the discriminant $\mathcal{D}_1(F)$ is the union of two surfaces $S1, S2$, with $S1$ parametrised by

$$(x, y) \mapsto (\pm y^2 - 2ax - 3bx^2, \pm y, \mp y^4 - ax^2 - 2bx^3)$$

and $S2$ parametrised by

$$(a_2, t) \mapsto (\mp t^2 + 12at^2 - 108bt^4 + 2a_2t, a_2, \pm 3t^4 - 36at^4 + 432bt^6 - 4a_2t^3).$$

The first surface $S1$ is regular and its tangent plane at the origin is $w = 0$. The second surface $S2$ is singular along the curve parametrised by $(\pm t^2 - 12at^2 + 324bt^4, \pm t - 12at + 216bt^3, \mp t^4 + 12at^4 - 432bt^6)$. Using Corollary 1.5 in [17] we prove that S_2 is a *cuspidal cross cap* (that is, it is $\mathcal{A}$-equivalent to the surface parametrised by (x, y^2, xy^3)).

The intersection between these two components S_1 and S_2 is a plane curve with a Z_{17}-singularity if $a = \mp\frac{1}{18}$ (that is, it is $\mathcal{R}$-equivalent to $x^3y + y^8 + \lambda xy^6$ for some $\lambda \in \mathbb{R}$) and a Z_{13}-singularity otherwise (that is, it is $\mathcal{R}$-equivalent to $x^3y + y^6 + \lambda xy^5$ for some $\lambda \in \mathbb{R}$) . Therefore, this intersection is the image by the parametrisation of the first component of two curves, which are, up to diffeomorphisms, a line ($y = 0$) and the zero-fiber of an E_{12} singularity $(x^3+y^7+\delta xy^5=0)$ if $a=\mp\frac{1}{18}$ and a line $(y=0)$ and the zero-fiber of an E_8 singularity $(x^3+y^5=0)$ otherwise.

The discriminant $\mathcal{D}_2(F)$ is the union of the plane $\{(a_1,a_2,0)\}$ and the surface S_2 of $\mathcal{D}_1(F)$.

Finally, $\mathcal{D}_3(F)=\{(a_1,a_2,0)\}\cup\{(\pm t^2+4at^2-12bt^4,a_2,\mp t^4-4at^4+16bt^6)\}$.

The discriminants $\mathcal{D}_2(F)$ and $\mathcal{D}_3(F)$ are illustrated in the Figure 4.3.

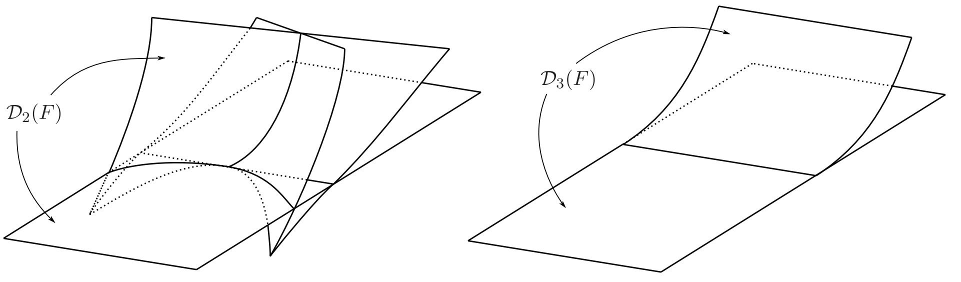

Figure 4.3: The discriminant $\mathcal{D}_2(F)$ (left) and the discriminant $\mathcal{D}_3(F)$ (right) of $F=\pm w+au^2+bu^3+a_1u+a_2v$.

The tangent plane to the fiber $g=0$ coincides with the tangent cone of the swallowtail at the origin. The contact of the tangential line with the fiber $g=0$ is measured by the singularities of $g(f(x,0))=ax^2+bx^3$ and is of type A_1.

5 The flat geometry of a swallowtail

We use here the classification in §3 to study the flat geometry of a geometric swallowtail M. The flat geometry is captured by the contact of the geometric swallowtail M with planes and is measured by the singularities of the height function $h_\nu(p)=p\cdot\nu$, with $\nu\in S^2$ orthogonal to the given plane. Varying ν locally in S^2 gives the family of height functions $H:M\times S^2\to\mathbb{R}$, given by $H(p,\nu)=h_\nu(p)$.

Let g be a parametrisation of a geometric swallowtail. Then g is $\mathcal{A}$-equivalent to f (the parametrisation of the standard swallowtail). That is, there exist germs of diffeomorphisms $\phi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ and $\psi : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ such that $g \circ \phi = \psi \circ f$.

We want to study the contact between the geometric swallowtail $\psi(X)$ and the plane $h_\nu^{-1}(0)$ for some $\nu \in S^2$. This contact is measured by the singularities of the function $h_\nu \circ g : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0)$, but these singularities are the same as those of the function $h_\nu \circ g \circ \phi = h_\nu \circ \psi \circ f$, which in turn measure the contact between the standard swallowtail $X = f(\mathbb{R}^2, 0)$ and the surface $(h_\nu \circ \psi)^{-1}(0)$.

Note that if there exist another germs of diffeomorphisms ϕ_1 and ψ_1 such that $g \circ \phi_1 = \psi_1 \circ f$, then $h_\nu \circ \psi_1 = h_\nu \circ \psi \circ (\psi^{-1} \circ \psi_1)$ and $\psi^{-1} \circ \psi_1(X) = \psi^{-1} \circ \psi_1 \circ f(\mathbb{R}^2, 0) = \psi^{-1} \circ g \circ \phi_1(\mathbb{R}^2, 0) = f \circ \phi^{-1} \circ \phi_1(\mathbb{R}^2, 0) = X$. The germ $\psi^{-1} \circ \psi_1$ is a germ of diffeomorphism which preserves the standard swallowtail X, that is, $\psi^{-1} \circ \psi_1 \in \mathcal{R}(X)$. Therefore, the function $h_\nu \circ \psi$ is well defined up to elements in $\mathcal{R}(X)$ (see [10]).

Following the transversality theorem in the Appendix of [10], for a generic swallowtail, the height functions h_ν, for any $\nu \in S^2$, can only have singularities of $\mathcal{R}_e^+(X)$-codimension ≤ 2 at the origin. Furthermore, as the height function $h_\nu : (\mathbb{R}^3, 0) \to (\mathbb{R}, 0)$ is a submersion, the function $h_v \circ \psi$ is also a submersion. Therefore $h_\nu \circ \psi$ is $\mathcal{R}(X)$-equivalent to one of the normal forms given in Theorem 3.6, that is, there exist a germ of diffeomorphism $\varphi : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ which preserves the standard swallowtail X such that $h_\nu \circ \psi = \widehat{g} \circ \varphi$, where $\widehat{g}$ is one of the normal forms given in Theorem 3.6. Hence the contact between a geometric swallowtail $\psi(X)$ and the plane $h_\nu^{-1}(0)$ coincide with the contact of the standard swallowtail X and the fiber $\widehat{g}^{-1}(0)$ (which is measured by the singularities of the function $\widehat{g} \circ f$).

We have the following consequences about the flat geometry of a generic swallowtail, where tangent/transverse to the swallowtail (resp. singular curve and double point curve) means tangent/transverse to its tangent cone (resp. the tangential line).

Proposition 5.1. *The possible singularities of $\widehat{g} \circ f$ have the following geometric interpretations:*

(i) $\pm u$ *: the corresponding plane is transverse to both the swallowtail,* *its singular curve and its double point curve. In this case the height function is regular;*

(ii) $v + au^2$ *: the plane is transverse to the swallowtail and is in the pencil of planes obtained as limiting tangents to the double point curve (which coincide with that of the singular curve). This submersion yields an* A_1*-singularity of the height function;*

(iii) $v + au^3$ *: the plane is transverse to the swallowtail and is in the pencil of planes obtained as limiting tangents to the singular curve and is the limiting osculating plane to the double point curve. This submersion yields an* A_1*-singularity of the height function;*

(iv) $\pm w + au^2 + bu^3$*: the plane is the tangent cone of the swallowtail. This submersion yields an* A_3*-singularity of the height function.*

Proof. The proof follows form the analysis made in §4 for each case. $\square$

We can compare these results with Theorem 2.11 in [25]. For example, in the (iv) case, the singularity of the height function is degenerate, ν is binormal but the curvature parabola is given by $(0, y, 1)$ (a line) and the umbilic curvature is 1 (non zero), so the singularity is not of corank 2.

Consider a generic swallowtail with a parametrisation g and let λ and γ be parametrisations of its singular curve and its double point curve, respectively. For the family of height functions H we define

$$\mathcal{D}_1(H) = \{(\nu, h_\nu \circ g(x,y)) \in S^2 \times \mathbb{R}\,;\, \frac{\partial h_\nu \circ g}{\partial x} = \frac{\partial h_\nu \circ g}{\partial y} = 0 \text{ at } (x,y,\nu)\};$$

$$\mathcal{D}_2(H) = \{(\nu, h_\nu \circ \lambda(t)) \in S^2 \times \mathbb{R}\,;\, \frac{\partial h_\nu \circ \lambda}{\partial t} = 0 \text{ at } (t,\nu)\};$$

$$\mathcal{D}_3(H) = \{(\nu, h_\nu \circ \gamma(t)) \in S^2 \times \mathbb{R}\,;\, \frac{\partial h_\nu \circ \gamma}{\partial t} = 0 \text{ at } (t,\nu)\}.$$

The sets $\mathcal{D}_1(H), \mathcal{D}_2(H)$ and $\mathcal{D}_3(H)$ corresponds to the duals of the swallowtail, the singular curve and the double point curve, respectively.

As discussed in the beginning of §5, the contact between a swallowtail and a plane $h_\nu^{-1}(0)$ is described by that of the fiber $\widehat{g}=0$ with the standard swallowtail, with $\widehat{g}$ as in Theorem 3.6. Using this fact we can show that $\mathcal{D}_i(H)$ is diffeomorphic to $\mathcal{D}_i(F)$, for $i=1,2,3$, where F is an $\mathcal{R}^+(X)$-versal deformation of $\widehat{g}$ with 2-parameters. Therefore, the calculations and figures in §4 give models, up to diffeomorphisms, of $\mathcal{D}_i(H)$ for $i=1,2,3$.

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